PROVABILITY IN PREDICATE PRODUCT LOGIC

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ABSTRACT. We sharpen Hájek's Completeness Theorem for theories extending predicate product logic, $\Pi\forall$. By relating provability in this system to embedding properties of ordered abelian groups we construct a universal BL-chain $\mathbf{L}$ in the sense that a sentence is provable from $\Pi\forall$ if and only if it is an $\mathbf{L}$-tautology. As well we characterize the class of lexicographic sums that have this universality property.

1. INTRODUCTION

Predicate product logic is a variant of first-order logic wherein sentences are assigned a truth value in the closed interval $[0, 1]$. In product logic the truth value of a conjunction $\varphi \& \psi$ of sentences is equal to the product of the truth values of φ and ψ, which is natural in certain applications (e.g., if φ and ψ describe independent events).

In [H] Petr Hájek laid the groundwork for a proof theory for this logic. He defined an axiom system $\Pi\forall$, which consists of a (recursive) list of 'Basic Logic' axioms BL$\forall$, together with two additional axiom schema:

$$(\varphi \wedge \neg\varphi) \rightarrow \bar{0} \qquad \text{and} \tag{1}$$

$$\neg\neg\chi \rightarrow ((\varphi \& \chi \rightarrow \psi \& \chi) \rightarrow (\varphi \rightarrow \psi)) \tag{2}$$

The basic logic axioms are valid in many different predicate fuzzy logic systems, while the two additional axioms are valid in Product Logic (as well as conventional two-valued logic) but distinguish it from Gödel logic and Lukasiewicz logic.

As for semantics, Hájek defined a BL-chain as a linearly ordered residuated lattice satisfying certain properties. Here, however, we follow the treatment set out in [LS1] and [LS2] and define a *BL-chain* to be an ordered abelian semigroup $(L, +, \leq, 0, 1)$ in which 1 is both the maximal element of the ordering and the identity element of the

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semigroup, 0 is the minimal element of the semigroup, and whenever $a < b$ there is a maximal c such that $b+c = a$. By defining $b \Rightarrow a$ to be 1 whenever $b \leq a$ and the maximal c such that $b+c = a$ otherwise, one readily sees that this definition is equivalent to the definition proposed by Hájek in [H].

Following Hájek, fix a predicate language τ. For a given BL-chain $\mathbf{L}$, an $\mathbf{L}$-*structure* $\mathbf{M}$ is a triple $\langle M, (m_c)_c, (r_P)_P\rangle$, where M is a nonempty set, m_c is an element of M for each constant symbol $c \in \tau$, and $r_P : M^n \to \mathbf{L}$ is a function for every n-ary predicate symbol $P \in \tau$. Given an $\mathbf{L}$-structure $\mathbf{M}$, one recursively defines a function $||\varphi||_{\mathbf{M}}$ to every $\tau(M)$-sentence φ by demanding that $||\varphi\&\psi|| = ||\varphi||+||\psi||$; $||\varphi \to \psi|| = ||\varphi|| \Rightarrow ||\psi||$; $||\exists x\varphi(x)|| = \sup\{||\varphi(a)|| : a \in M\}$; and $||\forall x\varphi(x)|| = \inf\{||\varphi(a)|| : a \in M\}$. (In the clauses above, and whenever it is clear, we suppress the subscript on $||\varphi||$.) In general, there is no reason why the requisite suprema and infima need exist. Again, following Hájek, we call an $\mathbf{L}$-structure $\mathbf{M}$ *safe* if in fact all the suprema and infima needed to compute $||\varphi||$ do exist for all $\tau(M)$-sentences φ.

A *theory* T is simply a set of formulas. We say that an $\mathbf{L}$-structure $\mathbf{M}$ *is a model of* T if $||\varphi(\bar{a})|| = 1$ for every $\varphi \in T$ and every $\bar{a}$ from M of the requisite length. In [H] Hájek proves the following results that we use freely:

Theorem 1.1 (Deduction Theorem)**.** *For any theory T extending $BL\forall$ and sentences φ, ψ, $T \cup \{\varphi\} \vdash \psi$ if and only if $T \vdash \varphi^k \to \psi$ for some positive integer k (where φ^k denotes the k-fold conjunction $\varphi\& \ldots \&\varphi$).*

Theorem 1.2 (Completeness Theorem)**.** *$T \vdash \varphi$ if and only if $||\varphi||_{\mathbf{M}} = 1$ for all BL-chains $\mathbf{L}$ and all safe $\mathbf{L}$-structures $\mathbf{M}$ that model T.*

In this paper we obtain several strengthenings of Hájek's completeness theorem for theories extending $\Pi\forall$, the strongest of which is Theorem 2.9. As is usual in the study of provability, in order to understand the logical consequences of the theory $\Pi\forall$, one must include nonstandard models as well. In this context, this means defining a class of BL-chains that are generated from ordered abelian groups.

Definition 1.3. Let $(G, +, \leq)$ be any ordered abelian group. Let $N(G)$ be the subsemigroup with universe $\{a \in G : a \leq 0\}$. Let $L(G)$ denote the BL-chain with universe $N(G) \cup \{-\infty\}$ in which $+$ and $\leq$ are inherited from G with the additional stipulations that $-\infty$ is the minimal element of $L(G)$ and that $a + (-\infty) = -\infty$ for all $a \in L(G)$. The 'top' element of $L(G)$ is the zero element of G, and the 'bottom' element of $L(G)$ is $-\infty$.

