Orbit equivalence, flow equivalence and ordered cohomology

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ABSTRACT. We study self-homeomorphisms of zero dimensional metrizable compact Hausdorff spaces by means of the ordered first cohomology group, particularly in the light of the recent work of Giordano, Putnam, and Skau on minimal homeomorphisms. We show that flow equivalence of systems is analogous to Morita equivalence between algebras, and this is reflected in the ordered cohomology group. We show that the ordered cohomology group is a complete invariant for flow equivalence between irreducible shifts of finite type; it follows that orbit equivalence implies flow equivalence for this class of systems. The cohomology group is the (pre-ordered) Grothendieck group of the C*-algebra crossed product, and we can decide when the pre-ordering is an ordering, in terms of dynamical properties.

Let T be a homeomorphism of a compact metrizable zero-dimensional space X. Let $C(X, \mathbf{Z})$ denote the continuous integer-valued functions on X. Denote the subgroup of coboundaries by $\operatorname{cobdy}(T) = \{\, f - (f \circ T) \mid f \in C(X, \mathbf{Z}) \,\}$. The quotient group $C(X, \mathbf{Z})/\operatorname{cobdy}(T)$ will be abbreviated G^T. Define

$$G^T_+ = \left\{ [f] \in G^T \;\middle|\; \text{there exists nonnegative } f_0 : X \to \mathbf{Z} \text{ such that } [f_0] = [f] \right\} .$$

Let $\mathcal{G}^T$ denote the unital preordered group $(G^T, G^T_+, [1])$.

Building on earlier work [**V, P, HPS**], Giordano, Putnam, and Skau [**GPS**] recently proved that the groups $\mathcal{G}^T$ modulo their infinitesimals classify the minimal homeomorphisms of the Cantor set up to orbit equivalence. The possible $\mathcal{G}^T$ in this case are precisely the unital simple dimension groups [**HPS**]. This remarkable classification in the minimal case provides more than ample justification for a general investigation of $\mathcal{G}^T$. This paper is devoted to laying down some early foundations for this investigation.

Systems with a unique minimal set were studied in [**HPS**]. Poon [**Po**] showed $\mathcal{G}^T$ is an unperforated ordered group when T is topologically transitive (but not in general), pointed out that there one can recover the zeta function of T as an invariant of the abstract unital group $\mathcal{G}^T$, and began a study of $\mathcal{G}^T$ when T is an irreducible shift of finite type. It should be emphasized that the order structure is crucial in all of this.

This paper has five sections. In the first, we review some basics of preordered groups and suspensions, show the isomorphism class of (G^T, G^T_+) is determined by the flow equivalence class of T, and in a refinement of Poon's result, show that the set $\mathcal{Z}(T)$ of zeta functions of homeomorphisms flow equivalent to T is an invariant of the abstract unital preordered group $\mathcal{G}^T$ modulo its infinitesimals.

The latter is an invariant of orbit equivalence and therefore so is $\mathcal{Z}(T)$. In the second section we show that when T is an irreducible shift of finite type, $\mathcal{Z}(T)$ is a complete invariant of flow equivalence. This implies our main result (Theorem 1.12), which asserts that for irreducible shifts of finite type, orbit equivalence implies flow equivalence, and the isomorphism class of the ordered group (G^T, G^T_+) is a complete invariant of flow equivalence.

(In particular, the ordered group structure of $\mathcal{G}^T$ already yields new, computable invariants of orbit equivalence for shifts of finite type.) In the section 3, we study the group $\mathcal{G}^T$ in the light of chain recurrence, graphical groups and some elementary zero-dimensional dynamics. In section 4, we measure the failure of the preordered group (G^T, G^T_+) to be ordered by the first cohomology of the gradient-like flow space Conley associated to the suspension flow of T, and consider the relation of the order G^T_+ to the natural "winding order" arising from the identification of G^T with the first cohomology of the suspension space of T. In Section 5 we discuss the identification of $\mathcal{G}^T$ as K_0 of the crossed-product C*-algebra arising from T.

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Section 1. The general framework

1.1 *Suspensions.* Throughout this paper, T represents a homeomorphism of a compact metrizable zero-dimensional space X, and a flow means a continuous action of the reals on a compact metrizable space. If f is a continuous strictly positive function from X to $\mathbf{R}$, then a $\mathbf{Z}$-action on the product space $X \times \mathbf{R}$ is generated by the map $(x, t) \mapsto (Tx, t - f(x))$. The **suspension** of T by f is the quotient space Y under the map π which collapses the orbits of this $\mathbf{Z}$ action to points. The natural action of the reals on $X \times \mathbf{R}$ given by translation in the $\mathbf{R}$ coordinate pushes down to a real flow on the suspension space. We denote this real action by $\alpha^{f,T}$ or α^T or α, suppressing notation as permitted by context; so, t in $\mathbf{R}$ acts by $[(x, s)] = y \mapsto \alpha_t(y) = [(x, s + t)]$. The flow $\alpha^{f,T}$ is also called the flow under the function f with base map T. If f is the constant function 1, then the suspension space is called the **standard suspension** of T, and we denote it Y_T. For concreteness, we will identify Y_T with the quotient of $\{\, (x, s) \mid x \in X, 0 \le s \le 1 \,\}$ under the identifications $[(x, 1)] = [(Tx, 0)]$.

A **cross section** to a flow α is a closed set C such that $\alpha\colon C \times \mathbf{R} \to Y$ is a local homeomorphism onto Y [**Sch**]. It follows that if C is a cross section, then every orbit hits C in forward time and in backward time, the return time r_C is continuous and strictly positive on C, and the return map R_C is a homeomorphism from C to C. We say that a homeomorphism T is a section to a flow if T is isomorphic (i.e., topologically conjugate) to R_C for some cross section C. For example, T is a section to its standard suspension flow. An **equivalence** between two flows is a homeomorphism whose restriction to any orbit is an orientation-preserving homeomorphism onto some orbit of the range flow. For example, all the suspensions of T are equivalent. A **semiequivalence** of flows is a continuous surjection whose restriction to any domain orbit is an orientation preserving local homeomorphism onto some orbit of the range flow. If α is a flow on Y, then a subset C of Y is a cross section if and only if there exists a semiequivalence ϕ onto the unit speed counterclockwise flow on the circle such that $C = \phi^{-1}(1)$ [**Sch**; 7]. It follows by considering compositions that under any semiequivalence, the inverse image of a cross section is a cross section. Two homeomorphisms are **flow equivalent** if they are sections to a common flow (equivalently, if one is a section to a suspension of the other). If S and T are flow equivalent, then S is a section to every flow to which T is a section.

1.2 *Preordered groups.* A **preordered group** is a pair (G, G_+), where G is an abelian group and the **positive cone** G_+ is a submonoid of G which generates G (i.e., G_+ is a subsemigroup containing 0, and every element of G is a difference of elements in G_+). All groups G in this paper will be countable. The preordered group is an **ordered group** if in addition $G_+ \cap -G_+ = \{0\}$. If (G, G_+) is a preordered group, we set $J \equiv J(G) := (G_+ \cap -G_+)$; then J is a subgroup and $(G/J, G_+ + J)$ is an ordered group; by abuse of notation, we often denote this $(G, G_+)/J$. An **order unit** in a preordered group is an element u of G_+ such that for all g in G, there exists n in $\mathbf{N}$ such that $(nu - g) \in G_+$. A **unital preordered group** is a triple (G, G_+, u) where (G, G_+) is a preordered group and u is an order unit. An isomorphism of preordered unital groups is an isomorphism of groups under which the positive sets and distinguished order units correspond.

1.3 *Ordered cohomology.* Let $C(X, \mathbf{Z})$ denote the continuous maps from X into the integers. Let cobdy$\,(T)$ denote $\{\, f - f \circ T \mid f \in C(X, \mathbf{Z}) \,\}$, the group of coboundaries of T in the group $C(X, \mathbf{Z})$; it is the range of the operator $\mathrm{id} - T$:

$C(X,\mathbf{Z}) \to C(X,\mathbf{Z})$. Let G^T denote the quotient group $C(X,\mathbf{Z})/\mathrm{cobdy}\,(T)$, which we make a preordered group by setting

$$G^T_+ = \left\{ [f] \in G^T \;\middle|\; \text{there exists nonnegative } f_0 : X \to \mathbf{Z} \text{ such that } [f_0] = [f] \right\} .$$

Let $\mathcal{G}^T$ denote the unital preordered group $(G^T, G^T_+, [\mathbf{1}])$, where $\mathbf{1}$ denotes the function on X that is identically 1.

1.4 *A theorem of Parry and Sullivan.* Suppose S is a self-homeomorphism of a zero dimensional compact metric space W. By a discrete suspension of S (or a finite tower over S) we will mean a homeomorphism S_f constructed as follows from a continuous map f from W into the positive integers. Let $C_j = \{w \mid f(w) = j\}$. Let $W_f = \bigcup_{0\le i<j} (C_j \times \{i\})$. Let $S_f \colon (x,i) \mapsto (x, i+1)$ if $(i+1) < f(x)$, otherwise let $S_f \colon (x, f(x)-1) \mapsto (Sx, 0)$. The **base** of the discrete suspension is the set $W \times \{0\}$. A clopen set C is a **discrete cross section** for the homeomorphism T if there is isomorphism (topological conjugacy) of T to a discrete suspension under which the image of C is the base. Equivalently, C is a clopen set finitely many iterates of which cover the entire domain X. Equivalently, a set C is a discrete cross section for T iff there exists an order unit $[f]$ in G^T, with $f \ge 0$, for which $C = \{x \mid f(x) > 0\}$.

In [**PS**], Parry and Sullivan proved among other things the following result. (For an alternative approach, see [**PT**].)

Theorem. *([**PS**]) Suppose C_1 and C_2 are cross sections to some suspension flow of a homeomorphism on a zero dimensional compact metrizable space. Let T_1 and T_2 denote their respective return map homeomorphisms. Then there exists a third cross section C_3, with return map homeomorphism T_3, such that T_1 and T_2 are isomorphic to discrete suspensions of T_3.*

Theorem 1.5. *The isomorphism class of the preordered group (G^T, G^T_+) is an invariant of the flow equivalence class of T.*

Proof. By the Parry-Sullivan result, if T_1 and T_2 are flow equivalent, then we can realize them as discrete suspensions over a common base map B. So it suffices to check that when (X,T) is a discrete suspension over a base system (B,S), there is an isomorphism $(G^S, G^S_+) \to (G^T, G^T_+)$. Here B is a subset of X and S is the return map to B under T. Given a function f in $C(B,\mathbf{Z})$, let f' be the function in $C(X,Z)$ which agrees with f on B and is zero elsewhere. We claim that $h \colon [f] \mapsto [f']$ defines the desired isomorphism of preordered groups.

To see this, given g on B let g'' in $C(X, \mathbf{Z})$ be defined by $g''(x) = g(T^i x)$, where $i \equiv i(x)$ is the least nonnegative integer such that $T^i x$ belongs to B. It is easy to check that a coboundary $g - gS$ is sent by h to the coboundary $g'' - g''T$. Therefore h is well defined. The map h is surjective because f in $C(X, \mathbf{Z})$ is cohomologous to the function f''', where f''' is zero off B and for x in B, $f'''(x)$ is the sum of the $f(T^j x)$ as j varies from zero to one less than the first return time to the base. This correspondence also shows that h maps G_+^S onto G_+^T. Finally, if f belongs to $C(B, \mathbf{Z})$ and f' is a coboundary $k - kT$, then because f' vanishes off the base, it must be that $k = g''$ for some g in $C(B, \mathbf{Z})$. Then $f = g - gT$ and it follows that h is injective, and hence an isomorphism of preordered groups. []

Note that the isomorphism above does not respect the distinguished order unit. In general, the **unital** preordered group $\mathcal{G}^T$ is **not** an invariant of flow equivalence.

1.6 *Measures and traces.* A **positive** homomorphism of ordered groups is a homomorphism which sends the positive cone of the domain into the positive set of the range. For the additive groups $\mathbf{R}$ and $\mathbf{Z}$ we always use the usual positive cones. A **trace** (often called a **state**) on a unital preordered group is a positive homomorphism into the reals which sends the distinguished order unit to 1. For the unital preordered group $(G^T, G_+^T, [\mathbf{1}])$, there is a well known bijection between the set of traces and the elements of M_T, the compact metrizable space of T-invariant Borel probabilities on X, as follows. It is easy to see that an element of M_T defines a trace by integration, since by T-invariance the integral of a coboundary is zero. Conversely, a trace τ defines a nonnegative, additive functional on the algebra of clopen sets which sends X to 1, and on general principles [**Roy**] this functional extends to a measure if within that algebra it is countably additive. This last is trivial here, because the only way a compact set can be a countable union of disjoint nonempty closed open sets is by being a finite union. (For more on traces on preordered groups (there called states), see the book [**G**].) If μ is an invariant measure, then we can define a positive linear functional $\tau : C(X) \to \mathbf{C}$ in the usual way ($\tau(f) = \int_X f\, d\mu$ where $C(X)$ denotes the continuous complex valued functions on X), as well as the corresponding trace, also called τ, from G^T to the reals.

By a **discrete** homomorphism, we will mean a homomorphism into the reals whose image is a discrete subgroup of the reals (c.f., [**GH**]). An **extremal** trace (also known as a **pure** trace) is a trace which is not a nontrivial convex combination of other traces. Thus for $(G^T, G^T_+, [\mathbf{1}])$, an extremal trace corresponds to an extreme point in the compact convex space of T-invariant Borel probabilities, i.e., to an ergodic measure. Thus the discrete extremal traces are in one-to-one correspondence with the finite orbits of T. The range of such a trace is $(1/n)\mathbf{Z}$, where n is the cardinality of the orbit. ¿From this observation, Poon [**Po**] recovered the zeta function of T as an invariant of $\mathcal{G}^T$.

Instead of normalizing the discrete homomorphisms on (G^T, G^T_+) by requiring them to be 1 on a special order unit, we can get a unit-independent normalization by requiring the range to be $\mathbf{Z}$. We let E_T denote the set of extremal positive discrete homomorphisms from (G^T, G^T_+) into the reals whose image is the group $\mathbf{Z}$. Now these are again in bijective correspondence with the finite orbits of T. Explicitly, the correspondence is that a finite orbit $\mathbf{O}$ determines the homomorphism $\beta_{\mathbf{O}}\colon [f] \mapsto \sum_{x\in\mathbf{O}} f(x)$.

