A Classification of BL-Algebras

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Abstract

We give an algebraic classification of BL-chains, which generalizes to a classification of BL-algebras. Using this, we classify the finite and the finitely universal BL-chains. From these results we strengthen Hájek's Completeness Theorem for BL-logic and give a new proof of the decidability of the set of BL-tautologies.

1 Introduction

BL-algebras arise naturally in the analysis of the proof theory of propositional fuzzy logics. Indeed, in [Ha 1], Hájek introduces the system of Basic Logic (BL) axioms for propositional logic and defines the class of BL-algebras (see Definition 1.1). He shows that a propositional formula φ is provable from the BL-axioms if and only if φ is an $\mathcal{M}$-tautology for every BL-algebra $\mathcal{M}$.

In this paper we give a complete characterization of the restricted class of BL-chains (Theorem 3.5). Since every BL-algebra is a subalgebra of a product of BL-chains (see Lemma 2.3.16 of [Ha 1]), this in turn yields a

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classification of all BL-algebras. We use our characterization to classify the *finite* BL-chains (Proposition 5.3) and prove that collectively the class of finite BL-chains has a certain universality property. Specifically, for every finite subset X of every BL-chain there is a finite embedding (see Definition 5.4) of X into some finite BL-chain. It follows that every existential sentence that is true in some BL-chain is true in some finite BL-chain. This result implies the decidability of the universal theory of BL-chains (Corollary 5.10).

Finally, we consider the class of all *finitely universal* BL-chains (i.e., BL-chains that embed every finite BL-chain). Theorem 5.13 enumerates a number of characterizations of this class of chains. It follows that this class is elementary and it is straightforward to construct a decidable finitely universal BL-chain. The existence of such an object immediately yields a new proof of a known proof-theoretic result, namely that the set of BL-tautologies is decidable. (In [BHMV], Baaz et.al. show that the set of BL-tautologies is in co-NP).

It should be noted that the classification of BL-chains given here is very similar to Aglianó and Montagna's representation [AM] of BL-chains by an ordinal sum of a family of Wajsberg hoops. As well, certain finitely universal BL-chains have been presented (see e.g. [BHMV]). However, to our knowledge the characterization of the class of finitely universal BL-chains presented here is new. Additionally, while other approaches have concentrated on the Mostert-Shields representation [MS] of continuous t-norms on $[0, 1]$, our arguments rely on some classical results about ordered Abelian groups.

Some of the ideas in this paper were gleaned from the work of Hájek [Ha 2] and Cignoli et.al. [CEGT], who characterize the class of 'saturated' BL-chains. However, our approach is somewhat different, and we do not use their classification here. Instead, we first identify a class of very desirable ordered Abelian semigroups (the 'basic forms' of Definition 3.1) and show that every BL-chain has an associated 'tower of basic forms.' Conversely, every tower of basic forms naturally yields a BL-chain (Proposition 3.4). What is noteworthy is that unlike the classification of the saturated BL-chains due to Hájek and Cignoli et.al., the basic building blocks of BL-chains are not BL-chains themselves. Example 4.6 describes a wide family of BL-chains that have only two idempotents (namely 0 and 1), and hence cannot be decomposed into smaller BL-chains. However, each of these BL-chains can be decomposed into smaller algebraic components.

Section 2 is devoted to the analysis of certain ordered Abelian semigroups that will be used in Section 3. Section 3 contains the main theorem of the paper, Theorem 3.5, which classifies the BL-chains. Section 4 is devoted to examples and a discussion of the uniqueness of the decomposition given in Section 3. In Section 5 we use Theorem 3.5 to classify the finite and the finitely universal BL-chains and derive the corollaries mentioned above.

We close the introduction by defining the principal objects of this paper:

Definition 1.1 A *BL-algebra* is a structure $\mathcal{M} = (M, \star, \Rightarrow, \leq, \cap, \cup, 0, 1)$ that satisfies the following:

1. $(M, \leq, \cap, \cup, 0, 1)$ is a lattice with smallest element 0 and largest element 1 (with respect to the lattice ordering $\leq$).
2. $(M, \star, 1)$ is an Abelian semigroup with the unit element 1.
3. Residuation: For all $x, y, z \in M$, $z \leq (x \Rightarrow y)$ if and only if $x \star z \leq y$.
4. For all $x, y \in M$, $x \star (x \Rightarrow y) = x \cap y$.
5. For all $x, y \in M$, $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$.

A *BL-chain* is a BL-algebra with the additional property that $\leq$ is a linear order.

An *MV-chain* is a BL-chain that satisfies 'double negation,' i.e., $x = [(x \Rightarrow 0) \Rightarrow 0]$ for all x.

2 Results on Ordered Abelian Semigroups

We begin this section by setting our notation.

Definition 2.1 An *ordered Abelian semigroup* $(G, \star, \leq)$ is a linear order $\leq$ and a commutative and associative operation $\star$ which satisfies

$$x \leq y \quad \text{implies} \quad x \star z \leq y \star z$$

for all $x, y, z \in G$. (We do *not* assume that G has an identity element).

An *ordered Abelian group* $(G, \star, \leq)$ is an ordered Abelian semigroup with an identity element such that every element has an inverse. In other words, $(G, \star)$ is a group. We will assume throughout the paper that ordered Abelian groups are infinite. That is, we will exclude the case of $G = \{0\}$.

Let $(G, \star, \leq)$ be any ordered Abelian group. Let d be a nonzero element of G. Let n be a positive integer. We say that d is *divisible by* n if there is a $g \in G$ such that $ng = d$ (where ng denotes $g \star g \star g \star \ldots \star g$, n copies).

Let d be a nonzero element of G. We say that d is *divisible* if d is divisible by n for all positive integers n. We say that G is a *divisible ordered Abelian group* if every nonzero element of G is divisible.

The following types of ordered Abelian semigroups will be used throughout the paper:

Definition 2.2 *Let $(G, \star, \leq)$ be any ordered Abelian group.*

1. The *negative cone* of G is the substructure $(N(G), \star, \leq)$ of G with universe $\{x \in G : x < 0_G\}$.
2. The *extended negative cone* $(N_{-\infty}(G), \star, \leq)$ is an extension of $N(G)$ with universe $N(G) \cup \{-\infty\}$, where $\star$ and $\leq$ are extended by the definitions:
   $x \star (-\infty) = (-\infty) \star x := -\infty$ for all $x \in N_{-\infty}(G)$,
   $-\infty < x$ for all $x \in N(G)$.
3. Choose any $d \in N(G)$. The *truncation of $N(G)$ at d* is the structure $(T(G, d), \star_T, \leq_T)$ with universe $\{x \in N(G) : x \geq d\}$, where $\leq_T$ is inherited from $\leq_G$ and $\star_T$ is defined by:

$$x \star_T y := \begin{cases} x \star_G y & \text{if } x \star_G y > d \\ d & \text{if } x \star_G y \leq d \end{cases}$$

Note: Our notation differs from the standard usage of the word 'cone' since the element 0_G is not included in $N(G)$.

The next two lemmas give algebraic characterizations of negative cones and truncations that will be used in Section 3.

Lemma 2.3 *Let $(S, \star, \leq)$ be an ordered Abelian semigroup that satisfies:*

1. *For all $x, y \in S, x > x \star y$.*
2. *For each $x, y \in S$ such that $x > y$, there is a $z \in S$ such that $x \star z = y$.*

Then $(S, \star, \leq) \cong (N(G), \star, \leq)$ for some ordered Abelian group G.