It is readily verified that $L(G)$ is a BL-chain for any ordered abelian group G. (In fact, for any $a < b$ in $L(G)$ there is a *unique* c such that $b + c = a$.) Furthermore, any $L(G)$-structure is necessarily a model of $\Pi\forall$. In order to describe a converse, we need to specify when two structures can be identified.

Definition 1.4. Fix a predicate language τ. An $\mathbf{L}$-structure $\mathbf{M}$ and an $\mathbf{L}'$-structure $\mathbf{M}'$ are *identified* if $M = M'$, $m_c^{\mathbf{M}} = m_c^{\mathbf{M}'}$ for all constant symbols $c \in \tau$, and

$$||\varphi||_{\mathbf{M}} = ||\varphi||_{\mathbf{M}'}$$

for all $\tau(M)$-formulas φ.

The intuition behind identifying such structures is that 'extra elements' of $\mathbf{L}$ or $\mathbf{L}'$ are irrelevant if they never appear in the computation of the truth of any $\tau(M)$-sentence. The following proposition is fundamental to our analysis.

Proposition 1.5. *Any $\mathbf{L}$-structure $\mathbf{M}$ that is a model of $\Pi\forall$ (i.e., a model of the empty theory over $\Pi\forall$) can be identified with an $L(G)$-structure $\mathbf{M}'$ for some ordered abelian group G.*

Proof. Let $B = \{||\varphi|| : \varphi \in \tau(M)\}$ and let $S = B \setminus \{0, 1\}$. It is readily checked that B is closed under $+$ and $\Rightarrow$, hence $(B, +, \leq, 0, 1)$ is itself a BL-chain. Additionally, it follows from $\mathbf{M}$ being a model of $\Pi\forall$ that the ordered abelian semigroup $(S, +, \leq)$ is cancellative and satisfies $a + b < a$ for all $a, b \in S$. Thus, by either Lemma 2.3 of [LS1] or Theorem 4.1.8 of [H], S is the set of negative elements of an ordered abelian group G. Then $\mathbf{M}$ can be identified with an $L(G)$-structure for any such G.

Thus, a formula φ is provable from $\Pi\forall$ if and only if φ is an $L(G)$-tautology for every ordered abelian group G. With our eye on improving this result, we describe a certain class of ordered abelian groups.

Definition 1.6. For $(J, <)$ any linear order, the *lexicographic sum* $(\mathbb{R}^J, +, \leq)$ is the ordered abelian group of functions $f : J \to \mathbb{R}$ whose support is well-ordered (i.e., $\{j \in J : f(j) \neq 0\}$ is a well-ordered subset of J). Addition is defined componentwise and the ordering on $\mathbb{R}^J$ is lexicographic.

The Hahn embedding theorem states that every ordered abelian group naturally embeds into a lexicographic sum. Moreover, Clifford's proof of this result (see [C]) shows that the embedding can be chosen in a very desirable fashion, which we describe below.

Definition 1.7. Let G be any ordered abelian group. For $a, b \in G$, we say a and b are *equivalent*, written $a \sim b$, if $a = b = 0$, or $a, b > 0$ and there is a positive integer n such that $na \geq b$ and $nb \geq a$, or $a, b < 0$ and there is a positive integer n such that $na \leq b$ and $nb \leq a$ (the notation na is shorthand for adding a to itself n times.) For $a, b \in N(G)$ we write $a << b$ when $a < nb$ for all $n \in \omega$.

It is easily checked that $\sim$ is an equivalence relation on G and that the equivalence classes are convex subsets of G. The $\sim$-classes are called the *archimedean classes* of G. Since G is a group, the behavior of $\sim$ on the set of negative elements determines the behavior of $\sim$ on all of G. Our notation is slightly nonstandard, as here the elements a and $-a$ are in different archimedean classes whenever $a \neq 0$. Note that $<<$ defines a strict linear order on the archimedean classes of $N(G)$.

It is easily checked that if $f : G \to H$ is a strict order-preserving homomorphism of ordered abelian groups, then $a \sim b$ if and only if $f(a) \sim f(b)$ for all $a, b \in G$. However, if we want our embedding to preserve suprema and infima, we require an additional property. Specifically, call a strict, order-preserving homomorphism $f : G \to H$ an *archimedean surjection* if for every $b \in H$ there is $a \in G$ such that $f(a) \sim b$. Our interest in this notion is given by the following two results.

Lemma 1.8. *If $f : G \to H$ is an archimedean surjection, $X \subset G$, and $a, b \in G$ satisfy $a = \sup(X)$, $b = \inf(X)$, then $f(a) = \sup(f(X))$ and $f(b) = \inf(f(X))$. Moreover, if X is a cofinal (coinitial) subset of G, then $f(X)$ is a cofinal (coinitial) subset of H.*

Proof. By symmetry, inversion, and translation, it suffices to show that if X is a set of positive elements from G and $\inf(X) = 0_G$, then $\inf(f(X)) = 0_H$. So fix such a set $X \subseteq G$. Since f is strictly order-preserving every element of $f(X)$ is positive, so 0_H is a lower bound. Now choose $r \in H$, $r > 0_H$. It suffices to find some $a \in X$ such that $f(a) < r$. Before demonstrating this, we establish the following claim:

Claim. For every positive $b \in G$ there is $a \in X$ such that $2a < b$.

Proof. Fix $b > 0$. Choose any $c \in X$ such that $c < b$ and let $d = b - c$. There are now two cases. First, if $2d \leq b$ then take a to be any element of X less than d. Second, if $b < 2d$, then let $e = b - d$ and choose $a \in X$ with $a < e$. Then $2a < 2e = 2b - 2d \leq b$ and the claim is proved.