1.7 *Order units and measures.* Suppose that $[f]$ belongs to G^T. We show that the following are equivalent:

(a) $\tau([f]) > 0$ for every trace τ, i.e., f has strictly positive integral against every element of M_T.

(b) $[f]$ is an order unit.

(c) For some positive integer N, the function $S_N f := f + f\circ T + \cdots + F\circ T^{N-1}$ is strictly positive.

Proof. (a) $\implies$ (c) Define

$$X' = \left\{ x \in X \;\middle|\; \text{for all } i \text{ and } k \text{ with } i \le k,\ \sum_{j=i}^{k} f\left(T^j(x)\right) \le 0 \right\}.$$

The set X' is closed and T-invariant. If no $S_N f$ is strictly positive, then by compactness, X' is nonempty. Hence X' supports an invariant measure; this measure determines a trace τ such that $\tau([f]) \le 0$, and therefore $[f]$ cannot be an order unit (if N is a positive integer and $(Nf - 1) \in G^T_+$, then $\tau(f) \ge 1/N$).
(c) $\implies$ (b) Suppose $S_N f$ is strictly positive; as it is i Thus $[f]$ is an order unit.
(b) $\implies$ (a) If $n[f] \ge [1]$, then for any trace τ, $\tau([f]) \ge 1/n$. $\square$

(More generally, in any unperforated unital partially ordered group, an element which is strictly positive at every trace is an order unit [**GH1**; Theorem 4.1].)

1.8 *Zeta functions.* The (Artin-Mazur) zeta function of a homeomorphism T can be defined by $\zeta_T(z) = \prod_{\mathbf{O}} [1 - z^{|\mathbf{O}|}]^{-1}$, where the product is over the finite orbits

$\mathbf{O}$, and $|\mathbf{O}|$ is the cardinality of $\mathbf{O}$. This makes sense as a formal power series if for every positive integer n, T has only finitely many orbits of length (cardinality) n. If the growth rate of the periodic orbits is no larger than exponential, then this zeta function may also be regarded as an analytic function on some neighborhood of the origin. It is easy to see that if T has a well defined zeta function, then so does every homeomorphism flow equivalent to T. In this case we define $\mathcal{Z}(T)$ as the set of all zeta functions of homeomorphisms flow equivalent to T; otherwise $\mathcal{Z}(T)$ is the empty set.

Theorem 1.8. *The set $\mathcal{Z}(T)$ is an invariant of the abstract preordered group (G^T, G^T_+).*

To prove this theorem, we define the zeta function of an order unit u in a preordered group G with E the set of discrete extremal positive homomorphisms onto $\mathbf{Z}$ by means of $\zeta_u(z) := \prod_{\beta \in E}[1 - z^{\beta(u)}]^{-1}$. As u is an order unit, $\beta(u) \geq 1$; so the formula has a chance of making sense. This zeta function is well defined for u provided that for every $n \in \mathbf{N}$, there exist only finitely many β in E such that $\beta(u) < n$. For our groups, as we have seen each $\beta = \beta_{\mathbf{O}}$ for some finite orbit $\mathbf{O}$ for T, where $\beta_{\mathbf{O}}([f])$ is the sum of the values of f over the points of $\mathbf{O}$. We have also seen that for any order unit u, there exists a positive integer N such that the image under any trace on $(G^T, G^T_+, [\mathbf{1}])$ is at least $1/N$. This implies $\beta_{\mathbf{O}}(u) \geq (|\mathbf{O}|/N)$. Thus if T has a well defined zeta function, then every order unit in G^T has a well defined zeta function.

Now note that if f is the constant function 1, then $\zeta_{[f]} = \zeta_T$.

More generally, given an order unit $[f]$ in G^T with $f \geq 0$, consider the discrete cross section to T, $W = \{x \mid f(x) > 0\}$. Let S be the return map homeomorphism of W. Build the discrete suspension S_f of S using the function f on W. Then the zeta function of $[f]$ (as an order unit in $\mathcal{G}^T$) equals the zeta function of the homeomorphism S_f. Conversely, the Parry-Sullivan result shows that if we begin with the return map R to a cross section of the suspension flow, then we can find a discrete cross section W in X, with return homeomorphism S, and a positive function f on W, such that R is isomorphic to S_f. If we extend this f to X by declaring it to be zero off of W, then $\zeta_R = \zeta_{[f]}$. This shows that the zeta functions of flow equivalent homeomorphisms are the same as the zeta functions of order units in the abstract preordered group (G^T, G^T_+). So we can write $\mathcal{Z}(T) = \mathcal{Z}((G^T, G^T_+))$.

1.9 *Infinitesimals and orbit equivalence.* Let $\mathcal{G} = (G, G_+, u)$ be a preordered unital group. An **infinitesimal** in (G, G_+) is an element g of G such that for every integer n, $ng \leq u$. Note that g satisfies this condition for one order unit iff g satisfies this condition for every order unit. If g is an infinitesimal, then $u + g$ is still an order unit. The infinitesimals form a subgroup $\mathrm{Inf} \equiv \mathrm{Inf}(\mathcal{G})$, and the quotient $\mathcal{G}/\mathrm{Inf}$ is a well defined unital group. (The subgroup $J = (G_+ \cap -G_+)$ is contained in the infinitesimals.)

An element g is an infinitesimal if and only if it is annihilated by every trace. It is clear that an infinitesimal must be annihilated by every trace. Conversely, if $|\tau(g)| > 1/n$ for some trace τ and positive integer n, then $(u - ng)$ and $(u + ng)$ cannot both be in G_+. Thus any trace τ on $\mathcal{G}^T$ determines a trace on $\mathcal{G}/\mathrm{Inf}$ by the rule $[g] \mapsto \tau(g)$, and this correspondence defines a natural bijection of the trace spaces of $\mathcal{G}$ and $\mathcal{G}/\mathrm{Inf}$. It is also easy to check that g is an order unit in $\mathcal{G}$ if and only if $[g]$ is an order unit in $\mathcal{G}/\mathrm{Inf}$. It follows that the zeta function of an order unit g in (G^T, G^T_+) is equal to the zeta function of the order unit $[g]$ in $(G^T, G^T_+)/\mathrm{Inf}$. In particular the invariant $\mathcal{Z}(T)$ is actually an invariant of $(G^T, G^T_+)/\mathrm{Inf}$.

This discussion also describes natural correspondences for the trace and order unit structures of the ordered group $(G, G_+)/J$ which sits between (G, G_+) and $(G, G_+)/\mathrm{Inf}$. On (G^T, G^T_+), a trace is given by integration, so that the infinitesimals are precisely the elements $[f]$ of G^T that have zero integral with respect to every T-invariant Borel probability.

Two homeomorphisms S and T are **orbit equivalent** if there is some homeomorphism h taking every S-orbit onto a T-orbit. The homeomorphism h is called an orbit equivalence from S to T. Obviously h induces an automorphism, also called h, of the ordered groups of continuous functions into $\mathbf{Z}$. It need not be the case that this automorphism sends $\mathrm{cobdy}\,(S)$ to $\mathrm{cobdy}\,(T)$, and it need not be the case that the ordered groups (G^S, G^S_+) and (G^T, G^T_+) are isomorphic [**GPS**]. However, it is easy to check that h does induce an affine homeomorphism between the spaces of invariant probability measures of S and T respectively. Thus h does send $\mathrm{Inf}(T)$ onto $\mathrm{Inf}(S)$, and therefore induces an isomorphism of unital ordered groups

$$(G^S, G^S_+, [\mathbf{1}])/\mathrm{Inf}(\mathcal{G}^S) \to (G^T, G^T_+, [\mathbf{1}])/\mathrm{Inf}(\mathcal{G}^T).$$

Therefore the isomorphism class of the unital group $(G^T, G^T_+, [\mathbf{1}])/\mathrm{Inf}$ is an invariant of orbit equivalence for a homeomorphism T of a compact zero dimensional metrizable space. (The remarkable theorem of Giordano, Putnam and Skau [**GPS**] is that for minimal homeomorphisms of the Cantor set, this isomorphism class is a **complete** invariant.)

Corollary 1.9. *Suppose S and T are orbit equivalent homeomorphisms of zero dimensional compact metric spaces. Then $\mathcal{Z}(S) = \mathcal{Z}(T)$.*

1.10 *Shifts of finite type.* Suppose A is the adjacency matrix (over $\mathbf{Z}_+$) of a graph with edge set $\mathcal{E}$. Let X_A be the subspace of $\mathcal{E}^{\mathbf{Z}}$ consisting of the x such that for all integers n, the terminal vertex of $x(n)$ equals the initial vertex of $x(n+1)$. Then with the subspace topology from the product topology, X_A is compact metrizable and the shift map S_A defined by $(S_A x)(n) = x(n+1)$ is a homeomorphism from X_A onto X_A. S_A is the shift of finite type (SFT) defined by A. In general an SFT is any homeomorphism topologically conjugate to some S_A.

An SFT is **irreducible** if it has a dense forward orbit; equivalently, it is topologically conjugate to an SFT S_A defined by an irreducible matrix A over $\mathbf{Z}_+$. We will call an SFT trivial if it consists of a single periodic orbit. If A is irreducible, then S_A is trivial if and only if A is a permutation matrix. It is not difficult to check from the Parry-Sullivan theorem (1.4) that a homeomorphism flow equivalent to an SFT must be an SFT. It is clear that if two SFT's are flow equivalent, either both are irreducible or neither is irreducible. See [**B2**] for a quick introduction to shifts of finite type and further references. Also see [**DGS**], [**PT**], and the forthcoming introductory text [**LM**]. For future reference, we note that the zeta function of the shift of finite type associated to the matrix A is $\zeta_A = \det(\mathrm{I} - zA)$.

1.11 *Flow equivalence.* We have seen that if two homeomorphisms are flow equivalent, then their ordered groups (G^T, G^T_+) are isomorphic. The converse fails. First, the GPS classification shows that for minimal homeomorphisms of the Cantor set, it is not (G^T, G^T_+) but rather $(G^T, G^T_+)/\mathrm{Inf}$ which is invariant under orbit equivalence. However the converse in general still fails grossly even when $\mathrm{Inf} = \{0\}$. ¿From the Parry-Sullivan theorem (1.4) we can deduce that a given T can be flow equivalent to only countably many pairwise nonisomorphic systems, and also that the trichotomy of having zero, finite positive, or infinite entropy is respected by flow equivalence. However, there is an (uncountable) family of homeomorphisms of the Cantor set achieving all entropies in $[0, +\infty]$, all with (G^T, G^T_+) isomorphic to the dyadic rationals [**BH2**].

For shifts of finite type the situation is quite different. We have seen in general that $\mathcal{Z}(T)$ is an invariant of orbit equivalence and obviously it is an invariant of flow equivalence. Suppose $\mathcal{Z}(T)$ is a complete invariant of flow equivalence for homeomorphisms within some collection $\mathcal{C}$. Then in $\mathcal{C}$, orbit equivalence implies flow equivalence (FE), and we also have the implications

$$\text{flow equivalent} \Longrightarrow \text{isomorphic } (G^T, G^T_+) \Longrightarrow \text{same } \mathcal{Z}(T) \Longrightarrow \text{flow equivalent}$$

Thus in $\mathcal{C}$, the preordered group information is equivalent to the flow equivalence information. For irreducible shifts of finite type, we will show in the next section that in fact $\mathcal{Z}(T)$ is a complete invariant of flow equivalence. This will yield our main theorem, which we record now.

Theorem 1.12. *Suppose S and T are irreducible shifts of finite type. Then the following are equivalent.*

(a) S and T are flow equivalent.

(b) $(G^S, G^S_+) \simeq (G^T, G^T_+)$.

(c) $\mathcal{Z}(S) = \mathcal{Z}(T)$.

If S and T are orbit equivalent, then they are flow equivalent.

It should be noted that for an irreducible SFT, the unital ordered group $\mathcal{G}^T$ is the same as the invariant of orbit equivalence $\mathcal{G}^T/\text{Inf}$, since an irreducible shift of finite type has no nonzero infinitesimals (Proposition 3.13).

1.13 *Orbit equivalence of shifts of finite type.* We are especially interested in the orbit equivalence of shifts of finite type for two reasons. First, the SFT case is basic for relating $\mathcal{G}^T$ and orbit equivalence for general T (see section 3, for example). Second, the **unital** ordered group $\mathcal{G}^T$ is a strong and still poorly understood invariant of conjugacy which possesses considerable structure and complexity, and which is not determined in any obvious way by the shift equivalence class. Thus it is not at all unreasonable to suspect that a deeper study might have significant implications for the classification problem.

For irreducible shifts of finite type, there are complete and computable invariants for flow equivalence (which are recalled in section 2); by Theorem 1.12, these are invariants of orbit equivalence. We are aware of just two other computable invariants of orbit equivalence of shifts of finite type. Obviously, an orbit equivalence induces a bijection of finite orbits, and the zeta function is an invariant. For irreducible SFT's, the atomic measures μ_n equidistributed on points in orbits of length n are well known to converge to the unique measure of maximal entropy; therefore an orbit equivalence must take one maximal measure to the other. (We thank B. Weiss for pointing out this easy argument.) In particular, the countable set of numbers which are the measures of closed open sets is an invariant. See [**B1**] for more information on orbit equivalence of SFT's.

However, the zeta function and the range of the maximal measure on clopen subsets do not determine the flow equivalence class; so flow equivalence is a new invariant for orbit equivalence between irreducible shifts of finite type. We illustrate this with an example, for which we need the following proposition (which is of independent interest). For an SFT S, we let $\mathcal{R}(S)$ denote the range of the maximal measure on clopen sets.