Proof: This is similar to Lemma 1.6.9 of [Ha 1]. We begin by showing that for all $x, y, z \in S$, $x \star y = x \star z$ implies that $y = z$. Suppose that $y < z$. By Clause (2), there is some $u \in S$ such that $y = z \star u$. Then $x \star y = x \star (z \star u) = (x \star z) \star u < x \star z$ by Clause (1), which proves our claim. Let S^+ denote the ordered Abelian semigroup with universe $S \cup \{e\}$, where $x < e$ for all $x \in S$, and $x \star e = x$ for all $x \in S$. Then S^+ is cancellative and (negatively) naturally ordered. Thus, by Proposition 1, page 154 of [Fu], there is an ordered Abelian group G such that S^+ is isomorphic to the cone of non-positive elements of G. We leave it to the reader to verify that $(S, \star, \leq) \cong (N(G), \star, \leq)$.

Lemma 2.4 *Let $(S, \star, \leq)$ be the ordered Abelian semigroup with a $\leq$-least element d that satisfies the following conditions:*

1. *For any $x, y \in S$, $x \star y \leq x$.*
2. *For any $x, y \in S$, if $x \star y = x$, then $x = d$.*
3. *There are $x, y \in S$, $x, y > d$ such that $x \star y = d$.*
4. *For each $x, y \in S$ such that $x > y$, there is a largest $z \in S$ such that $x \star z = y$.*

Then $(S, \star, \leq) \cong (T(G, d), \star, \leq)$ for some ordered Abelian group G and some $d \in N(G)$.

Note that Clauses (2) and (4) together imply that for each $x, y \in S$ such that $x > y > d$, there is a unique $z \in S$ such that $x \star z = y$. Reason as follows: Assume by way of contradiction that $x, y, z, w \in S$ satisfy $x > y > d$, $z < w$, and $x \star z = y = x \star w$. Let $u \in S$ be the largest such that $w \star u = z$. Then $z \star x = (w \star u) \star x = (w \star x) \star u = (z \star x) \star u$, which contradicts Clause (2). The case of $z > w$ is symmetric.

Proof of 2.4: The lemma can be proven directly by explicitly exhibiting the group G. However, since the verification that G is associative is lengthy, we will use Chang's Classification of MV-chains, see [Ch] or Theorem 3.2.19 of [Ha 1]. Let $S^+ := S \cup \{c\}$, and define $c \star s = s \star c = s$ for all $s \in S^+$, and $s \leq c$ for all $s \in S^+$. Then S^+ can be expanded to an MV-chain by defining

$$x \Rightarrow y = \begin{cases} c & \text{if } x \leq y \\ \text{the unique } z \text{ such that } x \star z = y & \text{if } y < x \end{cases}$$

$$x \cap y = \min\{x, y\}, x \cup y = \max\{x, y\}, 0_{S+} = d, 1_{S+} = c$$

It is straightforward to verify that $(S^+, \star, \Rightarrow, \leq, \cap, \cup, 0, 1)$ is a BL-chain. In fact, more is true:

Claim: $(S^+, \star, \Rightarrow, \leq, \cap, \cup, 0, 1)$ is an MV-chain.

Proof of Claim: The arguments used here are similar to those found in Lemma 3.3 of [CEGT]. In order to prove the claim, we need to show that $\neg\neg a = a$ for all $a \in S^+$, where $\neg x$ is an abbreviation for $(x \Rightarrow d)$. To see this, first recall two technical facts that hold in any BL-chain:

1. For all $a \in S^+$ $\neg\neg a \geq a$ and $\neg\neg\neg a = \neg a$. (Proposition 1.3.14 (1.51, 1.55) of [Tur])
2. For all $a, b \in S^+$, $(a \star b) \Rightarrow d \ = \ b \Rightarrow (a \Rightarrow d)$. (Proposition 1.3.13 (1.47) of [Tur])

Subclaim: $\neg u \in S \setminus \{d\}$ for all $u \in S \setminus \{d\}$.

Proof of Subclaim: First fix $x, y \in S \setminus \{d\}$ such that $x \star y = d$. Let $w = \neg\neg x$. From Fact (1), $w \geq x > d$. Also, $w \neq c$ since $w \star \neg x = d$ but $c \star \neg x = \neg x \geq y > d$. Now assume, by way of contradiction, that $u \in S \setminus \{d\}$ and $\neg u = d$. Note that if $u \leq w$, then we have a contradiction since $u \star (x \Rightarrow d) = d$ by residuation, implying that $\neg u \geq (x \Rightarrow d) > d$. So we may assume $w < u$. Thus:

$$u \Rightarrow w \ = \ u \Rightarrow (\neg x \Rightarrow d) \ = \ \neg x \Rightarrow \neg u \ = \ \neg x \Rightarrow d \ = \ w$$

The second equality comes from two applications of Fact (2). Thus, $u \star w = u \star (u \Rightarrow w) = u \cap w = w$, but this contradicts Clause (2) of the hypothesis. So we have established the subclaim.

To complete the argument that S^+ is an MV-chain, fix any $a \in S^+$. Clearly, if $a = c$ or $a = d$ then $\neg\neg a = a$, so assume by way of contradiction that $a \in S \setminus \{d\}$ but $\neg\neg a > a$. Let $u = \neg\neg a \Rightarrow a$. Then $u \in S$, and by the subclaim, $\neg a \in S \setminus \{d\}$, but:

$$\begin{aligned}
\neg a &= (\neg\neg a \star (\neg\neg a \Rightarrow a)) \Rightarrow d \\
&= (\neg\neg a \star u) \Rightarrow d \\
&= u \Rightarrow (\neg\neg a \Rightarrow d) \text{ by Fact}(2) \\
&= (u \Rightarrow \neg\neg\neg a) \\
&= (u \Rightarrow \neg a) \text{ by Fact}(1)
\end{aligned}$$

Thus $u \star \neg a = u \star (u \Rightarrow \neg a) = (u \cap \neg a) = \neg a$. But $\neg a > d$, so this again contradicts Clause (2) of the hypothesis.

So by Chang's Theorem (Theorem 3.2.19 of [Ha 1]), there is an ordered Abelian group G and $d \in N(G)$ such that $(S^+, \star, \leq) \cong ([d, 0], \star, \leq)$. Hence $(S, \star, \leq) \cong (T(G, d), \star, \leq)$.

3 The Decomposition Theorem

In this, the main section of the paper, we prove a decomposition theorem for the class of BL-algebras. The decomposition of an arbitrary BL-algebra occurs in two stages, the first of which was done by Hájek [Ha 1]. He showed (Lemma 2.3.16 of [Ha 1]) that every BL-algebra is a subalgebra of a direct product of BL-chains. Conversely, the fact that every subalgebra of a direct product of BL-chains is a BL-algebra is immediate since the class of BL-algebras is a variety. Thus, in order to classify all BL-algebras, it suffices to classify the BL-chains. With this in mind, we begin by identifying a family of ordered Abelian semigroups, which we call the basic forms. The fundamental building blocks of an arbitrary BL-chain will come from this family.

Definition 3.1 Let $\mathcal{C} := (C, \star, \leq)$ be an ordered Abelian semigroup. We say that $\mathcal{C}$ is a *basic form* if one of the following holds:

1. $\mathcal{C}$ is a singleton $\{p\}$, where $p \star p = p$ and $p \leq p$.

2. $\mathcal{C} \cong N(G)$ for some ordered Abelian group G.