Since f is an archimedean surjection we can choose $b \in G$ such that $f(b) \sim r$. Fix a positive integer n so that $f(b) \leq nr$. By iterating the Claim several times we can find an $a \in X$ such that $na < b$. Hence

$nf(a) = f(na) < f(b) \leq nr$, so $f(a) < r$ as required. The 'moreover' clause is proved similarly.

Proposition 1.9. *If $f : G \to H$ is an archimedean surjection and $\mathbf{M}$ is any $L(G)$-structure, then there is a (unique) $L(H)$-structure $\mathbf{M}'$ with the same universe as $\mathbf{M}$ that satisfies*

$$f(||\varphi||_{\mathbf{M}}) = ||\varphi||_{\mathbf{M}'}$$

for all $\tau(M)$-sentences φ.

Proof. Take $\mathbf{M}'$ to have universe M and let $m_c^{\mathbf{M}'} = m_c^{\mathbf{M}}$ for all constants $c \in \tau$. For every predicate symbol P, let $r_P^{\mathbf{M}'} = f(r_P^{\mathbf{M}})$. One recursively checks that

$$f(||\varphi||_{\mathbf{M}}) = ||\varphi||_{\mathbf{M}'}$$

for all $\tau(M)$-sentences φ, using the previous Lemma to show that quantifiers are well behaved.

The following theorem can be read off from the main result in [C].

Theorem 1.10 (Clifford's proof of the Hahn embedding theorem)**.** *Let G be any ordered abelian group and let J consist of the archimedean classes of the negative elements of G with the induced ordering. Then there is an archimedean surjection $f : G \to \mathbb{R}^J$.*

One special case is worth noting. If $G = (0)$ is the trivial group, then $J = \emptyset$ and $\mathbb{R}^\emptyset$ is trivial as well. In this case $L(\mathbb{R}^\emptyset) = \{0, 1\}$, hence $L(\mathbb{R}^\emptyset)$-structures are classical two-valued structures.

In order to connect the results in this section we introduce the notion of *isomorphism* of structures in the same predicate language τ. The novelty is that a structure has two distinct sorts (its universe and the associated BL-chain) so an isomorphism itself should be a two-sorted object. Specifically, we say that an $\mathbf{L}$-structure $\mathbf{M}$ is *isomorphic* to an $\mathbf{L}'$-structure $\mathbf{M}'$ if there is a BL-chain isomorphism $g : \mathbf{L} \to \mathbf{L}'$ and a bijection $f : M \to M'$ such that

$$||\varphi(f(a_1), \ldots, f(a_n))||_{\mathbf{M}'} = g(||\varphi(a_1, \ldots, a_n)||_{\mathbf{M}})$$

for all τ-formulas $\varphi(x_1, \ldots, x_n)$ and all $(a_1, \ldots, a_n) \in M^n$. That said, the Corollary below follows immediately from our previous results.

Corollary 1.11. *Every model of $\Pi\forall$ is identified with a structure that is isomorphic to an $L(\mathbb{R}^J)$-structure for some (possibly empty) linear order $(J, <)$.*

Corollary 1.12. *A formula φ is provable from $\Pi\forall$ if and only if φ is an $L(\mathbb{R}^J)$-tautology for every linear order $(J, <)$.*

2. Closed models and lexicographic sums

Together with Hájek's Completeness Theorem, Corollary 1.11 tells us that if we want to semantically determine whether a formula is provable from a theory T extending $\Pi\forall$ it suffices to look at structures whose associated BL-chain arise from lexicographic sums. In this section we analyze the sets of tautologies for each of the lexicographic sums $\mathbb{R}^J$. In order to compare these sets of tautologies we need a suitable notion of embedding between lexicographic sums. Archimedean surjections are very nice, but unfortunately an archimedean surjection between lexicographic sums $\mathbb{R}^J$ and $\mathbb{R}^K$ exists only when the linear orders $(J,<)$ and $(K,<)$ are isomorphic. Thus, we both weaken our notion of embedding and strengthen our requirements on the class of 'suitable' structures.

Definition 2.1. A *BL-chain embedding* is a strict, order-preserving homomorphism $f : \mathbf{L} \to \mathbf{L}'$ of the BL-chains $\mathbf{L}$ and $\mathbf{L}'$. Such an embedding *respects 0* if, moreover, for every subset $X \subseteq \mathbf{L}$, if $\inf(X) = 0_{\mathbf{L}}$ then $\inf(f(X)) = 0_{\mathbf{L}'}$. If $\mathbf{M}$ is an $\mathbf{L}$-structure and $f : \mathbf{L} \to \mathbf{L}'$ is a BL-chain embedding, then $f(\mathbf{M})$ is the $\mathbf{L}'$-structure $\mathbf{M}'$ with universe M, $m_c^{\mathbf{M}'} = m_c^{\mathbf{M}}$, and $r_P^{\mathbf{M}'} = f(r_P^{\mathbf{M}})$.

The reader is cautioned that without extra conditions being placed on either the embedding or the structure, it is possible that the image of a safe $\mathbf{L}$-structure under a BL-chain embedding need not be safe. If $f : G \to H$ is a strict, order preserving homomorphism of ordered abelian groups, then f extends naturally to a BL-chain embedding (also called f) from $L(G)$ to $L(H)$ by positing that $f(-\infty) = -\infty$. It is easily checked that this induced embedding respects 0 if and only if the mapping of ordered abelian groups is coinitiality preserving.

