Kupka's Theorem [13]

Suppose S and T are homeomorphisms of a topological space X such that S and T have the same orbits and the complement of the periodic points is dense and path connected. (For example, X is a manifold of dimension greater than one, and S has at most countably many periodic points.) Then $S = T$ or $S = T^{-1}$.

Dye's dramatic theorem tells us that the orbit equivalence relation of I is trivial: the orbits are too "loose", everything is orbit equivalent. Kupka's theorem shows the opposite extreme holds for III: rigidity is total: an orbit conjugacy of S and T can only be a conjugacy of S to T or to T^{-1}. The object of our study, IV, is somewhere between these extremes. The rigidity of III is lost because the domains may be highly disconnected (hence the natural entrance of the totally disconnected spaces of symbolic dynamics). For example (see 4.2), uncountably many homeomorphisms share the orbits of the 3-shift. On the other hand, orbit equivalence in IV is far from trivial in the sense of Dye's theorem. We go on now to note some invariants of orbit equivalence in IV. From here, unless otherwise noted, S and T are orbit equivalent homeomorphisms on compact metric spaces.

Orbit properties

Clearly S and T must agree on any property which depends only on orbits: for example, periodic point counts, minimality and transitivity.

Preserved Measures

Let M(S) denote the space of probability measures which are completions of Borel measures and are preserved by S. If S and T have the same orbits, then M(S) = M(T). To see this, define a (measurable) function n(x) such that $Tx = S^{n(x)}x$. Let $A_k = \{x: n(x) = k\}$. Given m from M(S) and a measurable set E, compute that

$$\begin{aligned} mE &= \sum_k m(E\cap A_k) \\ &= \sum mS^k(E\cap A_k) \\ &= \sum mT(E\cap A_k) \\ &= mTE. \end{aligned}$$

So, if h is an orbit conjugacy of S and T, then h induces a homeomorphism of M(S) and M(T); a measure m is taken to a measure hm, where hm is defined on a measurable set E by $hm(E) = m(h^{-1}E)$. In particular, orbit equivalence respects the cardinality of the space of preserved measures. For example, an orbit conjugate of a uniquely ergodic homeomorphism must be uniquely ergodic.

The range of a measure on cylinders

Given m in M(S), let R(m) denote {mE: E is a clopen set}, the range of m on cylinder sets. If S is a subshift, then there is a countable basis of clopen sets; in particular, R(m) is nonempty. The additive group generated by R(m) is called the winding numbers group; for more on this group, see [15].

If h is an orbit conjugacy of S to T, then h induces a bijection of clopen sets, and R(m) = R(hm). In particular, if there is some way to specify a pair of measures which must correspond under h, then their common range gives an invariant of orbit equivalence. For example, suppose S is uniquely ergodic, preserving a unique measure m. Then for T to be orbit equivalent to S, it is necessary that T be uniquely ergodic; but also the measure p preserved by T must satisfy R(p) = R(m). This invariant alone probably determines uncountably many distinct equivalence classes of orbit conjugate uniquely ergodic subshifts.

The following observations do not give invariants of orbit equivalence, but address technical issues.

Orientation

Suppose S and T have the same orbits, and for any point x, the following conditions hold:

(1) $\{S^n x: n>0\} \cap \{T^n x: n<0\}$ is finite;

(2) $\{S^n x: n<0\} \cap \{T^n x: n>0\}$ is finite.

Then we say that S and T have the same orientation--it makes sense to think of them as sharing a common past and future. If S has the same orientation as either T or T^{-1}, then we say that the orbit conjugacy is orientable.

Orientability is the key to the topological analogue of Belinskaya's theorem we prove in the next section. (To see the analogy, replace "ergodic" with "transitive"--that is, replace "measure theoretically indecomposable" with "topologically indecomposable"--and replace "integrable" with "continuous".) It is also fundamental to some ergodic theorems which do not depend on a preserved measure ([12]). Alas, orientation in our category may fail miserably. Say that the orbit of a point x is disoriented it it fails to satisfy (1) and (2) above. In (4.9), we give a homeomorphism which shares orbits with the 2-shift, while the set of disoriented orbits is a dense G_δ of full measure.

Flows

We point out two significant differences between orbit equivalence for continuous flows (II) and orbit equivalence for homeomorphisms (IV).

For homeomorphisms, recall $M(S) = M(T)$. However, flows with the same orbits need not respect the same measures. For example, Marcus ([14]) constructed a continuous flow S whose only preserved measure rested on a fixed point, and reparametrized S to a flow T which was not uniquely ergodic.

An orbit of a continuous flow is in a natural sense orientable ([11]). The corresponding result for homeomorphisms fails badly.

Jumps are "typically" continuous

Suppose S and T are homeomorphisms of a compact metric space X with the same orbits, and their aperiodic orbits are dense. Then there is a function $n(x)$ with the following properties:

(1) $Tx = S^{n(x)}x$;

(2) the set of points on which $n(x)$ is continuous contains a dense open set;

(3) the set of points on which $n(x)$ is continuous contains a dense invariant G_δ.