3. $\mathcal{C} \cong N_{-\infty}(G)$ for some ordered Abelian group G.

4. $\mathcal{C} \cong T(G, d)$ for some ordered Abelian group G and some $d \in N(G)$.

Remark 3.2 For any basic form $\mathcal{C} = (C, \star, \leq)$, if $x, y \in C$ and $x > y$, then there is a largest $z \in C$ such that $x \star z = y$.

Definition 3.3 A *tower of basic forms* is a sequence $\mathcal{T} = \langle \mathcal{C}_i : i \in I \rangle$ indexed by a linearly ordered set $(I, \leq)$ with a first and a last element such that each $\mathcal{C}_i := (C_i, \star, \leq)$ is a basic form, $C_i \cap C_j = \emptyset$ for all $i, j \in I$ such that $i \neq j$, $\mathcal{C}_{\text{first}}$ has a least element, and $\mathcal{C}_{\text{last}}$ is a singleton.

Associated to any tower of basic forms is a canonical BL-chain $\mathcal{A}_{\mathcal{T}} := (A, \star, \Rightarrow, \leq, \cup, \cap, 0, 1)$ built from $\mathcal{T}$ defined by:

- $A := \bigcup\{C_i : i \in I\}$;
- For $x \in C_i$, $y \in C_j$, $x \leq_{\mathcal{T}} y$ if and only if $[i <_I j$ or $(i = j$ and $x \leq_{\mathcal{C}_i} y)]$;
- $0_{\mathcal{T}} :=$ the least element of C_{first}, and $1_{\mathcal{T}} :=$ the unique element of C_{last}
- For $x, y \in A$,

$$x \star_{\mathcal{T}} y = \begin{cases} x \star_{\mathcal{C}_i} y & \text{for } x, y \in C_i \text{ for some } i \in I \\ \min\{x, y\} & \text{for } x \in C_i, y \in C_j, i \neq j \end{cases}$$

  It follows from Remark 3.2 that for any $x, y \in A$ with $x >_{\mathcal{T}} y$, there is a largest $z \in A$ with $x \star_{\mathcal{T}} z = y$. So define:

- For $x, y \in A$,

$$x \Rightarrow_{\mathcal{T}} y = \begin{cases} 1_{\mathcal{T}} & \text{if } x \leq_{\mathcal{T}} y \\ \text{largest } z \in A \text{ such that } x \star_{\mathcal{T}} z = y & \text{if } x >_{\mathcal{T}} y \end{cases}$$

- For $x, y \in A$, $x \cap y = \min\{x, y\}$ and $x \cup y = \max\{x, y\}$

Note that $\mathcal{A}_{\mathcal{T}}$ is similar to the ordinal sum construction in Fuchs [Fu].

The verification of the following proposition is straightforward and left to the reader:

Proposition 3.4 *For any tower $\mathcal{T}$ of basic forms, the structure $\mathcal{A}_{\mathcal{T}}$ constructed above is a BL-chain.*

Conversely, we have the following theorem, which is the main result of the paper:

Theorem 3.5 *Every BL-chain $\mathcal{A}$ is isomorphic to $\mathcal{A}_{\mathcal{T}}$ for some tower $\mathcal{T}$ of basic forms.*

Before beginning the proof of Theorem 3.5, we define the manner in which we decompose any given BL-chain:

Definition 3.6 Let $\mathcal{A}$ be any BL-chain.

1. For any $x, y \in A$, we say that y *stabilizes* x if $y \star x = x$.

2. For any $x \in A$, let $S_x := \{y \in A : y \text{ stabilizes } x\}$.

3. We define an equivalence relation $\sim$ on $\mathcal{A}$ by: $a \sim b$ if and only if for all $x \in A$, a stabilizes x if and only if b stabilizes x.

4. For any $a \in A$, $C(a)$ is the equivalence class of a under $\sim$.

Lemma 3.7 *Let $\mathcal{A}$ be any BL-chain.*

1. *For any $x \in A$, S_x is upward closed; that is, if $y \leq z$ and $y \in S_x$, then $z \in S_x$.*

2. *If $a \leq b$, then $S_b \subseteq S_a$.*

3. *If x and y are in different $\sim$-equivalence classes, then $x \star y = \min\{x, y\}$.*

4. *If $x \sim y$ and y stabilizes x, then x is the $\leq$-least element of $C(x)$.*

5. *$C(1) = \{1\}$.*

6. *Every $\sim$-equivalence class is a convex subset of $\mathcal{A}$.*

7. *Every $\sim$-equivalence class is closed under $\star$.*

8. *If $x \sim y$ and $x < y$, then $x \sim (y \Rightarrow x)$.*

Proof of 1: Assume $y \in S_x$, and $y \leq z$. Then $x \star z \geq x \star y = x$, and we always have $x \star z \leq x$, so $x \star z = x$.

Proof of 2: Assume $a \leq b$ and $y \star b = b$. Then $y \star a = y \star (a \cap b) = y \star (b \star (b \Rightarrow a)) = b \star (b \Rightarrow a) = a \cap b = a$.

Proof of 3: Suppose $x < y$. Choose $b \in A$ such that y stabilizes b but x does not stabilize b. Let $u = x \star b < b$, and let $v = (b \Rightarrow u)$. Notice that $y \star b = b$, but $v \star b = (b \Rightarrow u) \star b = u$, so these two imply that $v < y$. We have that $(y \Rightarrow v) \star b = (y \Rightarrow v) \star (y \star b) = v \star b = u$. So by residuation, $(y \Rightarrow v) \leq (b \Rightarrow u) = v$. Then by monotonicity, $y \star (y \Rightarrow v) \leq y \star v$, which means that $v \leq y \star v$. Thus y stabilizes v. But $x \star b = u$ by definition, so by residuation, $x \leq v$. Thus y stabilizes x by (2).

Proof of 4: Choose any $b \in C(x)$. Since y stabilizes x, we must also have that b stabilizes x, so $b \star x = x$, hence $b \geq x$.

Proof of 5: Choose any $a \in C(1)$. Then a stabilizes every $b \in A$, hence a stabilizes 1. That is, $a \star 1 = 1$, so $a = 1$.

Proof of 6: Assume $b \sim c$ and $b < u < c$. We will show $u \sim b$. First, for any $x \in A$, if $u \in S_x$, then $c \in S_x$ since S_x is upward closed, hence $b \in S_x$ since $b \sim c$. Conversely, if $b \in S_x$, then $u \in S_x$ again since S_x is upward closed. Thus $b \sim u$.

Proof of 7: Assume $b \sim c$. We will show $b \star c \sim b$. First, for any $x \in A$, if $b \star c \in S_x$, then $b \in S_x$ since S_x is upward closed. Conversely, if $b \in S_x$, then $c \in S_x$ as well, so $b \star x = x$ and $c \star x = x$, hence $(b \star c) \star x = b \star (c \star x) = b \star x = x$.

Proof of 8: If $(y \Rightarrow x) = x$ then we are done. Otherwise, we have $(y \Rightarrow x) \star y = y \cap x = x$, which means in particular that $(y \Rightarrow x) \star y \neq \min\{y, (y \Rightarrow x)\}$. So y and $(y \Rightarrow x)$ must be in the same $\sim$-class by (3).