Definition 2.2. Fix a BL-chain $\mathbf{L}$ and a predicate language τ.

(1) An $\mathbf{L}$-structure $\mathbf{M}$ is *strongly closed*[1] if for all $\tau(M)$-formulas $\varphi(x)$ with one free variable, there are $a, b \in M$ such that $||\varphi(a)|| = ||\exists x\varphi(x)||$ and $||\varphi(b)|| = ||\forall x\varphi(x)||$.
(2) An $\mathbf{L}$-structure $\mathbf{M}$ is *closed* if for all $\tau(M)$-formulas $\varphi(x)$ with one free variable, there are $a, b \in M$ such that $||\varphi(a)|| = ||\exists x\varphi(x)||$ and **either** $||\varphi(b)|| = ||\forall x\varphi(x)||$ **or** $||\forall x\varphi(x) = 0||$.

Note that if $\mathbf{M}$ is strongly closed then it is closed, and if it is closed then it is safe. The following Lemma is proved by an easy induction on the complexity of φ (*cf.* Proposition 1.9).

[1] Strongly closed structures are also called *witnessed* in the literature, e.g., [H2].

Lemma 2.3. *If* $\mathbf{M}$ *is a closed* $\mathbf{L}$*-structure and* $f : \mathbf{L} \to \mathbf{L}'$ *respects 0, then* $||\varphi||_{f(\mathbf{M})} = f(||\varphi||_{\mathbf{M}})$ *for all* $\tau(M)$*-sentences* φ*. In particular,* $\mathbf{M}$ *models* T *if and only if* $f(\mathbf{M})$ *models* T *for any theory* T*.*

Similarly, if $\mathbf{M}$ is a strongly closed $\mathbf{L}$-structure, then the conclusions of Lemma 2.3 apply for any BL-chain embedding. This observation yields a slight strengthening of Corollary 1.11. At the end of this section we will achieve a far stronger result.

Lemma 2.4. *If* T *extends* $\Pi\forall$ *and* $T \not\vdash \sigma$*, then there is a* **nonempty** *ordering* $(J, <)$ *and an* $L(\mathbb{R}^J)$*-structure* $\mathbf{M}$ *that models* T*, yet* $||\sigma||_{\mathbf{M}} < 1$*. In particular, any such* $\mathbb{R}^J$ *is infinite and divisible.*

Proof. In light of Corollary 1.11 we need only consider what happens if there is an $L(\mathbb{R}^\emptyset)$-structure $\mathbf{M}$ that models T with $||\sigma||_{\mathbf{M}} < 1$ (hence equal to 0). Then (trivially) $\mathbf{M}$ is strongly closed. Thus, if $(J, <)$ is arbitrary and $f : L(\mathbb{R}^\emptyset) \to L(\mathbb{R}^J)$ is any BL-chain embedding (i.e., $f(0) = 0$ and $f(1) = 1$) then $f(\mathbf{M})$ models T and $||\sigma||_{f(\mathbf{M})} < 1$.

At first blush it seems like the notion of being strongly closed is more natural than that of being closed, but the example below, which exploits the fact that $\to$ is discontinuous at $(0, 0)$, indicates that one cannot prove Theorem 2.6 for such structures.

Example 2.5. *Take* σ *to be* $(\forall x R \to S) \to (\exists x(R \to S))$ *where* R *and* S *are unary predicate symbols. Then there are closed structures* $\mathbf{M}$ *in which* $||\sigma||_{\mathbf{M}} < 1$*, yet* $||\sigma||_{\mathbf{M}} = 1$ *for every strongly closed structure* $\mathbf{M}$*.*

Theorem 2.6. *Let* T *be any theory extending* $\Pi\forall$ *and let* σ *be any sentence. If* $T \not\vdash \sigma$ *then there is a closed model* $\mathbf{M}$ *of* T *such that* $||\sigma||_{\mathbf{M}} < 1$*. Furthermore, if the language is countable then* $\mathbf{M}$ *can be chosen to be countable as well.*

Before proving Theorem 2.6 we state and prove two proof-theoretic lemmas about the axiom system $\Pi\forall$. In keeping with the spirit of the paper, our proofs of these facts will be model theoretic in nature. However, in [M] the second author proves these lemmas directly from the axiom system.

Lemma 2.7. *Suppose that* T *is a theory extending* $\Pi\forall$*,* σ *is a sentence,* c *is a constant symbol that does not appear in either* T *or* σ*, and* $T \cup \{\exists x\varphi(x) \to \varphi(c)\} \vdash \sigma$*. Then* $T \vdash \sigma$*.*

Proof. Let τ denote the language of $T \cup \{\sigma\}$ and let $\tau_c = \tau \cup \{c\}$. Let $\mathbf{M}$ be an $L(\mathbb{R}^J)$-structure in the language of τ that models T in which $J \neq \emptyset$. In particular, $\mathbb{R}^J$ is infinite and divisible as an abelian group.

Because of Lemma 2.4, in order to show that $T \vdash \sigma$ it suffices to show that $||\sigma||_{\mathbf{M}} = 1$. Since $L(\mathbb{R}^J)$ is infinite and divisible, 1 is an accumulation point of $\mathbb{R}^J$, so it suffices to prove that $||\sigma||_{\mathbf{M}} \geq \epsilon$ for every $\epsilon \in L(\mathbb{R}^J))$, $\epsilon < 1$.