Proposition 1.13. *Suppose A is an irreducible matrix over $\mathbf{Z}_+$, with spectral radius λ a positive rational integer. Let l, r denote the positive left(row) and right(column) eigenvectors for λ, each of whose entries are positive rational integers with g.c.d. 1. Let M be the largest positive integer which is relatively prime to λ and divides lr.*

Then

$$\mathcal{R}(S_A) = [0,1] \cap \left\{ (k/M)\lambda^{-n} \;\middle|\; k \in \mathbf{Z}_+ \right\} .$$

Proof. The rows and columns of A are indexed by the set of vertices of the edge set $\mathcal{E}$ used to define the SFT S_A (see section 1.10). Let μ denote the measure of maximal entropy for S_A, and suppose $e_1 \dots e_n$ is an allowed sequence of edges, where e_1 has initial vertex u_1 and e_n has terminal vertex v_n. Then

$$\mu \{ x \mid x_i = e_i, 1 \le i \le n \} = (1/lr) l(u_1) \lambda^{-n} r(v_n).$$

Since any clopen set is a finite union of shifts of such sets, this shows that the left side is contained in the right side. Now we show the converse. As a special case of a theorem of Trow [T], we know that the number M depends only on the topological conjugacy class of S_A. Marcus [M] proved that after passing to a topologically conjugate system, we may assume that the matrix A has all row sums and all column sums equal to λ. In this case, all entries of the eigenvectors l, r must equal one, so $lr = MJ$ for some integer J, and for a point x we must have

$$\mu \{ y \mid y_i = x_i, 1 \le i \le n \} = (1/JM) \lambda^{-n}.$$

For a positive integer n, there must be exactly $JM\lambda^n$ such sets (since the sum of their measures is 1). For $0 \le k \le M\lambda^n$, the union of kJ disjoint such sets has measure $(k/M)\lambda^{-n}$. $\square$

Example. A pair of mixing subshifts of finite type with the same zeta function and the same range of the maximal measure on clopen subsets, which are not orbit equivalent.

Define the following matrices:

$$A = \begin{bmatrix} 9 & 4 & 2 \\ 8 & 3 & 4 \\ 10 & 4 & 1 \end{bmatrix} \qquad A' = \begin{bmatrix} 15 & 4 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 10 & 4 & 1 \\ 5 & 3 & 7 \\ 11 & 4 & 0 \end{bmatrix} \qquad B' = \begin{bmatrix} 15 & 4 & 1 \\ 0 & -1 & 6 \\ 0 & 0 & -1 \end{bmatrix}$$

Then it is not difficult to check that $A = UA'U^{-1}$ and $B = VB'V^{-1}$ for some matrices U and V in $\mathrm{GL}(3, \mathbf{Z})$: adding the first row to the second and third row, then subtracting the second and third columns from the first converts A' to A and B' to B. Thus the mixing subshifts determined by A and B have the same zeta function. For the large eigenvalue (15), A' and B' have the same left and right eigenvectors, specifically $(32\ 8\ 5)$ and $(1\ 0\ 0)^T$. Therefore the inner product of the left and right eigenvectors for A, and for B, is also 32. It follows immediately from Proposition 1.13 that the shifts of finite type corresponding to A and B have the same range of values on clopen sets.

However, $\mathrm{cok}\,(\mathrm{I} - A) \simeq \mathrm{cok}\,(\mathrm{I} - A')$, and the latter is easy to compute as $\mathbf{Z}/7\mathbf{Z} \oplus (\mathbf{Z}/2\mathbf{Z})^3$; also $\mathrm{cok}\,(\mathrm{I} - B) \simeq \mathrm{cok}\,(\mathrm{I} - B')$, and the latter is easily calculated as $\mathbf{Z}/7\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z}$.

Hence the corresponding subshifts are not flow equivalent, and by our Theorem 1.12, they cannot be orbit equivalent. $\square$

Questions. Within the class of irreducible shifts of finite type, we ask three questions.

Is the isomorphism class of $\mathcal{G}^T$ a complete invariant of orbit equivalence? If $\mathcal{G}^S$ is isomorphic to $\mathcal{G}^T$, must S be shift equivalent to T or T^{-1}? If $\mathcal{G}^S$ is isomorphic to $\mathcal{G}^T$, must S be conjugate to T or T^{-1}?

If we only insist that one of S and T are mixing shifts of finite type, then there is a resounding negative answer to the corresponding questions. Specifically, in his thesis, Boyle [**B1**] gave a construction that yields for any non-trivial shift of finite type, a strongly orbit equivalent system that is not expansive (hence not even a subshift). In this case, there results two systems, one a shift of finite type, the other not, with isomorphic unital ordered groups. However, it is still open whether the following is true: if S is a subshift and T is a mixing shift of finite type with $\mathcal{G}^S \simeq \mathcal{G}^T$, then S must be a shift of finite type itself.

Section 2. Zeta functions of flow equivalent shifts of finite type

Recall for a homeomorphism T that $\mathcal{Z}(T)$ denotes the set of all zeta functions of homeomorphisms flow equivalent to T. In this section, we show that if S and T are irreducible SFT's, then they are flow equivalent if and only if $\mathcal{Z}(S) = \mathcal{Z}(T)$. In fact we will show something finer. For an irreducible SFT T, let $\mathcal{Z}_{\mathbf{Z}}(T)$ be the subset of $\mathcal{Z}(T)$ consisting of the zeta functions of mixing SFT's all of whose poles are rational. (These are the zeta functions of the flow equivalent SFT's which are defined by primitive matrices all of whose eigenvalues are rational integers.) We will prove

Theorem 2.1. *If S and T are irreducible SFT's, then they are flow equivalent if and only if $\mathcal{Z}_{\mathbf{Z}}(S) = \mathcal{Z}_{\mathbf{Z}}(T)$.*

An irreducible SFT S is trivial if and only if $\mathcal{Z}_{\mathbf{Z}}(S) = \{(1-z)^{-1}\}$. To prove Theorem 2.1, we have to construct families of flow equivalent nontrivial irreducible SFT's, rich enough that their zeta functions separate the flow equivalence classes. For this we rely on two results. The first (of course) is Franks' classification theorem [**F**]. Two nontrivial irreducible SFT's S_A and S_B, are flow equivalent if and only if $\det(\mathrm{I}-A) = \det(\mathrm{I}-B)$ and the abelian groups $\mathrm{cok}(\mathrm{I}-A)$, $\mathrm{cok}(\mathrm{I}-B)$ are isomorphic (where the matrices are considered as endomorphisms of the integer lattices of the appropriate size). The necessity of these conditions for flow equivalence was established by Parry and Sullivan [**PS**] for the determinant and by Bowen and Franks [**BF**] for the group. Franks [**F**] then invented constructions to show these necessary conditions are sufficient. Note that $|\det(\mathrm{I}-A)|$ above is nonzero if and only if the group $\mathrm{cok}(\mathrm{I}-A)$ is finite, in which case $|\det(\mathrm{I}-A)|$ equals its cardinality. Thus in place of $\det(\mathrm{I}-A)$ we can use its sign, which we take to assume values 0, 1, and -1.

The other major ingredient in the proof is the following realization result [**BH1**]. Suppose B is a square integral matrix all of whose eigenvalues are rational (which entails that they be integers). Then B is algebraically shift equivalent (i.e., shift equivalent by matrices whose entries are integers but need not be nonnegative) to a primitive matrix if and only if the following two necessary conditions hold.

Perron condition: The spectral radius is an eigenvalue of algebraic multiplicity one and all other eigenvalues have smaller absolute value.

Trace condition: For all n, $\mathrm{tr}_n(B) \geq 0$.

Here $\mathrm{tr}_n(B)$ is the nth net trace of B, $\mathrm{tr}_n(B) = \sum_{d|n} \mu(n/d)\mathrm{tr}(B^d)$, where μ is the Möbius function, $\mu\colon \mathbf{N} \to \{-1, 0, 1\}$. If B is nonnegative, then $\mathrm{tr}_n(B)$ is the number of points in orbits of cardinality n for the SFT S_B [**Sm**]. The Perron and trace conditions together will be referred to as the "spectral conditions for primitive realization". Specifying the zeta function of an SFT S_A is equivalent to specifying the list of nonzero eigenvalues of A with algebraic multiplicities. The algebraic shift equivalence class is a finer invariant. If B is algebraically shift equivalent to a matrix A, then $\det(\mathrm{I}-A) = \det(\mathrm{I}-B)$ and $\mathrm{cok}(\mathrm{I}-A) \simeq \mathrm{cok}(\mathrm{I}-B)$. Combining the classification theorem and the realization theorem will give us a more than adequate supply of examples. We remark that it is quite possible to have $\mathcal{Z}_{\mathbf{Z}}(S_A) \subset \mathcal{Z}_{\mathbf{Z}}(S_B)$ with S_A and S_B not flow equivalent; this will occur, for example, if $\mathrm{cok}(\mathrm{I}-A) \simeq \mathbf{Z}^2$, but $\mathrm{cok}(\mathrm{I}-B) \simeq \mathbf{Z}$.

Lemma 2.2. *A primitive matrix A is flow equivalent to a primitive matrix all of whose eigenvalues are integers, if and only if $\det(I-A) \neq 1$.*

Proof. We can assume $A \neq [1]$ and use Franks' classification. It is easy to see that if $\det(\mathrm{I}-B) = 1$ and B has only integer eigenvalues, they must consist of zeroes and an even number of 2's, which cannot be realized by a primitive or even an irreducible matrix.

If $\mathrm{cok}(\mathrm{I}-A) \simeq \mathbf{Z}^k$ with $k > 0$ or if both $k = 0$ and $\det(\mathrm{I}-A) = -1$, realize the shift equivalence class of $\mathrm{diag}(2, 1, \ldots, 1)$ (with k ones appearing). If the torsion part of $\mathrm{cok}(\mathrm{I}-A)$ is not zero, we may write it as $\oplus_i(\mathbf{Z}/n_i\mathbf{Z})$, with $1 < n_1|n_2|\ldots|n_I$. It is routine to verify that for all k, for all sufficiently large t, the list of integers $(1+n_I, 1-n_{I-1}, \ldots, 1-n_1, 1, 1, \ldots, 1, 2, 2, \ldots, 2)$ with k ones and t twos, satisfies the spectral conditions for primitive realization. If there is no torsion-free part, we set $k = 0$ and by altering the parity of t, we can arrange that $(-n_I \cdot (-1)^t)$ matches the sign of $\det(\mathrm{I}-A)$. If the torsion-free part is of rank $k > 0$, then there is no parity condition required, since $\det(\mathrm{I}-A) = 0$. □

(We can realize flow equivalence classes with nonnegative integer eigenvalues if $n_I > n_{I-1}$, since then we can replace all the $1-n_j$ terms by $1+n_j$ and still satisfy the Perron condition.)

Let $\mathcal{C}$ denote the collection of lists of integers of the form,

$$c = (1-n_1, 1-n_2, \ldots, 1-n_{I-1}, 1+n_I, \overbrace{1,1,\ldots,1}^{F \text{ times}}, \underbrace{2,2,\ldots,2}_{\text{any number of times}})$$

where $F \geq 0$ and either $I = 0$ and there is a single 2, or the n_i are positive integers satisfying $1 < n_1|n_2|\ldots|n_I$.

Define a partial order on $\mathcal{C}$ (which will be a total order, except that the multiplicity of 2 will be ignored) as follows: $c \prec c'$ if

$F < F'$, or

$F = F'$ and $I < I'$, or

$F = F'$, $I = I'$, and $(n_1, n_2, \ldots, n_I) < (n'_1, n'_2, \ldots, n'_I)$ lexicographically from the left.

(For example, $(2,8) < (4,4)$.) To make $\prec$ into a total order, say $c \sim c'$ if both $c \preceq c'$ and $c' \preceq c$. Finally, given a square integral matrix A, there exists c in $\mathcal{C}$ such that

$$\mathrm{cok}(\mathrm{I} - A) \simeq \mathbf{Z}^F \oplus \left(\bigoplus_{i=1}^{I} \mathbf{Z}/n_i\mathbf{Z} \right).$$

Note that c is unique up to the multiplicity of 2. This gives a distinguished class, $[c]_A$ determined by A with respect to $\sim$.

For irreducible A, define $\mathcal{C}_A$ to be the set of lists $c \in \mathcal{C}$ for which there exists a primitive B with nonzero spectrum c such that S_A and S_B are flow equivalent.

Lemma 2.3. *If the irreducible matrix A is not a permutation matrix and cok$(I - A) \neq 0$, then $[c]_A$ is the unique minimal element of $\mathcal{C}_A/\sim$.*

Proof. Let $c = (1-n_1, 1-n_2, \ldots, 1-n_{I-1}, 1+n_I, 1, 1, \ldots, 1)$ (with $F \geq 0$ ones) represent $[c]_A$. If $I \neq 0$, then adjoining enough 2's will guarantee that the spectral conditions for primitive realization will hold, so there will exist a primitive matrix B which is algebraically shift equivalent to the diagonal matrix having c with enough 2's adjoined, as its diagonal part (as in the proof of Lemma 2.2). We can also arrange that $\det(\mathrm{I} - B)$, if nonzero, has the same sign as $\det(\mathrm{I} - A)$ by possibly increasing the number of 2's by one. Hence S_B is flow equivalent to S_A, so in particular, $[c]_A$ belongs to $\mathcal{C}_A/\sim$. If $I = 0$, we may realize $\mathrm{diag}(2, 1, \ldots, 1)$ as before.

It remains to verify that $[c]$ is minimal in $\mathcal{C}_A/\!\sim$ (since the ordering is total, there can be only one minimal element). Suppose $[c'] \leq [c]$ in $\mathcal{C}_A/\!\sim$. Then $F = F'$, since F is the rank of $\mathrm{cok}(\mathrm{I} - A)$ and this is less than or equal to F'. Also, $I' = I$, because I is the minimal number of generators of the torsion part of $\mathrm{cok}(\mathrm{I} - A)$. Finally, let A' be a primitive matrix with nonzero spectrum c' (counting algebraic multiplicities),

$$c' = (1 - n'_1, 1 - n'_2, \ldots, 1 - n'_{I-1}, 1 + n'_I, \overbrace{1, 1, \ldots, 1}^{F \text{ times}}, 2, 2, \ldots, 2).$$

Let B' be a nonsingular matrix algebraically shift equivalent to A'. Set $C' = \mathrm{I} - B'$, and note that C' is similar (with respect to $\mathrm{GL}(\mathbf{Z})$) to an upper triangular matrix with the block form

$$U = \begin{bmatrix} N' & V & W \\ 0 & X & Y \\ 0 & 0 & -\mathrm{I} \end{bmatrix}.$$

where the diagonal blocks are square, X is nilpotent, and N' is upper triangular with diagonal $(n'_1, \ldots, n'_I)$. By minimality of F, the matrix X must be a zero matrix. Thus U is equivalent to $\mathrm{diag}(n''_1, \ldots, n''_I, 0, \ldots, 0, -1, \ldots, -1)$, where the n''_i are positive integers such that $n''_1 | \ldots | n''_I$, and $\mathrm{diag}(n''_1, \ldots, n''_I, 0, \ldots, 0)$ is equivalent to the matrix $Z = \begin{bmatrix} N' & V \\ 0 & X \end{bmatrix} = \begin{bmatrix} N' & V \\ 0 & 0 \end{bmatrix}$.