Proof of Theorem 3.5: Let $\mathcal{A}$ be any BL-chain. Since the $\sim$-equivalence classes of $\mathcal{A}$ are convex, the ordering on $\mathcal{A}$ induces an ordering $(I, \leq)$ on the $\sim$-equivalence classes. For $i \in I$, let C_i be the i^{th} $\sim$-equivalence class. By Lemma 3.7 (7), each C_i is closed under $\star$, hence each $(C_i, \star, \leq)$ (with $\star$ and $\leq$ inherited from $\mathcal{A}$) is an ordered Abelian semigroup which we denote by $\mathcal{C}_i$. The bulk of the proof of the theorem is devoted to establishing the following claim:

Claim: The sequence $\mathcal{T} = \langle \mathcal{C}_i : i \in I \rangle$ is a tower of basic forms.

Proof of Claim: Since $C(0_{\mathcal{A}})$ and $C(1_{\mathcal{A}})$ are the first and last equivalence classes, I has a first and a last element. Clearly $C(0_{\mathcal{A}})$ has a least element (namely $0_{\mathcal{A}}$) and by Lemma 3.7 (5), $C(1_{\mathcal{A}}) = \{1_{\mathcal{A}}\}$. Thus, proving the claim amounts to showing that each $\mathcal{C}_i$ is a basic form. Fix any equivalence class $\mathcal{C} = (C, \star, \leq)$. The argument splits into four cases:

Case 1 $\mathcal{C}$ *is a singleton.*

There is only one ordered Abelian semigroup of size 1 up to isomorphism.

Case 2 $\mathcal{C}$ *does not have a $\leq$-least element.*

We will show that $(C, \star, \leq) \cong (N(G), \star, \leq)$ for some ordered Abelian group G by applying Lemma 2.3 to $\mathcal{C}$. We have already proven that $\mathcal{C}$ is closed under $\star$. Thus $\mathcal{C}$ inherits the property of being an ordered Abelian semigroup directly from $\mathcal{A}$. Since $\mathcal{C}$ does not have a $\leq$-least element, it follows from Lemma 3.7, Clause (4) that $x \star y < x$ for all $x, y \in C$. Towards verifying Clause (2) of Lemma 2.3, choose $x, y \in C$, $x > y$. Let $z := x \Rightarrow y$. Then $z \in C$ by Lemma 3.7, Clause (8), and $x \star z = y$.

Case 3 *$\mathcal{C}$ has a $\leq$-least element d, $|C| > 1$, and for all $x, y \in C$ such that $x, y > d$, we have $x \star y > d$.*

We will show that $(C, \star, \leq) \cong (N_{-\infty}(G), \star, \leq)$ for some ordered Abelian group G. Let $S := C \setminus \{d\}$. Since $x \star y > d$ whenever $x, y > d$, S is closed under $\star$, hence $(S, \star, \leq)$ is a nonempty ordered Abelian semigroup. So $S \cong N(G)$ for some ordered Abelian group G by the same argument as that given in Case 2. Extend this isomorphism by mapping d to $-\infty$, so that $\mathcal{C} \cong N_{-\infty}(G)$.

Case 4 *$\mathcal{C}$ has a $\leq$-least element d, and there are $x, y \in C$, $x, y > d$ such that $x \star y = d$.*

We will show that $(C, \star, \leq) \cong (T(G, d), \star, \leq)$ for some ordered Abelian group G. We do this by showing that $\mathcal{C}$ satisfies the hypotheses of Lemma 2.4: Clause (2) holds by Lemma 3.7 (4), and Clause (3) holds by assumption. So we need only verify Clause (4). Assume $x, y \in C$ and $x > y$. Let $z := (x \Rightarrow y)$. By Lemma 3.7, Clause (8), $z \in C$. Since $\mathcal{A}$ is a BL-chain, z is the largest element of A (and hence of C) such that $x \star_{\mathcal{A}} z = y$. Since $\star_{\mathcal{C}}$ is inherited from $\star_{\mathcal{A}}$, z is the largest element of C such that $x \star_{\mathcal{C}} z = y$.

To complete the proof of the theorem, let $\mathcal{A}_{\mathcal{T}}$ be the BL-chain associated to the tower $\mathcal{T} = \langle \mathcal{C}_i : i \in I \rangle$. Clearly, the universe of $\mathcal{A}_{\mathcal{T}}$ is $\bigcup\{C_i : i \in I\}$, which is precisely the universe of our original $\mathcal{A}$. We claim that the identity map between $\mathcal{A}$ and $\mathcal{A}_{\mathcal{T}}$ is an isomorphism. We show this by establishing that the functions, relations, and constants correspond. The verification of these is routine. As an example, we establish the correspondence for $\star$: When x, y are in the same $\mathcal{C}$, $x \star_{\mathcal{T}} y$ is inherited from $\mathcal{A}$, so there is nothing to show. When x, y are from distinct $\sim$-equivalence classes, it follows that $x \star_{\mathcal{T}} y = \min\{x, y\} = x \star_{\mathcal{A}} y$ by Lemma 3.7 (3). The other cases are also easily established.

The following corollary follows immediately since every BL-algebra is a subalgebra of a direct product of BL-chains.

Corollary 3.8 *For any BL-algebra $\mathcal{M}$, there is a collection of basic forms $\{\mathcal{C}_{ij} : i \in I_j, j \in J\}$ so that $\mathcal{M} \subseteq \prod\{\mathcal{A}_j : j \in J\}$ and each $\mathcal{A}_j \cong \mathcal{A}_{\mathcal{T}_j}$ where $\mathcal{T}_j = \langle \mathcal{C}_{ij} : i \in I_j \rangle$.*

4 On the Uniqueness of Decompositions

We begin this section by giving examples of how the three 'classical' BL-chains decompose into $\sim$-equivalence classes via the construction in Theorem 3.5.

In what follows, $\mathbb{R}$ denotes the additive ordered Abelian group of the reals.

Example 4.1 *1.* The standard Łukasiewicz algebra:
$[0,1]_{Ł}$ denotes the BL-chain with universe $[0,1]$ and operations defined by:

$$x \star y = \max\{0, x+y-1\}$$

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ 1-x+y & \text{if } x > y \end{cases}$$

Then $[0,1]_{Ł} \cong \mathcal{A}_{\mathcal{T}_{Ł}}$, where $\mathcal{T}_{Ł} = \langle \mathcal{C}_0, \mathcal{C}_1 \rangle$, where $(C_0, \star, \leq) \cong (T(\mathbb{R}, -1), \star, \leq)$ and C_1 is a singleton.

2. The standard Gödel algebra:
$[0,1]_{G}$ denotes the BL-chain with universe $[0,1]$ and operations defined by:

$$x \star y = \min\{x, y\}$$

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$$

Then $[0,1]_{G} \cong \mathcal{A}_{\mathcal{T}_{G}}$, where $\mathcal{T}_{G} = \langle \mathcal{C}_i : i \in [0,1] \rangle$, where every C_i is a singleton.

3. The standard Product algebra:
$[0,1]_{\prod}$ denotes the BL-chain with universe $[0,1]$ and operations defined by:

$$x \star y = xy$$

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{if } x > y \end{cases}$$

Then the decomposition of $[0,1]_{\prod}$ given by $\sim$-classes yields a decomposition $[0,1]_{\prod} \cong \mathcal{A}_{\mathcal{T}_{\prod}}$, where $\mathcal{T}_{\prod} = \langle \mathcal{C}_0, \mathcal{C}_1 \rangle$, where $(C_0, \star, \leq) \cong (N_{-\infty}(\mathbb{R}), \star, \leq)$ and C_1 is a singleton. However, note that $[0,1]_{\prod}$ is also isomorphic to $\mathcal{A}_{\mathcal{T}'_{\prod}}$, where $\mathcal{T}'_{\prod} = \langle \mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2 \rangle$, where $(C_1, \star, \leq) \cong (N(\mathbb{R}), \star, \leq)$ and both C_0 and C_2 are singletons.