For each $a \in M$, let $\mathbf{M}_a$ denote the expansion of $\mathbf{M}$ to a structure in the language τ_c formed by setting $m_c^{\mathbf{M}_a} = a$. As notation, let $\theta(y)$ abbreviate $\exists x\varphi(x) \to \varphi(y)$. Since $T \cup \theta(c) \vdash \sigma$, it follows from the Deduction theorem that we can fix a positive integer k such that $T \vdash \theta(c)^k \to \sigma$. Since $\mathbf{M}$ is a model of T, each $\mathbf{M}_a$ is a model of T, hence $||\theta(c)^k \to \sigma||_{\mathbf{M}_a} = 1$ for all $a \in M$. Reflecting back to $\mathbf{M}$, this implies $||\theta(a)^k \to \sigma||_{\mathbf{M}}$ for all $a \in M$, hence

$$k||\theta(a)||_{\mathbf{M}} \leq ||\sigma||_{\mathbf{M}} \quad \text{for all } a \in M$$

Now fix an $\epsilon < 1$. From our comments above, it suffices to show that $k||\theta(a)||_{\mathbf{M}} \geq \epsilon$ for *some* $a \in M$.

But $||\exists x\varphi(x)||_{\mathbf{M}} = \sup\{||\varphi(a)||_{\mathbf{M}} : a \in M\}$, so there is some $a \in M$ so that $||\varphi(a)||_{\mathbf{M}} \geq ||\exists x\varphi(x)||_{\mathbf{M}} + \epsilon/k$ (this makes sense since $\mathbb{R}^J$ is divisible). It follows that $k||\theta(a)||_{\mathbf{M}} \geq \epsilon$ and the lemma is proved.

Lemma 2.8. *Suppose that T is a theory extending $\Pi\forall$, σ is a sentence, c is a constant symbol that does not appear in either T or σ, and* **both** *$T \cup \{\forall x\varphi(x) \to 0\} \vdash \sigma$ and $T \cup \{\varphi(c) \to \forall x\varphi(x)\} \vdash \sigma$. Then $T \vdash \sigma$.*

Proof. Fix τ, τ_c and $\mathbf{M}$ as in the proof of the Lemma 2.7. As before, it suffices to prove that $||\sigma||_{\mathbf{M}} = 1$. First, if $||\forall x\varphi(x)||_{\mathbf{M}} = 0$, then $\mathbf{M}$ would be a model of $T \cup \{\forall x\varphi(x) \to 0\} \vdash \sigma$ and $||\sigma||_{\mathbf{M}} = 1$ by our hypothesis. So we assume that $||\forall x\varphi(x)||_{\mathbf{M}} \neq 0$. As notation, let $\psi(y)$ abbreviate $\varphi(y) \to \forall x\varphi(x)$ and, for each $a \in M$, let $\mathbf{M}_a$ denote the expansion of $\mathbf{M}$ satisfying $m_c^{\mathbf{M}_a} = a$. As in the proof of Lemma 2.7 choose k so that $T \vdash \psi(c)^k \to \sigma$. As in that argument, by considering each of the $\mathbf{M}_a$'s and reflecting back to $\mathbf{M}$, we obtain that

$$k||\psi(a)||_{\mathbf{M}} \leq ||\sigma||_{\mathbf{M}} \quad \text{for all } a \in M$$

Now fix an $\epsilon < 1$.

Since $||\forall x\varphi(x)||_{\mathbf{M}} = \inf\{||\varphi(a)|| : a \in M\}$ and $||\forall x\varphi(x)||_{\mathbf{M}} \neq 0$, there is some $a \in M$ so that $||\forall x\varphi(x)||_{\mathbf{M}} \geq ||\varphi(a)|| + \epsilon/k$ (again $\mathbb{R}^J$ is divisible). As before, this implies that $k||\psi(a)||_{\mathbf{M}} \geq \epsilon$. Since $||\sigma||_{\mathbf{M}} \geq ||\psi(a)||_{\mathbf{M}}$ and $\epsilon < 1$ was arbitrary, it follows that $||\sigma||_{\mathbf{M}} = 1$.

Proof of Theorem 2.6. We follow the proof of Hájek's Completeness Theorem (Theorem 5.2.9 of [H]). Therein, given T and σ in the language τ, he first augments τ by adding one new constant symbol for each formula in the original language. Then, in Lemma 5.2.7 he iteratively constructs an extension T' of T in this expanded language that is *complete* (i.e., for every pair (φ, ψ) of sentences, either $T \vdash (\varphi \to \psi)$ or $T \vdash (\psi \to \varphi)$) and *Henkin* (i.e., for every formula $\varphi(x)$ with one free variable **if** $T \not\vdash \forall x \varphi(x)$ then there is a constant symbol c such that $T \not\vdash \varphi(c)$) *while still preserving that* $T' \not\vdash \sigma$. In our context, we do precisely the same thing, but by iteratively using Lemmas 2.7 and 2.8 to handle each formula $\varphi(x)$ we additionally require that T' satisfy two additional requirements:

- For all $\varphi(x)$ in the expanded language there is a constant symbol c such that $T' \vdash \exists x \varphi(x) \to \varphi(c)$ and
- For all $\varphi(x)$ in the expanded language **either** $T' \vdash \forall x \varphi(x) \to \bar{0}$ **or** there is a constant symbol c such that $T' \vdash \varphi(c) \to \forall x \varphi(x)$.