Since S_A and S_B are flow equivalent, $(n''_1, \ldots, n''_I) = (n_1, \ldots, n_I)$. Since $n_1 = n''_1 | n'_1 \leq n_1$, we get $n''_1 = n'_1 = n_1$. Thus Z is equivalent to the matrix Z' which equals Z except that every off diagonal term in the first row is zero. Thus n''_2 is the gcd of the entries of Z below the first row. Now the obvious induction argument yields $n''_i = n'_i = n_i$ for all i. So $c = c'$. ∎

We can now complete the proof of Theorem 2.1. Suppose A and B are irreducible and S_A, S_B are nontrivial irreducible SFT's which are not flow equivalent. We will show $\mathcal{Z}_{\mathbf{Z}}(S_A) \neq \mathcal{Z}_{\mathbf{Z}}(S_B)$.

If $\det(\mathrm{I} - A) = 1$, then $\det(\mathrm{I} - B) \neq 1$ (else their flow equivalence invariants would be the same and they would be flow equivalent.) Then by Lemma 2.2, $\mathcal{Z}_{\mathbf{Z}}(S_A)$ is empty and $\mathcal{Z}_{\mathbf{Z}}(S_B)$ is not. So we may assume $\det(\mathrm{I} - A)$ and $\det(\mathrm{I} - B)$ are both not 1. We recover $\det(\mathrm{I} - A)$ from any zeta function, so we may assume $\mathrm{sign}\ (\det(\mathrm{I} - A)) = \mathrm{sign}\ (\det(\mathrm{I} - B))$. If now $\det(\mathrm{I} - A) = -1$, then A is flow equivalent to B (for the cokernels are both zero and the signs are the same), a contradiction. Hence, $|\det(\mathrm{I}-A)| \neq 1$, whence $\mathrm{cok}(\mathrm{I}-A)$ and $\mathrm{cok}(\mathrm{I}-B)$ are not trivial. By Lemma 2.3, $\mathcal{C}_A \neq \mathcal{C}_B$. ▯

Section 3. Graphical groups and chain recurrence

In this section, we explain how the groups $\mathcal{G}^T$ are generated in terms of graphs.

Our goal here is a thorough and elementary introduction. Much of this is implicit or explicit in Poon's paper [**Po**]. The main differences are that we spell things out in a general setting (i.e., without transitivity constraints on T); in the "graph groups", we use vertex coboundaries; and we relate the order structure to chain recurrence and suspension flows.

Graph groups. By a **graph**, we mean a finite directed graph (Γ, say) which is nondegenerate—every vertex has at least one incoming edge and at least one outgoing edge. Let $\mathcal{E} = \mathcal{E}(\Gamma)$ and $\mathcal{V} = \mathcal{V}(\Gamma)$ denote the edge and vertex sets, let $C = C(\Gamma, \mathbf{Z})$ denote the group of functions from $\mathcal{E}$ into $\mathbf{Z}$, let C_+ denote the elements of C with range in $\mathbf{Z}_+$, let $\mathbf{1}$ denote the constant function from $\mathcal{E}$ to 1. If v is a vertex, define γ_v as the function in C which assigns output 1 to edges with initial vertex v, -1 to edges with terminal vertex E, and zero to the other edges. (If E is an edge from v to itself, we define $\gamma_v(E) = 0$.) Let $B = B(\Gamma, \mathbf{Z})$ be the subgroup of C generated by the γ_v. A **vertex coboundary** (or more briefly a coboundary) is an element of B. By a **graph group** we will mean a preordered unital group $\mathcal{G}(\Gamma) = (C/B, C_+ + B, \mathbf{1} + B)$ defined from some graph Γ as above.

By a **path** we mean a finite sequence $E_1 \dots E_k$ of edges such that for $1 \leq i < k$ the terminal vertex of E_i is the initial vertex of E_{i+1}. It is a path from v to v' if the initial vertex of E_1 is v and the terminal vertex of E_k is v'. An edge E is **wandering** if no path of at least two edges which begins with E can end with E. The graph is **recurrent** if every edge is nonwandering. Equivalently, the adjacency matrix for the graph is a direct sum of irreducible matrices.

We say a function f in C is zero/nonnegative on cycles if the sum of $f(E)$ over the edges E of any cycle is zero/nonnegative respectively. The next lemma in a slightly different form comes from Poon [**Po**].

Theorem 3.1. *[**Po**] Suppose a graph is recurrent and f belongs to $\mathcal{C}$. Then*

(1) f belongs to B if and only if f is zero on cycles.

(2) There exists g in B such that $f + g \geq 0$ if and only if f is nonnegative on cycles.

(3) The graph group is an ordered group.

Proof. If $[f]$ is both positive and negative, then clearly f is zero on cycles. Thus (3) follows from (1) and (2).

In both (1) and (2), the forward implications are clear. For the converse, fix a vertex v_0. Given a vertex $v \neq v_0$, let $a_v = \min\{\sum_i f(E_i)\}$, where the minimum is over all paths $E_1 \dots E_k$ from v_0 to v. Let $a_{v_0} = 0$. Define $g = \sum_v a_v \gamma_v$. If E is an edge from v to v', then $(f(E) + a_v)$ is in the set of sums $\sum f(E_i)$ minimized by $a_{v'}$, so

$$(f+g)(E) = f(E) + a_v - a_{v'} \geq 0.$$

Thus $f + g \geq 0$, and $f + g = 0$ if f is zero on cycles. ∎

If Γ is a graph (which in our context means a directed graph), we can associate to it an undirected graph $\tilde{\Gamma}$ in a natural way. The vertex sets are identical, and to each directed edge e in Γ from i to j, there is a distinct undirected edge $\tilde{e}$ between i and j. An **undirected path** in Γ is a sequence of edges $\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_k$ such that for each j, $\tilde{e}_j$ is an edge joining i_{j-1} to i_j; the definition of **undirected cycle** is obvious. Given such a path, we define $\sigma(\tilde{e}_j)$ to be 1 if e_j goes from i_{j-1} to i_j, and define it to be -1 if it goes in reverse. If f is a real valued function on Γ, then the directed sum of f on the undirected path is the sum of terms of the form $f(e_j)\sigma(\tilde{e}_j)$, summed over e_j in the path. We say that f **vanishes on undirected cycles** if the directed sum of f on every undirected cycle is zero.

Lemma 3.2. *For a graph Γ, $f : \Gamma \to \mathbf{Z}$ is a vertex coboundary if and only if f vanishes on undirected cycles.*

Proof. It is easy to verify that every vertex coboundary vanishes on undirected cycles. Conversely, suppose f vanishes on undirected cycles. Extend the graph Γ to a larger graph $\overline{\Gamma}$ in the following manner. For each edge e, add a new edge $\overline{e}$ whose initial vertex is the terminal vertex of e, and whose terminal vertex is the initial one of v. Extend f to $\overline{f} : \overline{\Gamma} \to \mathbf{Z}$ by setting $\overline{f}(\overline{e}) = -f(e)$. Now $\overline{\Gamma}$ is recurrent. The weight of $\overline{f}$ on any cycle of $\overline{\Gamma}$ is the directed sum of f on some undirected cycle of Γ. Thus $\overline{f}$ vanishes on cycles, and by Theorem 3.1, $\overline{f}$ is a vertex coboundary on $\overline{\Gamma}$. By restriction, f is a vertex coboundary on Γ. ∎

Recall that a preordered group (G, G_+) is **unperforated** if for every g in G and positive integer n, if ng belongs to G_+ then g belongs to G_+. Sometimes authors build unperforation into the definition of a (pre)ordered group (we have not).

Proposition 3.3.

(1) *An edge E in a graph is wandering if and only if there exists a nonnegative function in B which is positive on E.*

(2) *An element $[f]$ in a graph group G is in G_+ if and only if f is nonnegative on cycles. In particular, every graph group is unperforated.*

(3) *Every graph group is torsion free.*

(4) *If $(G, G_+, [1])$ is a graph group and $(R, R_+, [1])$ is the group derived from the maximal recurrent subgraph of the given graph, then the map which restricts a function in C to that subgraph induces an isomorphism of ordered groups $(G, G_+, [1])/J \mapsto (R, R_+, [1])$.*

Proof. (1) Any nonnegative function in B must vanish on a nonwandering edge, because a nonwandering edge lies on a cycle. For the converse, partition the vertex set into nonempty sets $\mathcal{V}_0, \dots \mathcal{V}_M$ as follows. $\mathcal{V}_0$ is the set of vertices in the "initial" irreducible components, those which are not accessible from other irreducible components. Given $\mathcal{V}_i$ for $i < j$, define $\mathcal{V}'_j$ to be the set of vertices v which are not in $\mathcal{V}_i$ for $i < j$ and for which there is an arc from $\mathcal{V}_{j-1}$ to v. Then define $\mathcal{V}_j$ to be $\mathcal{V}'_j$ together with any vertices which are in the same irreducible component as some element of $\mathcal{V}'_j$. In particular, $\mathcal{V}_M$ consists entirely of vertices in some set of "terminal" irreducible components.

Define $f_i = \sum_{v \in \mathcal{V}_i} \gamma_v$ and let $g_0 = f_0$. For $0 < i < M$, define $g_i = 2g_{i-1} + f_i$. By induction on i, each g_i is a nonnegative function in $\mathcal{B}$ which is positive exactly on the wandering edges out of the $\mathcal{V}_j$ with $j \leq i$. Thus g_{M-1} is a nonnegative element of B which is positive exactly on the wandering arcs.

(2) and (4) follow easily from (1) and Theorem 3.1.

To prove (3), it suffices to see that whenever n is a positive integer and nf is a coboundary, then so is f. This is clear from Lemma 3.2. (Alternatively, we could just note that the group C/B for the graph is exactly the first integral simplicial cohomology group of the graph, and such a group is well known to be torsion free.) ∎

While we consider functions from edges of a graph into $\mathbf{Z}$, with vertex coboundaries, Poon ([**Po**]) considered only functions from the vertex set of a graph. Each approach has its advantages and in the end, they are equivalent. We chose the edge function view (as in [**PT**]) since then graphs with multiple edges are permitted, and also because a cohomology group is obtained which coincides with its traditional simplicial cohomology group. (In this light, presumably parts of Theorem 3.1–Proposition 3.3 are well known.)

Graphical groups. A **graphical group** is a countable unital preordered abelian group isomorphic to the direct limit of graph groups $\mathcal{G}_n$, where the bonding maps $\mathcal{G}_n \mapsto \mathcal{G}_{n+1}$ are induced by graph homomorphisms $\Gamma_{n+1} \mapsto \Gamma_n$ of the corresponding graphs.

By a graph homomorphism, we mean a homomorphism of directed graphs—a map of edges respecting direction and inducing a well defined vertex map. It is easy to check that if $h\colon \Gamma \to \Gamma'$ is a graph homomorphism and f belongs to $B(\Gamma')$, then $f \circ h \in B(\Gamma)$. So h does induce a well defined homomorphism $h^*\colon \mathcal{G}(\Gamma') \to \mathcal{G}(\Gamma)$. Without loss of generality, we may assume that the bonding maps above which give the direct limit group are surjective: we get the same direct limit group by replacing the nth graph with the subgraph whose edges are in the images of compositions of bonding maps from Γ_{n+k} for all $k > 0$. We will see below that for any homeomorphism T of a zero-dimensional compact metric space, $\mathcal{G}^T$ is a graphical group. Of course, dimension groups are graphical groups. We defer a further study of graphical groups to a sequel. To dispel any impression that dimension groups are ubiquitous, we remark without proof that $(G^T, G^T_+, [1])$ is *not* a dimension group if T is an irreducible shift of finite type which contains at least two orbits. (In contrast, if T is minimal, then $\mathcal{G}^T$ is a simple dimension group, and all possible simple dimension groups can be realized in this manner [**GPS**].)

Proposition 3.4. *A graphical group is torsion free and unperforated.*

Proof. A direct limit of torsion free unperforated groups is torsion free and unperforated. A graphical group is a direct limit of graph groups, which are torsion free and unperforated. ∎

To understand $\mathcal{G}^T$ as a graphical group, it will be convenient to have a few general results on zero-dimensional dynamics.

Zero dimensional dynamics.

Proposition 3.5. *Suppose S_A is an edge SFT. Suppose f is a coboundary in $C(X_A, \mathbf{Z})$ defined by $f(x) = f'(x_0)$, where $f' \in C(\Gamma, \mathbf{Z})$ and Γ is the graph with* *adjacency matrix A. Then f' is a vertex coboundary.*

Proof. If f' were not a vertex coboundary, then by Lemma 3.2, f would not vanish on all undirected cycles. Then for any f'' induced by f on a higher block presentation of the SFT, f'' would also not vanish on some undirected cycle. This would be a contradiction, since for an SFT, any coboundary can be presented as arising from a vertex coboundary for a sufficiently high block presentation. □

Suppose for a positive integer n, S_n is a self homeomorphism of a compact metric space Y_n. Suppose for $n > 1$, $\pi_n : Y_n \to Y_{n-1}$ is a continuous map intertwining S_n and S_{n-1}. Define the inverse limit homeomorphism as the restriction S of the product of the S_n to the set

$$Y := \left\{ (y_1, y_2, \dots) \in \prod Y_n \;\middle|\; \text{for all } n > 1,\ \pi_n(y_n) = y_{n-1} \right\} .$$

We say the inverse limit system is **surjective** if each of the bonding maps π_n is surjective.