Although Theorem 3.5 ensures us that any BL-chain $\mathcal{A}$ is isomorphic to $\mathcal{A}_{\mathcal{T}}$, where $\mathcal{T}$ is the tower of $\sim$-equivalence classes, the last example indicates that different towers can generate isomorphic BL-chains. Specifically, a singleton and a copy of $N(G)$ can be fused together to produce a copy of $N_{-\infty}(G)$. We argue that this phenomenon is the only obstruction to the uniqueness of a decomposition. More formally, we have the following definitions:

Definition 4.2 Given a tower of basic forms $\langle \mathcal{C}_i : i \in I \rangle$ and an ordered pair $(i, i') \in I^2$, we say that $(\mathcal{C}_i, \mathcal{C}_{i'})$ is a *reducible pair* if i' is the immediate successor of i, C_i is a singleton, and $\mathcal{C}_{i'} \cong N(G)$ for some ordered Abelian group G. A tower of basic forms is *irreducible* if it has no reducible pairs.

Clearly, for every tower $\mathcal{T}$ of basic forms, there is an irreducible tower $\mathcal{T}'$ of basic forms such that $\mathcal{A}_{\mathcal{T}} = \mathcal{A}_{\mathcal{T}'}$. The tower $\mathcal{T}'$ is obtained by replacing each reducible pair $(\mathcal{C}_i, \mathcal{C}_{i'})$ by a single basic form $\mathcal{D}_i = \mathcal{C}_i \cup \mathcal{C}_{i'}$. Also, it is easy to see that for any BL-chain $\mathcal{A}$, the tower of basic forms given by the $\sim$-equivalence classes is irreducible.

Definition 4.3 Two towers of basic forms $\langle \mathcal{C}_i : i \in I \rangle$ and $\langle \mathcal{D}_j : j \in J \rangle$ are *equivalent* if there is a bijection $g : I \to J$ such that $\mathcal{C}_i \cong \mathcal{D}_{g(i)}$ for each $i \in I$.

In the proof of the next lemma, we will repeatedly use the following fact, which is verified by inspecting each of the four types of basic forms:

Fact: If $\mathcal{C}$ is any basic form, then for all $a, b \in C$,

$$a \star b = b \quad \text{if and only if} \quad b = \min C.$$

In particular, a is an idempotent (that is, $a \star a = a$) if and only if $a = \min C$.

Lemma 4.4 *Suppose that the BL-chain $\mathcal{A}$ is canonical for each of the irreducible towers of basic forms $\mathcal{T} = \langle \mathcal{C}_i : i \in I \rangle$ and $\mathcal{T}' = \langle \mathcal{D}_j : j \in J \rangle$. Then there is a bijection $g : I \to J$ such that $\mathcal{C}_i = \mathcal{D}_{g(i)}$ for all $i \in I$.*

Proof: It suffices to show that if $\mathcal{C}_i$ is a basic form in $\mathcal{T}$ and $\mathcal{D}_j$ is a basic form in $\mathcal{T}'$ and $C_i \cap D_j \neq \emptyset$, then $C_i = D_j$.

Towards proving this, we first show that

$$C_i \setminus \min C_i = D_j \setminus \min D_j \tag{1}$$

whenever $(C_i \setminus \min C_i) \cap D_j \neq \emptyset$ or $C_i \cap (D_j \setminus \min D_j) \neq \emptyset$. Since the cases are symmetric, assume $a \in (C_i \setminus \min C_i) \cap D_j$. Since a is not an idempotent, $a \neq \min D_j$. So, for any $b \in C_i \setminus \min C_i$, $a \star b < \min\{a, b\}$, so $b \in D_j$. Again, since b is not an idempotent, b is not the minimal element of D_j, hence $b \in D_j \setminus \min D_j$. Conversely, if $b \in D_j \setminus \min D_j$, then $a \star b < \min\{a, b\}$, so $b \in C_i \setminus \min C_i$, which establishes (1).

Now, to prove the lemma, suppose that $C_i \cap D_j \neq \emptyset$. We argue that C_i is a singleton if and only if D_j is a singleton. Since the cases are symmetric, assume by way of contradiction that $C_i = \{a\}$, $a \in D_j$, and $|D_j| > 1$. Note that since a is an idempotent, $a = \min D_j$. Since D_j is not a singleton, $\mathcal{D}_j$ must be one of the three other basic forms. Clearly, $D_j \ncong N(G)$ for any ordered Abelian group G since D_j has a minimal element. If $\mathcal{D}_j \cong T(G, d)$ for some ordered Abelian group G and some $d \in N(G)$, then there is some $b \in D_j$ such that $b > a$ and $b \star b = a$. But then since every basic form is closed under $\star$, it must be that $b \in C_i$, which is a contradiction. Thus, it must be that $\mathcal{D}_j \cong N_{-\infty}(G)$ for some ordered Abelian group G. So $\mathcal{D}_j \setminus \min \mathcal{D}_j \cong N(G)$. Now choose any $b \in D_j \setminus \min D_j$. Let's say $b \in C_{i'}$ for some $i' \in I$. For any c in the interval (a, b), $a < c \star b < c$, hence $c \in C_{i'}$. Since the entire interval $(a, b) \subseteq C_{i'}$, i' must be an immediate successor of i. Moreover, $C_{i'}$ cannot have a minimal element. That is, $(\mathcal{C}_i, \mathcal{C}_{i'})$ is a reducible pair, which contradicts $\mathcal{T}$ being irreducible.

So we now assume that neither C_i nor D_j is a singleton. If, in addition, neither C_i nor D_j has a minimal element, then $C_i = D_j$ follows from Equation (1). If both C_i and D_j have minimal elements and are not singletons and $C_i \cap D_j \neq \emptyset$, then there is some $b \in (C_i \setminus \min C_i) \cap D_j$, so $C_i \setminus \min C_i = D_j \setminus \min D_j$. But $\min C_i$ is the greatest element of A below $C_i \setminus \min C_i$, and dually for D_j, so $\min C_i = \min D_j$. Hence $C_i = D_j$ as required.

Finally, assume that neither C_i nor D_j is a singleton, but one of these, say C_i, has a minimal element, and D_j does not. We argue that these conditions imply $C_i \cap D_j = \emptyset$. To see this, assume by way of contradiction that some $b \in C_i \cap D_j$. Since D_j does not have a minimal element, $b \star b < b$, hence $b \neq \min C_i$. So, it follows from (1) that $C_i \setminus \min C_i = D_j \setminus \min D_j = D_j$. Furthermore, since D_j is a basic form with no minimal element, $\mathcal{D}_j \cong N(G)$ for some ordered Abelian group G. Look at $a = \min C_i$. Choose $j' \in J$ such that $a \in D_{j'}$. Since a is an idempotent, a is the minimal element of $D_{j'}$. But, since a is the greatest element below $C_i \setminus \min C_i$, a is the greatest element below D_j. Hence $(\mathcal{D}_{j'}, \mathcal{D}_j)$ is a reducible pair, contradicting our assumption about $\mathcal{T}'$.