Then, just as in Hájek, one can canonically construct a structure $\mathbf{M}$ whose universe consists of the constant symbols of the expanded language. In his context $\mathbf{M}$ was safe, $\mathbf{M}$ model of T' (hence the reduct to the original language is a model of T) and $||\sigma||_{\mathbf{M}} < 1$. It is easy to see that with the addition of the two extra properties noted above, $\mathbf{M}$ is closed.

We close this section by indicating a general construction and then an application of it which asserts the existence of a 'universal' BL-chain in the context of predicate product logic.

Fix a language τ. Given any BL-chain $\mathbf{L}$ and any $\mathbf{L}$-structure $\mathbf{M} = (M, m_c, r_P)$, we form a first-order, two-valued structure that encodes all of this data. Specifically, let

$$\mathfrak{M} = (M, L, +, \leq, 0, 1, m_c, r_P)$$

be the two-sorted structure in which $\{+, \leq, 0, 1\}$ refer to the BL-chain, each m_c points to an element in the M-sort, and $r_P : M^n \to L$ is the map described by $\mathbf{M}$ for each n-ary $P \in \tau$.

Many of the notions discussed in this paper are first-order in this language. If $\mathfrak{M}' = (M', L', \dots)$ is elementarily equivalent to the structure $\mathfrak{M}$ defined above, then $\mathbf{L}' = (L', +, \leq, 0, 1)$ is a BL-chain and $\mathfrak{M}'$ describes an $\mathbf{L}'$-structure $\mathbf{M}'$ with universe M'. One can check that $\mathbf{M}$ is safe (resp. closed) if and only if $\mathbf{M}'$ is safe (closed). Using the algebraic characterization of such semigroups in Proposition 1.5, the BL-chain $\mathbf{L}$ is equal to $L(G)$ for some ordered abelian group if and only if $\mathbf{L}' = L(G')$ for some group G'. Furthermore, if $\mathfrak{M}$ is an elementary substructure of $\mathfrak{M}'$ (in the usual first-order sense) one can inductively argue that $||\varphi||_{\mathbf{M}} = ||\varphi||_{\mathbf{M}'}$ for all $\tau(M)$-sentences φ.

Theorem 2.9. *If τ is countable, $T \supseteq \Pi\forall$ and $T \not\vdash \sigma$, then there is a countable, closed $L(\mathbb{R}^{\mathbb{Q}})$-structure $\mathbf{M}$ that models T but does not model*

σ. In particular, a sentence σ is an $L(\mathbb{R}^{\mathbb{Q}})$-tautology if and only if σ is provable from $\Pi\forall$.

Proof. Fix τ, T, and σ as in the hypotheses. By Lemma 2.4 there is a nonempty $(J,<)$ and a closed $L(\mathbb{R}^J)$-structure $\mathbf{M}$ that models T, but $||\sigma||_{\mathbf{M}} < 1$. Form the first-order structure $\mathfrak{M} = (M, L(\mathbb{R}^J), \dots)$ described above. Let $\mathfrak{M}_0$ be any countable elementary substructure of $\mathfrak{M}$. Note that by elementarity the BL-chain associated to $\mathfrak{M}_0$ is equal to $L(G)$ for some infinite ordered abelian group G.

We now form an increasing elementary chain of countable (first-order) structures $\mathfrak{M}_0 \preceq \mathfrak{M}_1 \preceq \dots$ as follows: Given $\mathfrak{M}_k$ consider the type $p(x)$ stating that x is in the L-sort, $x > 0$, but for any element c in the L-sort of $\mathfrak{M}_k$, $x < nc$ for all positive integers n. p is clearly consistent since the L-sort of $\mathfrak{M}_k$ has the form $L(G_k)$ for some infinite ordered abelian group G_k. So choose $\mathfrak{M}_{k+1}$ to be any countable elementary extension of $\mathfrak{M}_k$ that realizes p.

Let $\mathfrak{M}^* = (M^*, L^*, \dots)$ be the union of the chain, let $\mathbf{L}^* = (L^*, +, \leq, 0, 1)$, and let $\mathbf{M}^*$ be the $\mathbf{L}^*$-structure coded by $\mathfrak{M}^*$. Since $\mathfrak{M}_0 \preceq \mathfrak{M}^*$, $\mathbf{M}^*$ is a closed model of T with $||\sigma||_{\mathbf{M}^*} < 1$. Also, $\mathbf{L}^* = L(G^*)$ for some ordered abelian group G^*. As well, our construction guarantees that *there is no smallest archimedean class of the negative elements of G^**. By Theorem 1.10 we can choose $(J,<)$ and an archimedean surjection $f : G^* \to \mathbb{R}^J$. It follows from Proposition 1.9 that $f(\mathbf{M}^*)$ is closed, is a model of T, and $||\sigma||_{f(\mathbf{M}^*)} < 1$. Since G^* is countable, J is countable. Furthermore, J has no smallest element. Thus, there is an order-preserving coinitial map $g : J \to \mathbb{Q}$. This map induces a BL-chain embedding $g' : L(\mathbb{R}^J) \to L(\mathbb{R}^{\mathbb{Q}})$. Since $f(\mathbf{M}^*)$ is a closed $L(\mathbb{R}^J)$-structure, it follows from our construction and Lemma 2.3 that $g'(f(\mathbf{M}^*))$ is a closed $L(\mathbb{R}^{\mathbb{Q}})$-structure that models T with $||\sigma|| < 1$.

3. Initially dense linear orderings

In this section we obtain a dichotomy among the sets of $L(\mathbb{R}^J)$-tautologies that is related to the order type of $(J,<)$.