A **graphical inverse limit** is an inverse limit of edge SFT's Z_n, where Γ_n is the graph defining the edge SFT Z_n; the bonding maps $Z_{n+1} \to Z_n$ are one-block codes given by graph epimorphisms $\pi_n : \Gamma_{n+1} \to \Gamma_n$ (**one-block codes** are maps between the sequence spaces determined by the effect on symbols at position 0); and the maximum over v in $\mathcal{V}(\Gamma_n)$ of the diameter of

$$\{z \in Z_n \mid z_0 \text{ has initial vertex } v\}$$

goes to zero with n.

Lemma 3.6. *Suppose T is a homeomorphism of zero-dimensional compact metric space.*

(a) T is isomorphic to a surjective inverse limit of subshifts T_n.

(b) T is isomorphic to a graphical inverse limit.

Proof. (a) Given a closed open partition $\mathcal{P}$ of X, one can define a subshift quotient of T, whose symbols are the elements of $\mathcal{P}$, as follows: for each x in X, define $\bar{x}$ by $T^i(x) \in \bar{x}_i$. If we take a refining sequence of closed open partitions $\mathcal{P}_n$, we get associated subshifts S_n; a surjective bonding map $z \mapsto \pi_n z$ from S_{n+1} to S_n is determined by $z_i \subset (\pi_n z)_i$ for positive integer i. If we require that the maximum diameter of an element in $\mathcal{P}_n$ go to zero with n, then the inverse limit homeomorphism is topologically conjugate to the original homeomorphism T on the zero dimensional space X.

(b) Let $\mathcal{P}_n$ and S_n be as in part (a). Define a graph Γ_n whose vertices are the elements of $\mathcal{P}_n$. There is an edge from P to P' in Γ_n if $T(P)$ intersects P'. Let PP' name such an edge. This graph defines an edge SFT Z_n. Now for each n define a conjugacy from S_n to a subsystem S'_n of Z_n by sending a point $x = (x_k)_{k\in\mathbf{Z}}$ to the point $x' = (x_k x_{k+1})_{k\in\mathbf{Z}}$. These embeddings conjugate the bonding maps $S_{n+1} \mapsto S_n$ to bonding maps $S'_{n+1} \mapsto S'_n$ determined by graph epimorphisms $\Gamma_{n+1} \mapsto \Gamma_n$. ▯

Lemma 3.7. *If T is a graphical inverse limit with associated graph groups Γ_n, then $\mathcal{G}^T$ is isomorphic to the direct limit of the graph groups $\mathcal{G}(\Gamma_n)$ under the induced homomorphisms π_n^*. In particular, for every homeomorphism T of a zero-dimensional compact metric space, $\mathcal{G}^T$ is a graphical group.*

Proof. Let Z_n, etc., be as in the definition of graphical inverse limit. We represent a point z in Z as a sequence $(z^{(n)})$, with $z^{(n)} \in Z_n$. If $f' \in C(\Gamma_n, \mathbf{Z})$, then f' naturally defines an element f of $C(Z, \mathbf{Z})$ by the rule $f\colon z \mapsto f'((z^{(n)})_0)$. This induces a well-defined, order preserving unital homomorphism $\phi_n\colon \mathcal{G}(\Gamma_n) \to \mathcal{G}^T$. For each n, $\phi_{n+1} = \phi_n \circ \pi_n^*$, so the maps ϕ_n induce a homomorphism ϕ from the unital direct limit group into $\mathcal{G}^T$. We need to check that ϕ is bijective and that the inverse image of a positive element is positive.

So suppose $f \in C(X, \mathbf{Z})$. Since f is constant on small clopen sets, f is defined by some f' on Γ_k. (By this we mean that f is determined on $\big(z^{(n)}\big)_{n\in\mathbf{Z}}$ by the symbol $(z^{(k)})_0$, which is an edge in the graph Γ_k; i.e., for some f' in $C(\Gamma_k, \mathbf{Z})$, $f(z) = f'((z^{(k)})_0)$.) Therefore ϕ is surjective. If $g \in C(X, \mathbf{Z})$ and $f = g\circ T - g$, then for some $m \geq k$, f and g are defined by f'', g'' on Γ_m. Since f defines a coboundary in $C(Z_m, \mathbf{Z})$, it follows by Proposition 3.5 that f'' is a vertex coboundary, so $\phi_m([f'']) = 0$. Since $\phi_m([f'']) = \phi_m \circ (\pi_{m-1}^* \circ \cdots \circ \pi_k^*)([f'])$, this shows ϕ is injective. A similar argument shows that if $f \geq 0$ in $\mathcal{G}^T$, then for some $m \geq k$, $[f''] \geq 0$ in (Γ_m). This finishes the proof. ▯

At this point the next result is an exercise, which we leave to the reader. In any case it follows a well-trodden path; we use it to transfer results on the ordered groups of mixing shifts of finite type, to those of more general chain recurrent systems. The use of graphs like the Γ_n to approximate zero-dimensional systems is not new (e.g., [AR]).

Proposition 3.8. *Let (X, T) be the inverse limit of the zero dimensional systems*

$$(X_n, T_n) \leftarrow (X_{n+1}, T_{n+1}).$$

Then $\mathcal{G}^T$ is naturally, unitally isomorphic to the corresponding direct limit,

$$\lim \mathcal{G}^{T_n} \to \mathcal{G}^{T_{n+1}}.$$

Here "naturally" means functorially.

Chain recurrence. Chain recurrence is an important idea in dynamical systems [**C, F1, Rob**]. An ϵ-chain from x to y is a finite sequence of points x_0, x_1, ... , x_n such that $x = x_0$, $x' = x_n$, and for $0 \le i < n$, $\mathrm{dist}(T(x_i), x_{i+1}) < \epsilon$. A point x is in the chain recurrent set $\mathrm{ch}(T)$ if for every $\epsilon > 0$, there is an ϵ-chain from x to x. (By compactness, this property is independent of the choice of a metric compatible with the topology.) T is chain recurrent if $\mathrm{ch}(T) = X$. The set $\mathrm{ch}(T)$ is closed and T-invariant, and the restriction of T to $\mathrm{ch}(T)$ is chain recurrent [**C, F1**].

If T is a subshift with domain X, then the **n-Markov approximation** T_n to T is the shift map on the set X_n consisting of all sequences x for which for all $i \in \mathbf{Z}$, the word $x_i \dots x_{i+n}$ occurs in some point in X. Note the system (X, T) is the nested intersection of the systems (X_n, T_n). Let Z_n be the $(2n+1)$-block presentation of X_{2n}. A block code ψ_n from X into Z_n is determined by $(\psi_n x)_0 = (x_{-n} ... x_n)$. Now $\psi = \prod_n \psi_n$ gives an isomorphism from T to the inverse limit.

The proofs of the next two propositions are routine given what has come before, and we leave them to the reader.

Proposition 3.9. *An SFT is chain recurrent if and only if it is the union of finitely many disjoint irreducible SFT's. If (X, T) is a subshift, then the following are equivalent.*

(i) T is chain recurrent.

(ii) Each Markov approximation T_n is chain recurrent.

(iii) T is the nested intersection of chain recurrent shifts of finite type.

Proposition 3.10. *Let T be a graphical inverse limit of edge SFT's, with Γ_k and $z^{(k)}$ as in the definition.*

(a) The chain recurrent set of T is the set of z such that for all k, $(z^{(k)})_0$ is a nonwandering edge of Γ_k.

(b) T is chain recurrent if and only if each Γ_k is a recurrent graph.
(c) If T is chain recurrent, then the preordered group $\mathcal{G}^T$ is an ordered group.

Below, by a nonnegative coboundary we mean a function of the form $g - g\circ T$, where g is a continuous map into the integers, and for all x, $(g - g\circ T)(x) \geq 0$.

Proposition 3.11. *Let T be a homeomorphism of a zero-dimensional compact metric space X. Then a point x of X is in the support of a nonnegative coboundary if and only if x is not a chain recurrent point for T. In particular, T is chain recurrent if and only if every nonnegative coboundary vanishes.*

Proof. Let T be a graphical inverse limit of edge SFT's Z_n given by graphs Γ_n. As in Lemma 3.7, f is a nonnegative coboundary in $C(X, \mathbf{Z})$ if and only if for some k, f is defined by $f(z) = f'((z^{(k)})_0)$, where f' is a nonnegative vertex coboundary in $C(\Gamma_k, \mathbf{Z})$. It then follows from Proposition 3.3(1) that z is in the support of a nonnegative coboundary in $C(X, \mathbf{Z})$ if and only if for some k, $(z^{(k)})_0$ is a wandering edge in Γ_k. By Proposition 3.10(a), this happens if and only if z is not a chain recurrent point for T.

∎

We now come to the main theorem of this section. Recall that $J \equiv J(G)$ denotes the subgroup $G_+ \cap -G_+$.

Theorem 3.12. *Let T be a homeomorphism of a zero-dimensional compact metrizable space X. Let R denote the restriction of T to the chain recurrent set $ch(T)$. Then the restriction map $f \mapsto f|ch(T)$ induces an isomorphism of unperforated unital ordered groups $(G^T, G^T_+, [1])/J \to (G^R, G^R_+, [1])$. Moreover,*

$$J(\mathcal{G}^T) = \left\{ [f] \in G^T \;\middle|\; f|ch(T) \text{ is a coboundary of } ch(T) \right\}$$

Proof. As R is chain recurrent, (G^R, G^R_+) is an ordered group. As a graphical group, it is unperforated. For $f \in C(X, \mathbf{Z})$, let $f_R = f|\mathrm{ch}(T)$. The map $C(X, \mathbf{Z}) \to C(R, \mathbf{Z})$ given by $f \mapsto f_R$ is a surjection (and therefore every coboundary in $C(R, \mathbf{Z})$ is the restriction of a coboundary in $C(X, \mathbf{Z})$). The restriction map induces a well-defined onto homomorphism $\rho\colon (G^T, G^T_+, [1]) \to (G^R, G^R_+, [1])$.

First we check that $\ker \rho = J$. Suppose $f|R = 0$. Using Proposition 3.11 and compactness, we find a nonnegative coboundary g which is strictly positive on the clopen set where f is nonzero. For some positive integer N, we have $(f + Ng) \geq 0$.

Therefore $[f]$ belongs to G^T_+, and similarly $-[f]$ belongs to G^T_+. Conversely, suppose $[f]$ belongs to $G^T_+ \cap -G^T_+$. Then $[f_R]$ belongs to $G^R_+ \cap -G^R_+ = \{0\}$ by chain recurrence, and f_R is a coboundary. Then there is some (not unique) coboundary g in $C(X, \mathbf{Z})$ such that $g_R = f_R$, so that $[f] = [f - g]$ where $(f - g)|R = 0$. Thus $J = \ker \rho$.

It remains to show that ρ^{-1} is order preserving. Suppose $[f_R] \geq 0$. Then there is a coboundary g on R such that $f_R + g \geq 0$; after subtracting from f a coboundary which restricts to g, we may assume that $f_R \geq 0$. There is a nonnegative coboundary g in $C(X, \mathbf{Z})$ which is strictly positive wherever f is strictly negative. For some N, $f + Ng \geq 0$. This finishes the proof. □

The homomorphism $\mathcal{G}^T \to \mathcal{G}^R$ above is a special case of a more general phenomenon. If Y is a chain recurrent subset (or sometimes merely a closed invariant subset), one may factor out $I(Y) := \left\{ [f] \in \mathcal{G}^T \mid f|Y \text{ is a coboundary of } Y \right\}$, and the quotient group (with the quotient ordering) is naturally order isomorphic to $\mathcal{G}^{T|Y}$. This will be discussed in a forthcoming sequel in considerable detail.

Remark. When T is not chain recurrent, the preordered group (G^T, G^T_+) is usually not ordered, but it can be. To restate an example appearing in [**Po**; Remark 1.12, p. 523], for a positive integer n, consider the shift of finite type T defined by the matrix

$$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

If $n = 1$, then the preordered group is ordered; if $n > 1$, then it is not.

The essence of part (a) of the following was announced by Poon [**Po**; p. 532, lines 1–2]. Recall from 1.6 the bijection between discrete pure traces on $\mathcal{G}^T$ and finite orbits of T.

Proposition 3.13. *Suppose (X, T) is an SFT. The following hold:*

(a) If $f : X \to \mathbf{Z}$ is continuous and for all discrete pure traces τ, $\tau(f) \geq 0$, then f is cohomologous to a nonnegative function.

(b) Suppose that (X, T) is irreducible. Then for f in $C(X, \mathbf{Z})$, $[f] = 0$ if and only if for all discrete pure traces τ, $\tau([f]) = 0$. In particular, $\mathcal{G}^T$ has no nonzero infinitesimals.

(c) Suppose (X, T) is irreducible. If U is a clopen set that has nontrivial intersection with every finite orbit, then there exists N such that $\cup_{0 \leq i \leq N} T^i U = X$.

(d) Suppose (X, T) is irreducible. If $f : X \to \mathbf{Z}$ is continuous and for all discrete pure traces τ, $\tau(f) > 0$, then $[f]$ is an order unit of $\mathcal{G}^T$.

Proof. (a) After passing to a higher block presentation, $X = X_A$ and $f(x) = f'(x_0)$, for some $f' \in C(\Gamma)$, where Γ is a graph with adjacency matrix A. The assumption on f guarantees that f' is nonnegative on cycles. By Proposition 3.3(2), $f' \in C(\Gamma, \mathbf{Z}_+)$, and (a) follows.

(b) This follows easily from Theorem 3.1, and is "well known" in dynamical systems.

(c) Passing to a higher block presentation, we may assume that there is a collection E' of edges in the graph presenting T such that U is the union of the points x such that $x_0 \in E'$. Suppose N exceeds the number of edges in the complement of E', $y \in X$, and the points y, Ty, $\dots$, $T^N y$ do not lie in U. Then the path $(y_0)(y_1)...(y_{N-1})$ avoids E' and for some i and j we have $0 \leq i < j < N$ with $y_i = y_j$. Then $(y_i)...(y_{j-1})$ is a loop which can be iterated to give a periodic point avoiding U, which is a contradiction.