To see this, let $B_k = \{x: Tx=S^k x\}$, a closed set. Define $n(x) = k$ if x is in the interior of B_k; elsewhere, define $n(x)$ arbitrarily to satisfy $Tx = S^{n(x)}x$. To see the definition is consistent, suppose j is not equal to k and the interiors of B_j and B_k have nonempty intersection. Then this open set contains an aperiodic point x. But then $S^k x = Tx = S^j x$, so $S^{j-k}x = x$, a contradiction.

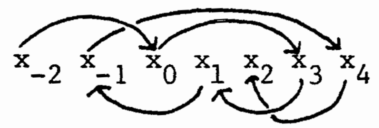
Clearly $n(x)$ is continuous on the union B of the interiors of the B_k. Let C_k denote the boundary of B_k, and let C be the union of the C_k. Then C is a countable union of closed, nowhere dense sets; Baire's theorem shows C has empty interior. Since C is the complement of B in X, B is a dense open set. The intersection of all translates of B is an invariant G_δ contained in B; by Baire's theorem, it is dense.

2. A Topological Analogue of Belinskaya's Theorem

The main goal of this section is theorem 2.6: if S and T are transitive homeomorphisms of a compact metric space orbit equivalent by continuous jumps, then S and T are flip conjugate. Moreover, the flip conjugacy may be defined to have the form $gx = T^{a(x)}x$, with $a(x)$ continuous. After proving (2.6), we state its natural generalization to the case where the jumps are only assumed to be bounded. Finally we show that a homeomorphism orbit equivalent to a subshift by bounded jumps must itself be a subshift.

To see why we need something like transitivity for (2.6), consider the following trivial example. Let S' be a transitive homeomorphism of X', not conjugate to its inverse. Let (S_1,X_1) and (S_2,X_2) be copies of (S',X'). Let S be the map defined by S_1 and S_2 on the disjoint union X of X_1 and X_2. Define T orbit equivalent to S on X by $T=S_1$ on X_1 and $T=(S_2)^{-1}$ on X_2. Then S and T cannot be flip conjugate.

The real work of the proof of (2.6) is done on subshifts. We will suppose through lemma 2.5 that S is not a finite permutation, S is a subshift, and T is a homeomorphism such that $Tx = S^{n(x)}x$, with n continuous. We let $N = \max |n(x)|$ and we let X denote the common domain of S and T. By recoding, we also assume through lemma 2.5 that n(x) depends only on x_o. Notice that $(Sx)_o=x_1$, while $(Tx)_o=x_{n(x)}$. So we may visualize the action of T on the orbit of a point $x = \dots x_{-1}x_o x_1 \dots$ as a succession of arrows which shows the succession of zero coordinates under T, for example



$$x \;=\; \ldots\; x_{-2}\; x_{-1}\; x_0\; x_1\; x_2\; x_3\; x_4\; \ldots$$

Although not strictly necessary, it is convenient for visualization to define a canonical subshift $\bar{T}$ conjugate to T. We will set the alphabet of $\bar{T}$ equal to the alphabet of S, and define a map h from X to $\bar{T}$, $h(x)=\bar{x}$, by $\bar{x}_i=(T^i x)_o$, for all i. For example, given the sample x with arrows above, we have

$$\bar{x} \;=\; \ldots\; \bar{x}_{-1}\; \bar{x}_0\; \bar{x}_1\; \bar{x}_2\; \bar{x}_3\; \bar{x}_4\; \bar{x}_5\; \ldots$$

$$=\; \ldots\; x_{-2}\; x_0\; x_3\; x_1\; x_{-1} x_4\; x_2\; \ldots \qquad .$$

Because n(x) depends only on x_o, h is injective and continuous; the image is compact, so $\bar{T}$ is a subshift; clearly $hT = \bar{T}h$, so h is a conjugacy.

(2.0) <u>Definition</u>

Let $f_k(x) = n(x) + n(Tx) + \ldots + n(T^{k-1}x)$, if k is nonnegative; if k is negative, let $f_k(x) = n(x) + n(T^{-1}x) + \ldots + n(T^{k+1}x)$. Then $f_k(x)$ is the coordinate of x used to define the k^{th} coordinate of $\bar{x}$: if $i=f_k(x)$, then $x_i = \bar{x}_k = (T^k x)_o$. Also, if $j(x)=f_1(x)$, then $Tx = S^{j(x)}x$. In the example above, $f_3(x) = -1$.

(2.1) <u>Lemma</u> (Bijection of coordinates)

For each x, the map $k \to f_k(x)$ is a bijection of the integers.