Proposition 4.5 *For any pair of irreducible towers $\mathcal{T}$ and $\mathcal{T}'$, $\mathcal{A}_{\mathcal{T}} \cong \mathcal{A}_{\mathcal{T}'}$ if and only if $\mathcal{T}$ and $\mathcal{T}'$ are equivalent.*

Proof: From right to left is immediate. On the other hand, suppose $f : \mathcal{A}_{\mathcal{T}} \to \mathcal{A}_{\mathcal{T}'}$ is an isomorphism of BL-chains. Then the tower $\mathcal{T}_f = \langle f(C_i) : i \in I\rangle$ (where $\mathcal{T} = \langle C_i : i \in I\rangle$) and $\mathcal{T}'$ satisfy the requirements of Lemma 4.4. Thus, there is a bijection $g : I \to J$ such that $f(C_i) = D_{g(i)}$ for all $i \in I$. This function g witnesses the equivalence of $\mathcal{T}$ and $\mathcal{T}'$.

Finally, we remark that unlike the situation for saturated BL-chains, a decomposition theorem for the full class of BL-chains cannot rely on idempotents. This is best demonstrated by exhibiting a wide class of BL-chains whose idempotents are trivial:

Example 4.6 Let I be any linear order and let $\langle G_i : i \in I\rangle$ be any sequence of ordered Abelian groups. Let $\mathcal{T} = \mathcal{C}_{\text{first}} \frown \langle \mathcal{C}_i : i \in I\rangle \frown \mathcal{C}_{\text{last}}$, where C_{first} and C_{last} are singletons and each $\mathcal{C}_i \cong N(G_i)$ for each $i \in I$. Then the canonical BL-chain $\mathcal{A}_{\mathcal{T}}$ has only two idempotents, namely $0_{\mathcal{T}}$ and $1_{\mathcal{T}}$.

5 Finite and Finitely Universal BL-Chains

Definition 5.1 A truncation $T(G, d)$ is *divisible by n* (respectively, *divisible*) if d is divisible by n (respectively, divisible) as an element of G.

For each positive integer m, let $\mathbb{Z}^\star_m$ denote the truncation of $N(\mathbb{Z}, +, \leq)$ at $-m$. The universe of $\mathbb{Z}^\star_m$ is $\{-1, -2, \ldots, -m\}$. Let $+_m$ denote $\star_{T(\mathbb{Z},-m)}$.

Note that $\mathbb{Z}^\star_1$ is the unique one-element ordered Abelian semigroup. For any positive integer m, it is clear that $\mathbb{Z}^\star_m$ is divisible by m. Further, if $\mathcal{C}$ is any truncation, then $\mathcal{C}$ is divisible by m if and only if there is an embedding of $\mathbb{Z}^\star_m$ into $\mathcal{C}$ as an ordered Abelian semigroup. We begin by classifying the finite BL-chains.

Lemma 5.2 *If $T(G, d)$ is any truncation with exactly m elements, then $\mathbb{Z}^\star_m \cong T(G, d)$ as ordered Abelian semigroups.*

Proof: Let $a \in T(G, d)$ be maximal. Let k be the least integer such that $ka = d$. Since $T(G, d)$ has m elements, we know $k \leq m$. However, if $k < m$, then by the pigeonhole principle there are an l and a b such that $1 \leq l < k$ and $la > b > (l+1)a$. Let $c := b - la$. Then $c \in T(G, d)$, contradicting the maximality of a. Hence $k = m$. It follows that the map $f : \mathbb{Z}_m^\star \longrightarrow T(G, d)$ defined by $f(-l) = la$ is an isomorphism of ordered Abelian semigroups.

Proposition 5.3 *If $\mathcal{A}$ is a finite BL-chain, then $\mathcal{A} \cong \mathcal{A}_\mathcal{T}$ for some tower $\mathcal{T}$ of the form $\langle \mathbb{Z}_{n_j}^\star : j \leq k \rangle$, where $n_k = 1$.*

Proof: By Theorem 3.5, $\mathcal{A} \cong \mathcal{A}_\mathcal{T}$ for some tower $\mathcal{T}$. Clearly, the number of basic forms must be finite, and each basic form must be finite. In particular, no basic form can be a negative cone or an extended negative cone. Thus, by viewing a singleton as isomorphic to $\mathbb{Z}_1^\star$, every basic form in $\mathcal{T}$ is a finite truncation. Thus, this lemma follows immediately from Lemma 5.2.

We next show that the class of finite BL-chains has a certain universality property with respect to the notion of finite embedding.

Definition 5.4 Let $\mathcal{A}, \mathcal{B}$ be BL-chains. Let X be a finite subset of B. A map $f : X \hookrightarrow \mathcal{A}$ is a *finite embedding* if it satisfies the following conditions:

1. For all $a, b \in X$ with $a \star_\mathcal{B} b \in X$, $f(a \star_\mathcal{B} b) = f(a) \star_\mathcal{A} f(b)$.
2. For all $a, b \in X$, if $a <_\mathcal{B} b$, then $f(a) <_\mathcal{A} f(b)$.
3. For all $a, b \in X$ with $a \Rightarrow_\mathcal{B} b \in X$, $f(a \Rightarrow_\mathcal{B} b) = f(a) \Rightarrow_\mathcal{A} f(b)$.
4. If $0_\mathcal{B} \in X$, then $f(0_\mathcal{B}) = 0_\mathcal{A}$.
5. If $1_\mathcal{B} \in X$, then $f(1_\mathcal{B}) = 1_\mathcal{A}$.

Note that the existence of a finite embedding from X into $\mathcal{B}$ is weaker than saying that the substructure generated by X embeds into $\mathcal{B}$. In particular, the substructure generated by a finite subset may be infinite.

Proposition 5.5 *For any BL-chain $\mathcal{B}$ and any finite $X \subseteq B$, there is a finite BL-chain $\mathcal{A}$ and a finite embedding $X \hookrightarrow \mathcal{A}$.*

Before proving Proposition 5.5, we begin by giving two algebraic results which will be used within the proof. The results are very old, but sketches of proofs are included since the authors do not know of an appropriate reference.

Lemma 5.6 *1. Every ordered Abelian group can be extended to a divisible ordered Abelian group.*

2. The extended theory of divisible ordered Abelian groups is a complete theory.

Proofs: Let $(G, \star, \leq)$ be any ordered Abelian group. Since the underlying Abelian group $(G, \star)$ is torsion-free, there is a minimal, divisible Abelian group $(D, \star)$ containing $(G, \star)$. (D is unique up to isomorphism fixing G). Expand $(D, \star)$ by calling an element $d \in D$ positive if and only if $nd = g$ for some positive integer n and some positive element $g \in G$. It is routine to verify that $(D, \star, \leq)$ is a divisible ordered Abelian group extending $(G, \star, \leq)$.

The proof of (2) is a standard exercise in model theory. The usual proof (see e.g., [RZ] or [Hod]) is to show that the theory of infinite, divisible, ordered Abelian groups admits elimination of quantifiers in the language of ordered Abelian groups $\{\star, \leq, 0\}$.

Definition 5.7 Let $\mathcal{S} := (S, \star, \leq)$ be an ordered Abelian semigroup. Let X be a subset of S. Let $m \in \omega \setminus \{0\}$. A map $g : X \hookrightarrow \mathbb{Z}_m^\star$ is *well-behaved* if:

1. For all $x, y \in X$, if $x \star_{\mathcal{S}} y \in X$, then $g(x \star_{\mathcal{S}} y) = g(x) +_m g(y)$.
2. For all $x, y \in X$, if $x <_{\mathcal{S}} y$, then $g(x) < g(y)$.
3. If $x, y, z \in X$ and z is the largest element of $\mathcal{S}$ such that $x \star_{\mathcal{S}} z = y$, then $g(z)$ is the largest element of $\mathbb{Z}_m^\star$ such that $g(x) +_m g(z) = g(y)$.