(d) With U taken as the support of f, it follows from part (b) that there exists N such that every $x \in X$, $\liminf_n \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) \geq 1/N$. By the ergodic theorem, the integral of f against any invariant measure must be at least $1/N$, so by 1.7, f is an order unit. ∎

Example. Suppose $(G^T, G^T_+, [1])$ is an ordered group. Recall that $[f] \in G^T$ is an order unit if and only if $\tau([f]) > 0$ for every trace τ, i.e., f has positive integral with respect to every T-invariant measure. It is natural but naive to hope that apart from the subtleties of the infinitesimals, the measures determine the positive set. Specifically, one might hope that if there are no infinitesimals, then an element $[f]$ of G^T would have to be in G^T_+ if $\tau([f]) \geq 0$ for every trace τ. Poon's result (Proposition 3.13(a)) showed this is true when T is a shift of finite type. However, the implication can fail even for simple unital dimension groups; these constitute the class occurring as $(G^T, G^T_+, [1])$ for minimal homeomorphisms T of the Cantor set [**HPS**].

As an easy example, give $\mathbf{Q} \times \mathbf{Q}$ with the strict ordering, i.e., $G_+ = \{(0,0)\} \cup \{(x,y) \mid x > 0, y > 0\}$. Here every trace is a positive linear combination of the coordinate projections. In particular, $(0,1)$ is nonnegative at all traces, but is not in G_+. This ordered group is simple (every nonzero element of G_+ is obviously an order unit), with no infinitesimals. It is obviously unperforated and satisfies the Riesz interpolation property, and therefore is a dimension group [**EHS**].

4. Order and Čech Cohomology

Recall (Section 1.1) that Y_T is the standard suspension space of T. It is well known [**PT**; Ch. IV, section 3] that the group G^T is isomorphic to $H^1(Y_T)$, the group of homotopy classes of continuous maps from Y_T into the circle (henceforth denoted H^1_T). (It also well known that H^1_T is isomorphic to the first Čech cohomology group of the suspension space Y_T.) In this section, we consider the order structure as it sits in the presentation H^1_T, and explain how the failure of the preordered group (G^T, G^T_+) to be ordered is measured by $H^1(\overline{Y}_T)$, where $\overline{Y}_T$ is the gradient-like flow space Conley associated to the suspension flow on Y_T.

As a preliminary remark, we note that if ϕ belongs to $C(Y_T, S^1)$ and y is an element of Y_T, then there is a continuous map $g\colon \mathbf{R} \to \mathbf{R}$ such that for all t in $\mathbf{R}$, $\phi\colon \alpha_t(y) \mapsto (\phi y)(\exp(2\pi i g(t))$. Given ϕ and y, the map g is unique up to translation by an integer. The rule $x \mapsto (g_y(1) - g_y(0))$, where $y = [(x, 0)]$, defines a continuous function from X to $\mathbf{R}$, which we denote by a_ϕ. If $\phi([(x,0)]) = \phi([(x,1)])$, then $a_\phi(x)$ is an integer which we regard as a winding number.

The standard isomorphism $G^T \to H^1(Y_T)$*.* We review the standard isomorphism given in [**PT**; Ch. 4, section 3]. Given $g \in C(X, \mathbf{R})$, we define $Sg \in C(Y_T, S^1)$ by the rule

$$Sg\colon [(x,s)] \mapsto \exp(2\pi i((1-s)(gx) + s(gTx))), \quad 0 \le s \le 1.$$

Then Sg is homotopic to the constant map 1. An explicit homotopy is $H_t = S(tf), 0 \le t \le 1$.

Given $f \in C(X, \mathbf{Z})$, let $\phi_f : Y_T \to S^1$ denote the map given by $[(x,s)] \mapsto \exp(2\pi i s f(x)), 0 \le s \le 1$. Given f and g in $C(X, \mathbf{Z})$, note that $\phi_{f+gT-g} = (\phi_f)(Sg)$. Therefore the rule $[f] \mapsto [\phi_f]$ gives a well-defined map $\iota\colon G^T \to H^1(T)$, which is clearly a homomorphism.

To see ι is injective, suppose $f \in C(X, \mathbf{Z})$ and H_t gives a homotopy such that $H_0 = \phi_f$ and H_1 is the constant map 1. For $x \in X$, let $g(x)$ be the integer n such that the map $[0,1] \to S^1$ given by $t \mapsto H_t([(x,0)])$ is homotopic to the map $t \mapsto \exp(2\pi i n t)$. Then $g \in C(X, \mathbf{Z})$ and $f = g - gT$, so ι is injective.

To obtain an inverse to ι, suppose ϕ belongs to $C(Y_T, S^1)$. As X is zero-dimensional, we may choose a continuous function $\rho\colon X \to \mathbf{R}$ such that for all x in X, we have that $\phi\colon [(x,0)] \mapsto \exp(2\pi i \rho(x))$. Then ϕ is homotopic to $(\phi/S\rho)$, and for all x in X, $(\phi/S\rho) : [(x,0)] \mapsto 1$. Thus the rule $x \mapsto a_{(\phi/S\rho)}(x)$ defines a function f in $C(X, \mathbf{Z})$, and ϕ_f is homotopic to ϕ. Therefore $[\phi] \mapsto [f]$ gives the desired inverse to ι.

The gradient-like flow space $\overline{Y}_T$. The space Y_T admits the suspension flow for T, under which a point $[(x,s)]$ advances in time t to $[(x, s+t)]$. Following Conley [**C**; Ch. II, section 6.3], define an equivalence relation on Y_T by setting two points to be equivalent if they belong to the same chain recurrent component under this flow. (The chain recurrent set $\mathrm{ch}(Y_T)$ for the suspension flow on Y_T is $\{[x,s] \mid x \in \mathrm{ch}(T),\ s \in \mathbf{R}\}$. Two points $[x,s]$ and $[x',s']$ are in the same chain component under the flow iff x and x' are in the same chain component for T; that is, for all $\epsilon > 0$, there exists in X an ϵ-chain from x to x' and an ϵ-chain from x' to x.) Let $\overline{Y}_T$ denote the quotient space.

Even when X is a compact metric space that is not zero dimensional, the flow on Y_T induces a well defined "gradient-like" flow on $\overline{Y}_T$, and the rest points in $\overline{Y}_T$ (which correspond to the chain components in Y_T) constitute a zero dimensional set [**C**]. We remark that in our setting this can be seen very concretely. Let T be given as a graphical inverse limit, with edge SFT's Z_n defined by graphs Γ_n and graph epimorphisms $\pi_n\colon \Gamma_{n+1} \to \Gamma_n$ inducing the direct limit. Let Γ'_n be the graph obtained from Γ_n as follows. Delete all nonwandering edges; for each irreducible component C of Γ_n (a graph subgraph consisting of the component, is irreducible, if there are paths from every point to all others; that is, the adjacency matrix is irreducible), collapse all vertices of C to a single vertex v_C; and let each wandering edge E of Γ_n give rise to a distinct edge E' in Γ'_n, where the initial and terminal vertices of E' are the images of those of E under the vertex-collapsing map. Now the graph epimorphisms π_n induce well defined graph epimorphisms $\pi'_n : \Gamma'_{n+1} \to \Gamma'_n$. The inverse limit of the graphs Γ'_n under these epimorphisms is a one-dimensional topological space X' homeomorphic to $\bar{Y}_T$. The rest points in X' of the gradient-like flow are the inverse limit of the distinguished vertices v_C.

The quotient map $q : Y_T \to \bar{Y}_T$ induces a corresponding map on the first cohomology groups, $H^1(q) : H^1(\overline{Y}_T) \to H^1(Y_T)$. Let $J^T = J(G^T) = (G^T) \cap (-G^T)$. Recall $J^T = \{[f] \in G^T \mid f|\mathrm{ch}(T) = 0\}$.

Proposition 4.1. *The standard isomorphism* $\iota\colon G^T \to H^1(Y_T)$ *induces an* *isomorphism of short exact sequences*

$$\begin{array}{ccccccccc}
\{0\} & \longrightarrow & H^1(\overline{Y}_T) & \xrightarrow{H^1(q)} & H^1(Y_T) & \xrightarrow{H^1(i)} & H^1(Y_R) & \longrightarrow & \{0\} \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\{0\} & \longrightarrow & J(G^T) & \longrightarrow & G^T & \longrightarrow & G^R & \longrightarrow & \{0\}
\end{array}$$

Here $R = T|ch(T)$, $q : Y_T \to \overline{Y}_T$ is the quotient map, and $i : Y_R \to Y_T$ is the inclusion. The two rightmost vertical maps are the standard isomorphisms on the corresponding groups, and the right vertical map is ι_R, the standard one for R. On the bottom line, $J(G^T) \to G^T$ is the inclusion, and $G^T \to G^R$ is induced by restriction, as in Theorem 3.12. In particular, $\mathcal{G}^T$ is an ordered group if and only if $H^1(\overline{Y}_T)$ is trivial.

Proof. First we show that the map $H^1(q) : H^1(\overline{Y}_T) \to H^1(Y_T)$ has image contained in $\iota(J(G^T))$. Let $\overline{\phi} : \overline{Y}_T \to S^1$ be a continuous map. Then $\overline{\phi}$ lifts to $\phi \in C(Y_T, S^1)$, where ϕ is constant on chain components of Y_T. The rest point set in $\overline{Y}_T$ is zero-dimensional, so it has a neighborhood which is the union of several disjoint closed sets $\overline{C}$ on each of which the diameter of the image of $\overline{\phi}$ is less than 1. Lift these to disjoint closed sets C in Y_T, and identify X with $\{\, [(x,0)] \in Y_T \mid x \in X \,\}$. After expanding the sets C slightly, we may assume they are still disjoint and that each set $C' := C \cap X$ is a closed open set in the relative topology on X. We may define the local lift $\rho : X \to \mathbf{R}$ (which is used to define f such that $\iota(f) = [\phi_f] = [\phi]$) independently on each set C'. As any chain component of X must be contained in some C', and ϕ is constant on chain components, this results in a function ρ which is constant on chain components.

This produces f in $C(X, \mathbf{Z})$ such that f vanishes on $\mathrm{ch}(T)$ and ϕ_f is homotopic to ϕ, which shows that the image of $H^1(q)$ is contained in $\iota(J(G^T))$.

Continuing, we show that $H^1(q)$ is injective. Suppose that the ϕ_f obtained above is null-homotopic. We can take (X, T) to be a graphical inverse limit, so that f can be identified with a vertex coboundary on some graph Γ_n. By construction, f must vanish on nonwandering edges of Γ_n, so the natural homotopy from ϕ_f to the constant function 1 is identically 1 on $\mathrm{ch}(T)$. This homotopy pushes down to $\overline{Y}_T$. This shows that $H^1(q)$ is injective.

Next, given an equivalence class $[f]$ in $J(G^T)$, we may assume f vanishes on $\mathrm{ch}(T)$. Then ϕ_f is identically 1 on each chain component, so induces a map $\tilde{\phi}_f : \overline{Y}_T \to S^1$, and obviously the image of the homotopy class of $\tilde{\phi}_f$ in $\mathcal{G}^T$ is f, so that the range of $H^1(q)$ is all of J.

At this point we know that the left square of the diagram commutes, and all vertical arrows are isomorphisms. It is straightforward to check that the right square commutes, so the whole diagram commutes. The bottom sequence is exact by Theorem 3.12, so $H^1(i)$ must be surjective and the top sequence must be exact as well. This finishes the proof. ∎

A **connecting orbit** between two chain components of T is an orbit which is forwardly asymptotic to one and backwardly asymptotic to the other. It is not hard to see from the isomorphism $J^T \simeq H^1\overline{Y}_T$ that J_T will be non-trivial if there are two chain components with more than one connecting orbit between them (the converse fails). With this in mind, we leave the following proposition as an exercise. In its statement, $C_i \to C_j$ means there is a connecting orbit from the component C_i to the component C_j.

Proposition 4.2. *Suppose T is a shift of finite type. Then $J^T = 0$ if and only if T is the disjoint union of SFT's which are irreducible or have the following structure:*

(i) each irreducible component is a single periodic orbit;

(ii) there is at most one connecting orbit between any two periodic orbits—in particular, there is no chain of connecting orbits $C_1 \to C_2 \to C_3$;

(iii) for $n > 0$, there is no loop of connecting orbits $C_0 \to C_1 \leftarrow C_2 \to C_3 \to \cdots \leftarrow C_{n-1} \to C_0$.

The winding order. Recall that if ϕ is an element of $C(Y_T, S^1)$ and y belongs to Y, then there is a continuous map $g_y \equiv g : \mathbf{R} \to \mathbf{R}$ such that for all t in $\mathbf{R}$, $\phi\colon \alpha_t(y) \mapsto (\phi y)(\exp(2\pi i g(t))$. We define $C_+(Y_T, S^1)$ to be the set of ϕ such that for all y, g_y is nondecreasing. (In other words, as a point moves forward under the suspension flow on Y_T, its image under ϕ is stationary or moves counterclockwise on the circle.) Then we define $H^1_{\oplus}(T) := \{[\phi] \mid |\phi \in C_+(Y, S^1)\}$, where $[\phi]$ is the homotopy class of ϕ. (The circle around the plus sign is meant to suggest winding number.) Now $\left(H^1(T), H^1_{\oplus}(T)\right)$ is a preordered group. We call this preorder the **winding order** on $H^1(T)$.

The winding order is geometrically natural. It also makes sense for any real flow on a compact metric space Y (Y may have nonzero dimension, and the flow need not have a cross section)–one still has a well defined group H^1 of continuous functions from Y into S^1 modulo homotopy, and the winding order still makes H^1 into a preordered group. In this language (translating Schwartzmann's important study of sections to flows [**Sch**]), one can say that a flow has a cross section if and only if there is an order unit in $\left(H^1(T), H^1_\oplus(T)\right)$, and following [**Sch**] again, one can relate the order units to the positivity of certain integrals.

Let $H^1_+(T)$ denote the standard order, i.e., the image of G^T_+ under the standard isomorphism ι. With this motivation, we want to compare the winding order to the standard order. Obviously, $H^1_+(T) \subset H^1_\oplus(T)$. The two orders are sometimes but not always the same. Let $G^T_\oplus$ denote the preimage of $H^1_\oplus(T)$ under the standard isomorphism. The next proposition gives a comparison of the two orders in terms of coboundaries.