Proof

If x is aperiodic, then the lemma is trivial. So, for the rest of the proof suppose $S^k x=x$. Then $T^k x=x$. So, $f_k(x)$ is a multiple of k. Each f_i is continuous. If j is not equal to i and $f_j(x)=f_i(x)$, then f_j and f_i agree in some neighborhood of x, hence on some aperiodic y: but then $T^j y = T^i y$ and $T^{j-i} y = y$, a contradiction. In particular, $f_k(x)$ is nonzero. It remains to show $f_k(x)$ is either k or -k. Given y, let $F_+(y)$ denote the set of coordinates of y used for positive coordinates of $\bar{y}$; that is,

$F_+(y) = \{f_k(y): k>0\}$; likewise, let

$F_-(y) = \{f_k(y): k<0\}$.

Suppose $f_k(x)$ is strictly greater than k (the argument is essentially the same if $f_k(x)$ is strictly less than -k). Notice that for all integers m, $f_{mk}(x) = mf_k(x)$; in particular, $F_+(x)$ contains infinitely many positive integers but only finitely many negative integers, while $F_-(x)$ contains infinitely many negative integers but only finitely many positive integers. Pick a positive integer b which dominates the absolute value of any negative integer in $F_+(x)$ and any positive integer in $F_-(x)$. By transitivity, since S is not a permutation, a sequence $\{y^m\}$ of aperiodic points converges to x. If a string of symbols periodic like x occurs in some y^m, then the periodic pattern must eventually be broken to the right and/or left of this string, since y^m is aperiodic. Therefore, appropriate translates of a subsequence of $\{y^m\}$ converge to an aperiodic y for which one of the following hold:

(1) $y_i = x_i$ for $i \geq -b$, or

(2) $y_i = x_i$ for $i \leq b$.

We suppose (1)--the argument is essentially the same for (2). Then $F_+(y) = F_+(x)$. But y is aperiodic, so every nonzero integer is included in the union of $F_+(y)$ and $F_-(y)$. So, $F_-(y)$ contains infinitely many positive integers and infinitely many negative integers. So, in the sequence $\{f_{-1}(y), f_{-2}(y), f_{-3}(y), \ldots\}$, there are infinitely many k such that $f_{-k}(y)$ is nonnegative while $f_{-(k+1)}(y)$ is nonpositive. Since $|f_{-(k+1)}(y) - f_{-k}(y)| = |n(T^{-k}y)| \leq N$, the map $k \to f_{-k}(y)$ cannot be injective, a contradiction. □

(2.2) Lemma (Roadblocks)

Given a positive integer M, there is a positive integer $\bar{M}$ such that for all x, $\{m: -M \leq m \leq M\}$ is contained in $\{f_k(x): -\bar{M} \leq k \leq \bar{M}\}$.

Proof

By (2.1), some $\bar{M}$ exists for each x; since the f_k are continuous, $\bar{M}$ applies in a neighborhood of x. Use compactness. □

Remark In other words, coordinates of x near zero are "used up" for coordinates of $\bar{x}$ at a rate uniform over X: for any x, if $-M \leq m \leq M$, then x_m is used for some $\bar{x}_k$, $-\bar{M} \leq k \leq \bar{M}$.

(2.3) Lemma (Orientation)

Let

$A_m = \{x: \forall n \geq m,\ f_n(x) > 0 \text{ and } f_{-n}(x) < 0\}$,

$B_m = \{x: \forall n \geq m,\ f_n(x) < 0 \text{ and } f_{-n}(x) > 0\}$.

Then there is a positive integer k such that $X = A_k$ or $X = B_k$.

Remark In other words, it makes sense to speak of T as preserving or reversing the orientation of S. This might not be possible if n(x) is allowed to be unbounded (example 4.9).

Proof

Using (2.2), choose k such that $\{m: -N \leq m \leq N\}$ is contained in $\{f_i(x): -k \leq i \leq k\}$. By (2.1), $A_k \cap B_k = X$. (For all m greater than k, $T^m x$ must have its zero coordinate on the same side of the "roadblock"; then for all m less than -k, $T^m x$ must have its zero coordinate on the other side.) Since n(x) is continuous, both A_k and B_k are open. Clearly they are invariant. By transitivity, one must be empty. □

Let

$P_k(x) = |\{f_i(x): f_i(x)>0, |i|\leq k\}|$,

$N_k(x) = |\{f_i(x): f_i(x)<0, |i|\leq k\}|$.

Then $P_k(x)$, for example, is the cardinality of the set of positive coordinates of x used in getting coordinates -k through k of $\bar{x}$. Pick k as in lemma 2.3: then for any $M \geq k$, the following definition is the same:

$a(x) = \frac{1}{2}[N_M(x) - P_M(x)]$.

Notice that a(x) is continuous.

We will define a conjugacy g from S to $\bar{T}$ (possibly after replacing T with T^{-1}) by $gx = \bar{T}^{a(x)}\bar{x}$. Visually, a(x) slides the block $\bar{B}$ defined by $\bar{x}_{-M}\ldots\bar{x}_M$ so that it sits about $\bar{x}_o$ in the same way that the block B defined by coordinates $f_{-M}(x), \ldots, f_M(x)$ sits about x_o. That is, $\bar{x}_{-M+a(x)}\ldots\bar{x}_{M+a(x)}$ also has $N_M(x)$