The following lemma is similar in spirit to the use of the Gurevich-Kokorin Theorem in the proof of the completeness of Łukasiewicz logic in Lemma 3.2.11 of [Ha 1].

Lemma 5.8 *If $\mathcal{C}$ is isomorphic to either of $T(G, d)$ or $N(G)$ for some ordered Abelian group G and X is any finite subset of C, then for some positive integer m there is a well-behaved embedding $g : X \hookrightarrow \mathbb{Z}_m^\star$.*

Proof: If $\mathcal{C} \cong T(G,d)$ for some ordered Abelian group G, then let $e := d$. Otherwise, if $\mathcal{C} \cong N(G)$ for some ordered Abelian group G, then let $e := 3\min(X)$. Let $Y := X \cup \{0, e\} \cup \{x \star_G y : x, y \in X\}$. By Lemma 5.6, there is a divisible ordered Abelian group D such that $G \subseteq D$. Since the theory of divisible ordered Abelian groups is complete, there is an order-preserving $f : Y \hookrightarrow \mathbb{Q}$ such that $f(x \star_G y) = f(x) + f(y)$ for all $x, y \in X$. Let H be the least common denominator of $\{f(y) : y \in Y\}$. Composing f with multiplication by H yields an order-preserving $g : Y \hookrightarrow \mathbb{Z}$ such that $g(x \star_G y) = g(x) + g(y)$ for all $x, y \in X$. Let $m := g(e)$. It is routine to verify that $g|_X$ is a well-behaved embedding of X into $\mathbb{Z}^\star_m$.

Proof of Proposition 5.5:
Let $\mathcal{B}$ be any BL-chain. Let $\mathcal{B} \cong \mathcal{B}_{\mathcal{T}'}$, where $\mathcal{T}'$ is the tower $\langle \mathcal{C}_i : i \in I\rangle$, by applying Theorem 3.5. By decomposing any extended negative cone into a negative cone with a point below it, we may assume that each $\mathcal{C}_i$ is either a singleton, or is isomorphic to some $T(G,d)$ or to some $N(G)$. Let X be any finite subset of B. We may assume that $1_{\mathcal{B}} \in X$. Let $i_0 < i_1 < \ldots < i_N$ from I be the finitely many indices i such that $X \cap C_i \neq \emptyset$. By Lemma 5.8 (and the fact that any one element ordered semigroup is isomorphic to $\mathbb{Z}^\star_1$), for each $k \leq N$, there is a positive integer n_k such that there is a well-behaved embedding $g_k : X \cap \mathcal{C}_{i_k} \hookrightarrow \mathbb{Z}^\star_{n_k}$. Let $\mathcal{T} := \langle \mathbb{Z}^\star_{n_k} : k \leq N\rangle$, where $n_N = 1$, and let $\mathcal{A}_{\mathcal{T}}$ be the canonical BL-chain built from $\mathcal{T}$. The map $g := \bigcup\{g_k : k \leq N\}$ is a finite embedding of X into $\mathcal{A}_{\mathcal{T}}$.

Corollary 5.9 *Every existential sentence consistent with the theory of BL-chains is true in some finite BL-chain.*

Proof: Let $\exists\vec{x}\theta(\vec{x})$ be an existential sentence consistent with the theory of BL-chains, where θ is quantifier-free. Choose $\mathcal{B}$ a BL-chain and $\vec{b}$ from $\mathcal{B}$ such that $\mathcal{B} \models \theta(\vec{b})$. By Proposition 5.5, there is a finite BL-chain $\mathcal{A}$ and a finite embedding $g : \vec{b} \hookrightarrow \mathcal{A}$. Then $\mathcal{A} \models \theta(g(\vec{b}))$, hence $\mathcal{A} \models \exists\vec{x}\theta(\vec{x})$.

The next corollary follows immediately:

Corollary 5.10 *The universal theory of BL-chains is decidable.*

We next seek to classify the collection of all BL-chains which embed every finite BL-chain. Examples of such chains follow from Lemma 2 of [BHMV].

Definition 5.11 Let $\mathcal{B}$ be a BL-chain. $\mathcal{B}$ is *finitely universal* if every finite BL-chain embeds into $\mathcal{B}$ as a BL-chain.

The property of being finitely universal will have consequences for universality with respect to provability from the BL-axiom system, as we will see in Corollary 5.15. We will show that there are several equivalent ways of characterizing a BL-chain as finitely universal.

Definition 5.12 Let $\mathcal{T} := \langle \mathcal{C}_i : i \in I \rangle$ be a tower of basic forms. We will say that $\mathcal{T}$ is *full* if for all $k \in \omega$ and all finite sequences $\langle n_j : j < k \rangle$ of positive integers, there are $i_0 < i_1 < \ldots < i_{k-1}$ from I such that $i_0 = \min(I)$ and for all $j < k$, $\mathcal{C}_{i_j}$ is a truncation divisible by n_j.

Note that if $\mathcal{T}$ is full, then $\mathcal{C}_{\text{first}}$ is a divisible truncation.

Theorem 5.13 *Let $\mathcal{A}$ be any BL-chain. The following conditions are equivalent:*

1. *$\mathcal{A}$ is finitely universal.*
2. *$\mathcal{A} \cong \mathcal{A}_{\mathcal{T}}$ for some full tower $\mathcal{T}$.*
3. *For every BL-chain $\mathcal{B}$, every finite subset of B finitely embeds into $\mathcal{A}$.*
4. *Every existential sentence consistent with the theory of BL-chains is true in $\mathcal{A}$.*

Proof: ($1 \Rightarrow 2$): Suppose that $\mathcal{A}$ is finitely universal. From Theorem 3.5, we know that $\mathcal{A} \cong \mathcal{A}_{\mathcal{T}}$ for some tower $\mathcal{T} = \langle \mathcal{C}_i : i \in I \rangle$. We wish to show that $\mathcal{T}$ is a full tower. Choose any $k \in \omega$ and any sequence of integers $\langle n_j : j < k \rangle$. We may assume each $n_j > 1$. Let $\mathcal{T}' := \langle \mathbb{Z}^\star_{n_j} : j \leq k \rangle$, where $n_k = 1$, and let g be an embedding of the canonical BL-chain $\mathcal{B}_{\mathcal{T}'}$ into $\mathcal{A}$. For each $j < k$, let $b_j := \min(\mathbb{Z}^\star_{n_j})$, and for each $j < k$, let $i_j \in I$ be such that $g(b_j) \in C_{i_j}$. Since $0_{\mathcal{B}_{\mathcal{T}'}} = b_0$ and $g(0_{\mathcal{B}_{\mathcal{T}'}}) = 0_{\mathcal{A}}$, we know that $i_0 = \min(I)$. Further, since distinct idempotents of $\mathcal{B}_{\mathcal{T}'}$ must map to distinct idempotents of $\mathcal{A}$ under g, we have that $g(b_j) = \min(C_{i_j})$ for each $j < k$. Thus $i_0 < i_1 < \ldots < i_{k-1}$ since g is order-preserving. As well, for any $j < k$ and any $b \in \mathbb{Z}^\star_{n_j}$, there is some $m \in \omega$ such that $mb = b_j$, hence $mg(b) = \min(C_{i_j})$. Since $n_j > 1$ for each $j < k$, $\mathcal{C}_{i_j}$ must be a truncation. Furthermore $g(b) \in C_{i_j}$ for all $b \in \mathbb{Z}^\star_{n_j}$.