Theorem 4.3. *Suppose f is an element of $C(X, \mathbf{Z})$.*
(1) $[f] \in G^T_+$ iff there exists $g \in C(X, \mathbf{Z})$ such that $f + g - gT \geq 0$.
(2) $[f] \in G^T_\oplus$ iff there exists $g \in C(X, \mathbf{R})$ such that $f + g - gT \geq 0$.

Remark. Note that in (2) the function $g - gT$ is allowed to have noninteger outputs.
Proof. (1) This is a definition.
(2) Let us say an element ϕ of $C(Y_T, S^1)$ is piecewise linear if for every $x \in X$, there is a constant $v = v(x)$ (the "velocity") such that for $0 < s < 1$, $\phi\colon [(x,s)] \mapsto \phi([(x,0)]) \cdot \exp(2\pi i v s)$. Suppose $h \in G^T_\oplus$, with ϕ_h homotopic to $\phi \in C_+(Y_T, S^1)$; we want to deduce there is $g \in C(X, \mathbf{R})$ such that $h + g - gT \geq 0$. It is clear that ϕ is homotopic to a piecewise linear map still in $C_+(Y_T, S^1)$, so without loss of generality we may suppose ϕ is piecewise linear. The construction of the inverse to the standard isomorphism produces f in $C(X, \mathbf{Z})$ and ρ in $C(X, \mathbf{R})$ such that ϕ is homotopic to $\phi_f(S\rho)$. As ϕ is piecewise linear, the choice of f actually gives $\phi = \phi_f(S\rho)$. Thus for $x \in X$ and $0 < s < 1$,

$$\phi\colon [(x,s)] \mapsto \exp\left(2\pi i \left(sf(x) + (1-s)\rho(x) + s\rho(Tx)\right)\right)$$

and the piecewise linear map ϕ has velocity $v(x) = (f + \rho T - \rho)(x)$. Since v is nonnegative and h is cohomologous to f, the desired g exists.

Conversely, suppose $g \in C(X, \mathbf{R})$ and $f + g - gT \geq 0$. Then ϕ_f is homotopic to $\phi_f(Sg)$, which is piecewise linear with nonnegative velocity. □

In two interesting cases we can show that the winding order is the standard order. If (G, G^+) is a partially ordered abelian group with an order unit, then we will call it **archimedean** if for g in G, $\alpha(g) \geq 0$ for all (pure) traces α implies

$g \geq 0$. This definition, suitable for our purposes, is equivalent to the strongest form of archimedeanness discussed in [**G**]. In particular, if (X, T) is a shift of finite type, then $\mathcal{G}^T$ is archimedean (Proposition 3.13). This also occurs for certain rather special sofic shifts but for other (also special) sofic shifts, there can be infinitesimals in $\mathcal{G}^T$. Another archimedean example occurs if (X, T) arises as the orbit space of mixing SFT under the action of a finite group. Both of these constructions will be discussed in the sequel to this paper.

Corollary 4.4. *If (X, T) is a zero dimensional dynamical system, the traces for both orderings on G^T, that is, G^T_+ and $G^T_\oplus$ (and normalized with respect to the constant function in both cases), are given by integration with respect to T-invariant measures on X. In particular, the group isomorphism $G^T \to H^1(T)$ induces an affine homeomorphism between the normalized trace spaces.*

Proof. The first part is a consequence of Theorem 4.3 and the discussion in Section 1.6, and the second follows from this and the definitions. ∎

Proposition 4.5.

In the following cases, the standard group isomorphism $\iota : G^T \to H^1(T)$ is an order isomorphism $\iota : (G^T, G^T_+) \to (H^1(T), H^1_\oplus(T))$.

(a) If (X, T) is such that $\mathcal{G}^T$ is archimedean; in particular, this applies if (X, T) is a shift of finite type.

(b) If (X, T) is minimal.

In general, ι restricts to a bijection between the sets of order units of (G^T, G^T_+) and of $(H^1(T), H^1_\oplus(T))$.

Proof. In general, we have $G^T_+ \subseteq G^T_\oplus$, so it suffices to show the reverse inclusion.
(a) Given $[f]$ in $G^T_\oplus$, we must have that $\alpha([f]) \geq 0$ for all traces α with respect to $G^T_\oplus$. By Corollary 4.4, this entails $\beta([f]) \geq 0$ for all traces β with respect to G^T_+, and by archimedeanness, this forces $[f] \in G^T_+$.
(b) Now suppose T is minimal. Without loss of generality suppose T has more than one orbit, so its domain is a Cantor set. If $f + gT - g$ is identically zero, then because f has integer outputs and T has a dense orbit, the fractional part of g must be constant; replacing g by its integer part, we see $[f] = 0$ in G^T, so $[f] \in G^T_+$. If $f + gT - g$ does not vanish identically, then by minimality there is a positive integer N such that for all x, the partial sum $S_N(f + gT - g)(x) > 2 \max\{|g|\}$, where $S_N(h) = h + h \circ T + \cdots + h \circ T^{n-1}$. Then $S_N(f) > 0$ and $[f] \in G^T_+$.

By Theorem 4.3, the constant function $[\mathbf{1}_X]$ is an order unit with respect to $G^T_\oplus$. It follows immediately that order units with respect to G^T_+ are also order units with respect to $G^T_\oplus$. Conversely, if $[f]$ is an order unit with respect to $G^T_\oplus$, then $\alpha([f]) > 0$ for all traces α with respect to $G^T_\oplus$. Hence $\beta([f]) > 0$ for all traces β with respect to G^T_+, and thus by 1.7 (or using the fact that $\mathcal{G}^T$ is unperforated and a general positivity principle), $[f]$ is an order unit. $\square$

Suppose $\epsilon > 0$. Recall that a sequence of points $(z_i)_{i\in I}$ is an **ϵ-pseudo-orbit** for T if for all $\{i, i+1\} \subset I$, $\text{dist}(Tz_i, z_{i+1}) < \epsilon$. (The set of integers I may be finite or infinite, but must contain more than one element.) It is a periodic ϵ-pseudo-orbit if $I = \mathbf{Z}$ and the sequence $(z_i)_{i\in I}$ is periodic. Let T be given as a graphical inverse limit X of SFT's Z_n, with projections $p_n : X \to Z_n$. Given $\epsilon > 0$, there is a minimal positive integer $n(\epsilon)$ such that for every $n > n(\epsilon)$, the set of images $p_n(z_i)$ of periodic ϵ-pseudo-orbits of T is the set of periodic orbits of Z_n. As $\epsilon \to 0$, $n(\epsilon) \to \infty$. Thus we can think of periodic pseudo-orbits of T as being the periodic orbits of the approximating Z_n.

Proposition 4.6. *Suppose $f \in C(X, \mathbf{Z})$. Then $[f] \in G^T_+$ iff there exists $\epsilon > 0$ such that f has nonnegative sum along every periodic ϵ-pseudo-orbit.*

Proof. Let T be given as a graphical inverse limit of SFT's Z_n with graphs Γ_n. As in Proposition 3.8, a function f in $C(X, \mathbf{Z})$ is given by a function F on edges of some Γ_N, which defines a function $f_N \in C(Z_N, \mathbf{Z})$ such that $f(x) = f_N(p_N(x))$. Suppose $[f] \in G^T_+$. Then F is nonnegative on cycles, so f_n has nonnegative sum along every periodic orbit of Z_N and f has nonnegative sum along every ϵ-pseudo-orbit of T (for sufficiently small ϵ). Conversely, given $\epsilon > 0$, we can choose Z_N above with N large enough that every periodic orbit of Z_N is the image of an ϵ-pseudo-orbit for T. If f has nonnegative sum along every ϵ-pseudo-orbit, then F must be nonnegative on cycles, so $[f_N] \in G^{Z_N}_+$ and thus $f \in G^T_+$. $\square$

Example 4.7. *A chain transitive T for which the preorder $G^T_\oplus$ is not an order.*

Since the preorder G^T_+ is an order when T is chain recurrent, this gives an example in which the standard and winding orders are different.

Let C be the middle-thirds Cantor set in [0,1]. Let $\{U_1, U_2, \dots\}$ be the collection of disjoint open intervals from $[0,1]$ whose union is $[0,1] \setminus C$. Choose an increasing continuous surjection $p : [0,1] \to [0,1]$ such that for all x, y in $[0,1]$, $p(x) = p(y)$ if and only if $\{x, y\} \subset \bar{U}_i$ for some i. ¿From each $U_i = (a_i, b_i)$, choose a strictly increasing bisequence $(x_n)_{n\in\mathbf{Z}}$ such that $\lim_{n\to+\infty} x_n = b_i$ and $\lim_{n\to-\infty} x_n = a_i$. Let X' be the union of C and all these bisequences. Define $T'\colon X' \to X'$ by $T'(x_n) = x_{n+1}$ on the bisequences and $T' = \mathrm{Id}$ on C. Choose an isolated point from X' and define $f \in C(X', \mathbf{Z})$ by setting $f(y) = -1$ and $f(x) = 0$ if $x \neq y$. Define $\rho \in C(X', \mathbf{R})$ by

$$\rho(x) = \begin{cases} p(x) & \text{if } 0 \leq x < Ty \\ p(x) - 1 & \text{if } Ty \leq x \leq 1. \end{cases}$$

Then $(f + \rho - \rho T)(x)$ vanishes for all x. Let X be the quotient of X' obtained by collapsing 0 and 1 to a single point. Then T', ρ, f induce corresponding objects on X, which we shall call T, ρ, f respectively. Clearly T is chain transitive. For every $\epsilon > 0$, f has negative sum on some periodic ϵ-pseudo-orbit, and therefore $[f] \neq 0$. However, $f+\rho-\rho T$ vanishes; so $[f]$ belongs to $(G^T_\oplus \cap -G^T_\oplus)$. This finishes the proof. (We remark that the map $\phi\colon [(x,s)] \mapsto \exp(2\pi i\rho(x)) = \exp(2\pi i p(x))$ is an element of $C_+(Y_T, S^1)$ homotopic to ϕ_f.) ∎

5. K-Theory

The unital ordered group $\mathcal{G}^T$ can be interpreted as K_0 of the crossed-product C*-algebra arising from the dynamical system T.

This was explained by Putnam et al [**Pu, GPS**] for minimal systems and by Poon [**Po**] for systems with a dense forward orbit.

In this section, we work out the connection in the general case that T is a homeomorphism of a zero-dimensional compact metric space. We also point out that K_1 has a natural interpretation, and observe that a "dimension shift" occurs in the connection between K-theory and cohomology. A reference for the K-theory that occurs in what follows would be [**Bla**].

Form the C*-algebra $C = C(X) \times_T \mathbf{Z}$ obtained from the action of T viewed as an algebra automorphism of $C(X)$.

The Grothendieck group of C, written $\mathrm{K}_0(C)$, is the abelian group generated by the equivalence classes of finitely generated projective modules under stable isomorphism (in the case of a C*-algebra, finitely generated projective modules can be replaced by projections in matrix algebras). It has a natural structure of a preordered abelian group, $\mathrm{K}_0(C)^+$, consisting of the equivalence classes that contain projective modules. It also has a natural choice of order unit, the equivalence class of the free module on one generator, often denoted [1]. The assignment $C \mapsto (\mathrm{K}_0(C), \mathrm{K}_0(C)^+, [1])$ is functorial (with respect to unital ring homomorphisms).

There is also a (topological) K_1 group, which for C*-algebras is simply the direct limit of the group of invertibles modulo its connected component, in matrix algebras over C. It normally has no natural preordered structure, although one occasionally arises.

For the specific case of crossed product C*-algebras that we are dealing with here (and more generally), the two K-groups can be computed using the Pimsner-Voiculescu exact sequence. Let $A = C(X)$ (although this diagram is valid for A any C*-algebra, and $C = A \times \mathbf{Z}$ with any action on A); then this diagram is exact.

$$\begin{array}{ccccc}
\mathrm{K}_0(A) & \xrightarrow{\mathrm{K}_0(T)-\mathrm{id}} & \mathrm{K}_0(A) & \xrightarrow{\mathrm{K}_0(i)} & \mathrm{K}_0(C) \\
\uparrow & & & & \downarrow \\
\mathrm{K}_1(C) & \xleftarrow[\mathrm{K}_1(i)]{} & \mathrm{K}_1(A) & \xleftarrow[\mathrm{K}_1(T)-\mathrm{id}]{} & \mathrm{K}_1(A)
\end{array}$$

Here $i : A \to C$ is the natural inclusion (we use functorial notation applied to mappings, rather than the usual asterisks). In our case, $A = C(X)$ where X is zero dimensional, so that $\mathrm{K}_0(A) = C(X, \mathbf{Z})$ (with the coordinatewise ordering—every projective module corresponds to a nonnegative integer valued function, determined by the trace of the corresponding projection evaluated at points of X), and $\mathrm{K}_0(T)$ is the map induced by T, i.e., $f \mapsto f \circ T$. Also $\mathrm{K}_1(A) = 0$, so we deduce that as abelian groups, $\mathrm{K}_0(C) \simeq C(X, \mathbf{Z})/\mathrm{cobdy}\,(T)$ (since $\mathrm{cobdy}\,(T)$ is precisely the range of $\mathrm{K}_0(T) - \mathrm{id}$) and $\mathrm{K}_1(C)$ is isomorphic to the kernel of $\mathrm{K}_0(T) - \mathrm{id}$, i.e., the fixed point subalgebra of $C(X, \mathbf{Z})$—the vector space subspace spanned by characteristic functions of clopen sets U with $TU = U$.

In particular, as abelian groups, we have identified G^T with $\mathrm{K}_0(C)$. As we will see shortly, this identification is a preorder isomorphism as well (the P-V sequence does not determine the preordering on K_0).

There is a direct natural map $\mathrm{K}_0(C) \to H^1(Y_T)$, given by $[p] \to [\exp 2\pi i t \cdot \mathrm{Tr}\, p(x)]$ (note the use of brackets [ ] to denote equivalence classes of different kinds of objects, projections on the left and homotopy classes on the right), where p is a projection in some matrix algebra over C and $\mathrm{Tr}\, p(x)$ is its trace making it into a function on X. This map is compatible with the isomorphism $H^1(Y_T) \simeq G^T$ and the identification of $\mathrm{K}_0(A)$ with G^T, as is readily verified.