Definition 3.1. A linear ordering $(J,<)$ is *initially dense* if there is a coinitiality preserving embedding $f : (\mathbb{Q},<) \to (J,<)$.

It is readily checked that if $(J,<)$ is countable, then J is not initially dense if and only if there is some $a \in J$ such that $\{b \in J : b \leq a\}$ is scattered. The following theorem indicate that the BL-chain $L(\mathbb{R}^J)$ is universal in a strong sense whenever J is initially dense.

Theorem 3.2. *Suppose that $(J,<)$ is initially dense. If σ is any sentence and $T \supseteq \Pi\forall$ is any theory that has a model that does not model σ, then there is a closed $L(\mathbb{R}^J)$-model of T that does not model σ. In particular, a sentence σ is an $L(\mathbb{R}^J)$-tautology if and only if σ is provable from $\Pi\forall$.*

Proof. Fix T and σ as in the hypothesis. By Theorem 2.9 there is a closed $L(\mathbb{R}^{\mathbb{Q}})$-structure $\mathbf{M}$ that models T but does not model σ. Fix a coinitiality preserving embedding of $(\mathbb{Q},<)$ into $(J,<)$. This embedding naturally induces a coinitiality preserving ordered group homomorphism $f : \mathbb{R}^{\mathbb{Q}} \to \mathbb{R}^J$ of lexicographic sums, which in turn yields a BL-chain embedding $f : L(\mathbb{R}^{\mathbb{Q}}) \to L(\mathbb{R}^J)$ that respects 0. Since $\mathbf{M}$ is closed, Lemma 2.3 implies that $f(\mathbf{M})$ is the desired structure.

By contrast, Montagna [Mo] proves that the set of tautologies of $L(\mathbb{R}^1)$ is not arithmetical. Here we extend his method to show that the set of $L(\mathbb{R}^J)$-tautologies are not arithmetical whenever J is countable but not initially dense. We begin our analysis with a construction that appears in [Mo].

Let τ be a finite, relational vocabulary (for simplicity we do not allow τ to have any constant symbols in this discussion) containing a distinguished binary relation E. For any τ-formula φ, let φ° denote the τ-formula in which every relation symbol $R \in \tau$ is replaced by $\neg\neg R$. Note that if $\mathbf{M}$ is any model of $\Pi\forall$ then $||R^\circ(a_1,\ldots,a_n)||^{\mathbf{M}} \in \{0,1\}$ for any $a_1,\ldots a_n$ from M and $||R^\circ(a_1,\ldots,a_n)||_{\mathbf{M}} = 1$ if and only if $||R(a_1,\ldots,a_n)||_{\mathbf{M}} > 0$. It follows by induction on the complexity of φ that $||\varphi(\bar{a})||_{\mathbf{M}} \in \{0,1\}$ for any τ-formula φ and any tuple $\bar{a}$ from M. Moreover, the interpretation of quantifiers is as in 2-valued logic, e.g., $||\exists x\varphi^\circ(x,\bar{a})||_{\mathbf{M}} = 1$ if and only if $||\varphi^\circ(b,\bar{a})||_{\mathbf{M}} = 1$ for some $b \in M$.

Let $Quot(\tau)$ denote the conjunction of (finitely many) axioms asserting that E° is an equivalence relation and that

$$\forall x_1,\ldots,x_n\forall y_1\ldots,y_n(\bigwedge_i E^\circ(x_i,y_i) \wedge R^\circ(x_1,\ldots,x_n) \to R^\circ(y_1,\ldots,y_n))$$

for each relation symbol $R \in \tau$. Note that $||\sigma||_{\mathbf{M}} \in \{0,1\}$ for each $\sigma \in Quot(\tau)$ and any model $\mathbf{M}$ of $\Pi\forall$. We call a model $\mathbf{M}$ of $\Pi\forall$ τ-*quotientable* (with respect to the distinguished relation E) if $||\sigma||_{\mathbf{M}} = 1$ for all $\sigma \in Quot(\tau)$. As the name suggests, it is easily checked that if $\mathbf{M}$ is τ-quotientable, then E° is an equivalence relation on M. As notation, for each $a \in M$, let $[a] = \{b \in M : ||E^\circ(a,b)||_{\mathbf{M}} = 1\}$ and let $M^\circ = \{[a] : a \in M\}$. Furthermore, we can define an $L(\{0,1\})$-structure $\mathbf{M}^\circ$ with universe M° by positing that $R^{\mathbf{M}^\circ}([a_1],\ldots,[a_n])$ holds if and only if $||R^\circ(a_1,\ldots,a_n)||_{\mathbf{M}} = 1$. (The axioms of $Quot(\tau)$ guarantee that this is well-defined.) It follows by induction on the complexity of φ that

$$\mathbf{M}^\circ \models \varphi([a_1], \dots, [a_n]) \quad \text{if and only if} \quad ||\varphi^\circ(a_1 \dots, a_n)||_{\mathbf{M}} = 1$$

for any choice of representatives for $[a_1], \dots, [a_n]$.

Let $\tau_a = \{E, L, S, Z, A, P\}$ denote the (relational) *language of arithmetic*. The relations E, L, S are binary and their intended interpretations are equality, strict less than, and the graph of the successor function. Z is unary and is intended to denote the class of zero elements, while A and P are ternary and are intended to represent the graphs of addition and multiplication. Let $\mathfrak{N}$ denote the standard $\{0, 1\}$-model of arithmetic in the vocabulary τ_a, i.e., the universe of $\mathfrak{N}$ is ω and each of the relations are given their intended interpretations.