Hence $g|_{\mathbb{Z}^\star_{n_j}}$ is an embedding of ordered Abelian semigroups into $\mathcal{C}_{i_j}$. That is, $\mathcal{C}_{i_j}$ is a truncation divisible by n_j.

($2 \Rightarrow 3$): Let $\mathcal{B}$ be any BL-chain, and let X be any finite subset of B. By Proposition 5.5, there is a finite BL-chain $\mathcal{M}$ and a finite embedding $g : X \hookrightarrow \mathcal{M}$. By Proposition 5.3, $\mathcal{M} \cong \mathcal{M}_{\mathcal{T}'}$ for some tower $\mathcal{T}'$ of the form $\langle \mathbb{Z}^\star_{n_k} : k \leq N \rangle$, where $n_N = 1$. Let $g : X \hookrightarrow \mathcal{M}_{\mathcal{T}'}$ be an embedding satisfying all of the conditions of Definition 5.4. Now we apply the definition of $\mathcal{T}$ being a full tower to the sequence of numbers $\langle n_k : k < N \rangle$ to say that there are $j_0 < j_1 < \ldots < j_{N-1}$ from J such that $\mathcal{D}_{j_k}$ is a truncation divisible by n_k for all $k < N$ and $j_0 = \min(J)$. For each $k < N$, let $f_k : \mathbb{Z}^\star_{n_k} \hookrightarrow \mathcal{D}_{j_k}$ be defined by $f_k(-m) := m(d/n_k)$, where $d = \min(D_{j_k})$. Let $f := \bigcup\{f_k : k < N\} \cup \{\langle 1_{\mathcal{M}}, 1_{\mathcal{A}} \rangle\}$. The composition $f \circ g$ is a finite embedding of X into $\mathcal{A}$.

($3 \Rightarrow 4$): This is analogous to the proof of Corollary 5.9.

($4 \Rightarrow 1$): Let $\mathcal{B}$ be any finite BL-chain. Let $\triangle_{\mathcal{B}}$ denote the atomic diagram of $\mathcal{B}$. Let the tuple $\vec{b}$ enumerate all elements of B, and choose $\theta(\vec{x})$ so that $\triangle_{\mathcal{B}} = \theta(\vec{b})$. By our construction, $\exists \vec{x} \theta(\vec{x})$ is true in $\mathcal{B}$, hence by (4), we have $\mathcal{A} \models \exists \vec{x} \theta(\vec{x})$. Choosing witnesses for θ in $\mathcal{A}$ gives our embedding of $\mathcal{B}$ into $\mathcal{A}$.

Note that Clause (4) implies that the finitely universal BL-chains form an elementary class.

As examples, if $\mathcal{T}$ is any tower of basic forms that contains infinitely many divisible truncations and whose minimal element is a divisible truncation, then the associated BL-chain $\mathcal{A}_{\mathcal{T}}$ will be finitely universal. In particular, there are many decidable ones. Let $\mathcal{U}$ be the BL-chain with universe $[0,1] \cap \mathbb{Q}$, where for each $k \geq 0$ the restriction of $\mathcal{U}$ to $[\frac{k}{k+1}, \frac{k+1}{k+2}) \cap \mathbb{Q}$ is isomorphic to $T(\mathbb{Q}, -1)$. Then $\mathcal{U}$ is finitely universal. Further, if we choose a recursive bijection between ω and $[0,1] \cap \mathbb{Q}$, then the graphs of $\leq$ and the operations $\star, \Rightarrow$ are recursive.

We conclude with some applications of the previous results to the Basic Logic (BL) proof system. In [Ha 1] Hájek introduces a (recursive) set of propositional statements, called the BL-axioms. He proves a completeness theorem for this logic, which involves evaluations into arbitrary BL-chains.

Definition 5.14 For a propositional formula φ, let $S(\varphi)$ denote the set of subformulas of φ (including $\bar{0}$ and $\bar{1}$). An *evaluation* into a BL-chain $\mathcal{A}$ is a function $e : S(\varphi) \longrightarrow \mathcal{A}$ satisfying:

- $e(\bar{0}) = 0$, $e(\bar{1}) = 1$;

- $e(\alpha \& \beta) = e(\alpha) \star e(\beta)$ for all $(\alpha \& \beta) \in S(\varphi)$;
- $e(\alpha \to \beta) = e(\alpha) \Rightarrow e(\beta)$ for all $(\alpha \to \beta) \in S(\varphi)$.

Theorem 2.3.19 of [Ha 1] shows that a propositional formula φ is provable from the BL-axiom system (i.e., is a BL-tautology) if and only if $e(\varphi) = 1_{\mathcal{A}}$ for all BL-chains $\mathcal{A}$ and all evaluations e into $\mathcal{A}$. We offer two strengthenings of this result. Note that for any propositional formula φ, one can construct an existential sentence θ_φ in the language of BL-chains such that for any BL-chain $\mathcal{B}$, $\mathcal{B} \models \theta_\varphi$ if and only if there is an evaluation $e : S(\varphi) \to \mathcal{B}$ such that $e(\varphi) < 1_{\mathcal{B}}$. Details of this coding can be found in Definition 2.3.17 of [Ha 1].

Corollary 5.15 *1. A propositional formula φ is a BL-tautology if and only if $e(\varphi) = 1_{\mathcal{A}}$ for all finite BL-chains $\mathcal{A}$ and all evaluations e into $\mathcal{A}$*

2. Let $\mathcal{A}$ be any finitely universal BL-chain. A propositional formula φ is a BL-tautology if and only if $e(\varphi) = 1_{\mathcal{A}}$ for every evaluation e into $\mathcal{A}$.

Proof: (1) Assume that φ is not provable from the BL-axioms. It follows from Theorem 2.3.19 of [Ha 1] that the existential sentence θ_φ associated to φ is consistent with the theory of BL-chains. Thus, by Corollary 5.9, some finite BL-chain $\mathcal{A} \models \theta_\varphi$. That is, there is an evaluation e satisfying $e(\varphi) < 1_{\mathcal{A}}$.

The proof of (2) is analogous, using Theorem 5.13 (4) in place of Corollary 5.9.

Finally, we are able to give a new proof of the decidability of the BL-axiom system. In [BHMV], Baaz et.al. not only proved decidability, but they showed that the set of BL-tautologies are co-NP complete.

Corollary 5.16 *The set of BL-tautologies is decidable.*

Proof: Recall that the (codes of) axioms of BL are recursive, so the set of BL-tautologies is recursively enumerable (r.e.). We will show that this set is co-r.e. as well by considering the finitely universal BL-chain $\mathcal{U}$ exhibited above. Suppose that φ is not a BL-tautology. Every mapping of the variables of φ into the universe of $\mathcal{U}$ extends uniquely to an evaluation $e : S(\varphi) \to \mathcal{U}$. Further, since the graphs of the basic operations on $\mathcal{U}$ are recursive, it follows that the question of determining whether $e(\varphi) = 1_{\mathcal{U}}$ for a particular map of the variables of φ into $\mathcal{U}$ is recursive. But, it follows from Corollary 5.15 (2) that since φ is not a BL-tautology, there is an evaluation $e : S(\varphi) \to \mathcal{U}$ satisfying $e(\varphi) < 1_{\mathcal{U}}$. Thus, since the set of finite sequences from $\mathcal{U}$ is recursive, the set of BL-tautologies is decidable.

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