There is also a natural map $H^0(Y_T) \to \mathrm{K}_1(C)$. Let $f : Y_T \to \mathbf{Z}$ be continuous.

Then the level sets, $f^{-1}(m)$ are clopen and invariant under the action of the reals. In particular, their intersection with the image of $X \times 0$ is T-invariant, and this yields a clopen set in X which is T-invariant. We thereby obtain a map from $H^0(Y_T) \equiv C(Y_T, \mathbf{Z})$ to the fixed point subgroup of $C(X, \mathbf{Z})$, i.e., to $\mathrm{K}_1(C(X) \times_T \mathbf{Z})$. It is routine to verify that this is a one to one group homomorphism because any $\mathbf{R}$-invariant clopen subset of Y_T is uniquely determined by its restriction. Finally, the map is onto since any clopen invariant set in X can be enlarged to a $\mathbf{R}$-invariant clopen subset of Y_T in the obvious way.

Note for example, that Y_T is connected if and only if the only clopen T-invariant subsets of X are the trivial ones; that is, the dynamical system is **indecomposable**.

In particular, we have isomorphisms $H^1(Y_T) \to \mathrm{K}_0(C)$ and $H^0(Y_T) \to \mathrm{K}_1(C)$. This dimension shift suggests a general phenomenon is at work, and this is indeed the case. We are indebted to George Elliott for the following explanation; see [**Co**; Corollary 6].

Let (for now) A be any unital C*-algebra on which a single automorphism, call it T, acts. Form the crossed product C*-algebra $C = A \times_T \mathbf{Z}$ and also the **mapping torus of** T**,**

$$A_T := \{\, f : [0,1] \to A \mid f(1) = Tf(0)\,\} \ .$$

If $A = C(X)$ and T is induced by a self-homeomorphism of X (also called T), it is easy to check that $A_T = C(Y_T)$. There is a natural action of the reals on A_T given by

$$(r \cdot f)(a) = \begin{cases} T^{[r]} f(\{r+s\}) & \text{if } \{r\} + s < 1 \\ T^{[r]+1} f(\{r+s\}) & \text{if } \{r\} + s \geq 1, \end{cases}$$

where brackets denote greatest integer and braces denote fractional part. Again if $A = C(X)$ and T is induced by a self-homeomorphism, this real action is induced by the natural real action on Y_T. In any event, we can form the C*-algebra crossed product, $A_T \times \mathbf{R}$ with this action. A standard theorem in this context asserts that $A_T \times \mathbf{R}$ is stably isomorphic to $A \times_T \mathbf{Z}$ (that is, they become isomorphic on tensoring with the algebra of compact operators); in particular, their K-theory is the same. By Connes' isomorphism theorem ([**Co**]), $\mathrm{K}_0(A_T) \simeq \mathrm{K}_1(A_T \times \mathbf{R})$ and $\mathrm{K}_1(A_T) \simeq \mathrm{K}_0(A_T \times \mathbf{R})$. Hence there are isomorphisms $\mathrm{K}_0(A_T) \simeq \mathrm{K}_1(A \times_T \mathbf{Z})$ and $\mathrm{K}_1(A_T) \simeq \mathrm{K}_0(A \times_T \mathbf{Z})$.

If $A = C(X)$, then $A_T = C(Y_T)$, and it is known that for suitable compact spaces, Z, $\mathrm{K}_0(C(Z))$ is the direct sum of the even cohomology groups and

$\mathrm{K}_1(C(Z))$ is the direct sum of the odd ones. In our case, X is zero dimensional, so that Y_T is locally a product of a zero dimensional space and a one dimensional space, and thus Y_T is one dimensional. Hence all the higher cohomology groups vanish, and thus $\mathrm{K}_0(A \times_T \mathbf{Z}) \simeq H^1(Y_T)$ and $\mathrm{K}_1(A \times_T \mathbf{Z}) \simeq H^0(Y_T)$. This proof does not give the preorder isomorphism on the level of K_0, however.

In [**Po**; Remark 3.10], Poon noted that if $\mathcal{G}^T$ is archimedean, then the two possible preorderings on $\mathcal{G}^T$, the first being the ordering yielding $\mathcal{G}^T$ (that is, the quotient ordering on $C(X, \mathbf{Z})/(T - \mathrm{id})C(X, \mathbf{Z})$) and the second induced from $\mathrm{K}_0(C(X) \times_T \mathbf{Z})$ via the Pimsner-Voiculescu exact sequence, are the same. We prove this here in the general zero dimensional situation.

Lemma 5.1. *Let A and B be preordered abelian groups, and suppose $\phi : A \to B$ is a group isomorphism such that $\phi(J(A)) = J(B)$ and moreover, the induced map $\overline{\phi} : A/J(A) \to B/J(B)$ is an order isomorphism. Then ϕ is an isomorphism of preordered groups.*

Proof. Select b in B^+. Then $b + J(B) \geq 0$ (as an element of $B/J(B)$—note that $b + J(B)$ consists entirely of "positive" elements). By assumption, there exists a coset $a' + J(A) \geq 0$ such that $\overline{\phi}(a' + J(A)) = b + J(B)$. If $a = \phi^{-1}(b)$, then a belongs to the coset $a' + J(A)$; however, every element of this coset lies in A^+, so that a belongs to A^+. Thus ϕ^{-1} is preorder preserving. Interchanging A with B yields that ϕ is preorder preserving, so that ϕ is a preorder isomorphism. □

Theorem 5.2. *Let (X, T) be a zero dimensional metrizable compact space together with a self-homeomorphism. Then $\mathcal{G}^T$ is isomorphic to $K_0(C(X) \times_T \mathbf{Z})$ (as preordered abelian groups).*

Proof. We note first that the identification of $\mathrm{K}_0(C(X) \times_T \mathbf{Z})$ with G^T given by the PV sequence includes within it some information on the positive cone. Explicitly, the map $\mathrm{K}_0(i) : C(X, \mathbf{Z}) \to G^T$ is induced by the inclusion $C(X) \to C(X) \times_T \mathbf{Z}$ and when viewed as a map on the Grothendieck groups, is preorder preserving. The upshot is that with respect to the ordering on G^T induced by $\mathrm{K}_0(C(X) \times_T \mathbf{Z})$, the equivalence class of every image of a projection is positive. Hence the positive cone for the second, induced ordering contains that of the quotient ordering. Moreover, it is easy to see that the pure traces with respect to either preordering can be identified with the invariant measures; hence both preorderings have the same set of pure traces.

Suppose (X,T) is an irreducible shift of finite type. Pick a in $\mathcal{G}^T$ positive with respect to the induced ordering. Then it is nonnegative at every pure trace, hence by Proposition 3.13, it is positive with respect to the quotient ordering. So in this case, the two orderings agree. Of course the same is also true for a finite disjoint union of irreducible shifts of finite type. It is routine to verify that both orderings commute with inverse limits, hence they agree on the class of inverse limits of finite disjoint unions of irreducible shifts of finite type—by Proposition 3.9, this is precisely the set of chain recurrent (zero dimensional) systems. Thus the orderings are identical for chain recurrent systems.

Now let (X,T) be an arbitrary zero dimensional system. We form $(\mathrm{ch}(T),\overline{T})$— the restriction to the chain recurrent subset—and observe that the identification of G^T with K_0 commutes with the inclusion map $\mathrm{ch}(T) \to X$; explicitly,

$$\begin{array}{ccc} \mathcal{G}^T & \twoheadrightarrow & \mathcal{G}^{\overline{T}} \\ \downarrow & & \downarrow \\ \mathrm{K}_0(C(X) \times_T \mathbf{Z}) & \twoheadrightarrow & \mathrm{K}_0(C(\mathrm{ch}(T)) \times_{\overline{T}} \mathbf{Z}). \end{array}$$

Here the arrows to the right are onto; the top one just factors out $J(\mathcal{G}^T)$ as we have seen, the bottom one is simply induced by the inclusion of spaces. The vertical maps are preorder preserving group isomorphisms (first paragraph), and the right vertical map is an order isomorphism. Since the left vertical map is at least order preserving, it sends $J(\mathcal{G}^T)$ to $J(\mathrm{K}_0(C(X) \times_T \mathbf{Z}))$. Anything in the kernel of the bottom map would have to be in the image of the left vertical map, since both vertical maps are group isomorphisms. Hence the image of $J(\mathcal{G}^T)$ is all of $J(\mathrm{K}_0(C(X) \times_T \mathbf{Z}))$. The conditions of the preceding lemma now apply to yield that the left vertical map is an isomorphism of preordered groups. □

The result, Theorem 3.12, asserting that $\mathcal{G}^T/J \simeq \mathcal{G}^{\overline{T}}$, i.e., that $\mathrm{K}_0(C(X) \times_T \mathbf{Z})/J \simeq \mathrm{K}_0(\mathrm{ch}(T) \times_{\overline{T}} \mathbf{Z})$ can be interpreted as an excision result for K_0, at least when $J = 0$. There is no counterpart for K_1, as simple examples show. For example, if (X,T) is the two point compactification of the shift on $\mathbf{Z}$ (equivalently, it is the subshift of finite type determined by the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$), then $J = 0$, $\mathrm{K}_1(C(X) \times_T \mathbf{Z}) \simeq \mathbf{Z}$ (there are no nontrivial invariant sets), but $\mathrm{ch}(T)$ consists of the two points at infinity and $\overline{T}$ fixes these, so $\mathrm{K}_1(\mathrm{ch}(T) \times_{\overline{T}} \mathbf{Z}) \simeq \mathbf{Z}^2$.

References

[AR] P. Arnoux & G. Rauzy, *Représentation géometrique des suites de complexité* $2n+1$ Bull. Soc. France **119** 199–215.

[**Bla**] B. Blackadar, *K-Theory for Operator Algebras* MSRI Publications **5** (1986).

[**Bw**] R. Bowen, *Equilibrium States and Ergodic Theory of Anasov Diffeormorphisms* Springer Lecture Notes in Mathematics **470** (1975)Springer-Verlag.

[**BF**] R. Bowen and J. Franks, *Homology for zero-dimensional basic sets* Annals of Math. **106** (1977) 73–92.

[**B1**] M. Boyle, *Topological orbit equivalence and factor maps in symbolic dynamics* Ph.D. Thesis, University of Washington (1983).

[**B2**] ———, *Symbolic dynamics and matrices* in Proceedings of a Conference on Linear Algebra, Graph Theory and Related Topics edited by R. Brualdi, S. Friedland and V. Klee I.M.A. volumes in Mathematics, Minneapolis **50** 1–38.

[**BH1**] M. Boyle and D. Handelman, *Algebraic shift equivalence and primitive matrices* Trans. Amer. Math. Soc. **336** (1993) 121–149.

[**BH2**] ———, *Entropy versus orbit equivalence for minimal homeomorphisms* Pacific J. Math. **164** (1994) 1–13.

[**C**] C. Conley, *Isolated Invariant Sets and the Morse Index* CBMS Regional Conference Series in Math. **38** (1978)AMS, Providence.

[**Co**] A. Connes, *An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of* $\mathbf{R}$ Advances *in Mathematics* **39** (1981) 31–55.

[**DGS**] M. Denker, C. Grillenberg, and K. Sigmund, *Ergodic Theory on Compact Spaces* Springer Lecture Notes in Mathematics **527** Springer-Verlag (1976).

[**EHS**] E. Effros, D. Handelman and C.-L. Shen, *Dimension groups and their affine representations* American J. Math. **102** (1980) 385–407.

[**F1**] J. Franks, *Homology and Dynamical Systems* CBMS Regional Conference Series in Math. **49** (1982) AMS, Providence.

[**F2**] ———, *Flow equivalence of subshifts of finite type* Erg. Th. Dyn. Syst. **4** (1984) 53–66.

[**Fr**] D. Fried, *The geometry of cross sections to flows* Topology **21** (1982) 353–371.

[**GPS**] T. Giordano, I. Putnam, and C. Skau, *Topological orbit equivalence and C^*-crossed products* (to appear).

[**G**] K.R.Goodearl, *Partially Ordered Abelian Groups with Interpolation* Mathematical Surveys and Monographs **20** AMS Publications .

[**GH**] K. R. Goodearl and D. E. Handelman, *Stenosis in dimension groups and AF C^*-algebras* Crelle's *Journal* **332** (1982) 1–98.

[GH1] K. R. Goodearl and D. E. Handelman, *Rank functions and K_0 of regular rings* J. of *Pure and Applied Algebra* **7** (1976) 195–216.

[Gr] P. A. Griffith, *infinite abelian group theory* The University of Chicago Press (1970).

[HPS] R. Herman, I. Putnam and C. Skau, *Ordered Bratteli diagrams, dimension groups and topological dynamics* Int. Math. J. **3** 827–864.

[LM] D. Lind and B. Marcus, An Introduction to Symbolic Dynamics Cambridge Press (to appear).

[M] B. Marcus, *Factors and extensions of full shifts* Monatshefte *für Math.* **88** 239–247.

[PT] W. Parry and S. Tuncel, *Classification problems in Ergodic Theory* LMS Lecture Notes **67** Cambridge Press (1982).

[Pi] M. Pimsner, *Embedding some transformation group C^*-algebras into AF-algebras* Erg. Th. Dyn. Syst. **3** (1983) 113–126.

[Po] Y. T. Poon, *A K-theoretic invariant for dynamical systems* Trans. *Amer. Math. Soc.* **311** (1989) 515–533.

[Pu] I. Putnam, *The C^*-algebras associated with minimal homeomorphisms of the the Cantor set* Pacific J. Math. **136** (1989) 329–353.

[Roy] H. Royden, *Real Analysis* MacMillan (1963).

[Rob] C Robinson, *Introduction to the Theory of Dynamical Systems* CRC Press, to appear .

[Sch] S. Schwartzman, *Asymptotic cycles* Annals of Math. **66** (1957) 270–284.

[Sm] S. Smale, *Differentiable dynamical systems* Bull. AMS (1967) **73** 747–817.

[T] P. Trow, *Degrees of constant to one factor maps* Proc. AMS (1988) **103** 184–188.

[V] A. M. Veršik, *A theorem on periodical Markov approximation in ergodic theory* J. of *Soviet Math.* **28** (1985) 667–673.

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