Next, let Q^* be the finite set of τ_a-sentences $Quot(\tau_a)$, together with modified versions of each axiom of Robinson's Q. The modifications are two-fold. First, they need to be written in our relational language τ_a. Second, we replace each relation symbol R by R°. So, for example, the axiom $\forall x(x < S(x))$ becomes $\forall x \forall y(S^\circ(x, y) \to L^\circ(x, y))$.

It is easily checked that if $\mathbf{M}$ is any model of Q^* then the two-valued structure $\mathbf{M}^\circ$ (which is well-defined since $\mathbf{M}$ is quotientable) is a model of Robinson's Q. Note that Robinson's Q is strong enough to determine the 'standard part' of any two-valued model of Q. In particular, if $\mathbf{M}^\circ \models Q$ is 'standard', i.e., every class is an iterated successor of the zero class, then $\mathbf{M}^\circ$ is τ_a-isomorphic to $\mathfrak{N}$.

Let $\tau_U = \tau_a \cup \{U\}$, where U is a unary predicate. We will be interested in structures $\mathbf{M}$ in the vocabulary τ_U and their reducts to τ_a.

Definition 3.3. A structure $\mathbf{M}$ modelling $\Pi\forall$ in the vocabulary τ_U has a *standard arithmetical part* if $||\sigma||_{\mathbf{M}} = 1$ for all $\sigma \in Q^*$ and the associated two-valued structure $\mathbf{M}^\circ$ is τ_a-isomorphic to $\mathfrak{N}$.

Let Ψ denote τ_U-sentence that is the conjunction of Q^* with the following five sentences:

- $\psi_1 := \forall x \neg\neg U(x)$;
- $\psi_2 := \neg \forall x U(x)$;
- $\psi_3 := \forall x \forall y(E^\circ(x, y) \wedge U(x) \to U(y))$;
- $\psi_4 := \forall x \forall y(L^\circ(x, y) \wedge U(y) \to U(x))$;
- $\psi_5 := \forall x \forall y(S^\circ(x, y) \wedge U(y) \to (U(x) \& U(x) \& U(x)))$.

Lemma 3.4. *If $(J, <)$ is countable then there is an $L(\mathbb{R}^J)$-model of Ψ.*

Proof. Fix any countable J. Since $\mathbb{R}^J$ has countable coinitiality, we can choose elements $\langle b_n : n \in \omega \rangle$ from $\mathbb{R}^J$ such that $\{b_n : n \in \omega\}$

are coinitial and such that $b_{n+1} < 3b_n$ for each $n \in \omega$. Let $\mathbf{M}$ be the structure in the vocabulary τ_U with universe ω in which each of the relations in L_a is given its 'standard' interpretation and $||U(n)||_{\mathbf{M}} = b_n$ for each $n \in \omega$.

Proposition 3.5. *Suppose that $(J, <)$ is countable but not initially dense and $\mathbf{M}$ is any $L_U(\mathbb{R}^J)$-model of $\Pi\forall$ such that $||\Psi||_{\mathbf{M}} > 0$. Then $\mathbf{M}$ has a standard arithmetical part.*

Proof. Fix such a $(J, <)$ and $\mathbf{M}$. To ease notation, we write $||\cdot||$ in place of $||\cdot||_{\mathbf{M}}$ throughout the proof of this lemma. Let $||\Psi|| = \gamma$. Since $\gamma \neq 0$, $\gamma \in N(\mathbb{R}^J)$. Since $||\sigma|| \in \{0, 1\}$ for each $\sigma \in Q^*$, it follows that $||\sigma|| = 1$ for each $\sigma \in Q^*$. Thus, the reduct of $\mathbf{M}$ is τ_a-quotientable. As well, since $||\psi_1|| > 0$ $||U(a)|| \neq 0$, hence $||U(a)|| \in N(\mathbb{R}^J)$ for all $a \in M$. Since $||\psi_2|| \neq 0$, $\{||U(a)|| : a \in M\}$ is coinitial in $N(\mathbb{R}^J)$. Since $||\psi_i|| \geq \gamma$ for $i \in \{3, 4, 5\}$, $||U(b)|| \leq ||U(a)|| - \gamma$ whenever either $E^\circ(a, b)$ or $L^\circ(a, b)$ hold, and

$$||U(b)|| \leq 3||U(a)|| - \gamma \tag{3}$$

whenever $S^\circ(a, b)$ holds. (Recall that γ is a negative element of $\mathbb{R}^J$.) Fix an element $c \in M$ such that $||U(c)|| \leq 2\gamma$. It follows from (3) that

$$||U(b)|| \leq 2||U(a)|| \tag{4}$$

whenever $L^\circ(c, a)$ and $S^\circ(a, b)$ hold.

Now assume by way of contradiction that $\mathbf{M}^\circ \not\cong \mathfrak{N}$. Thus $\mathbf{M}^\circ \models Q$, but has nonstandard elements. By iterating (4),

$$||U(b)|| << ||U(a)|| \tag{5}$$

whenever $L^\circ(c, a)$ and $L^\circ(a, b)$ hold, and $[b] - [a]$ is nonstandard. That is, $||U(b)||$ is in a strictly smaller archimedean class than $||U(a)||$. This fact, together with the fact that $\{||U(a)|| : a \in M\}$ is coinitial in $\mathbb{R}^J$ imply that $\mathbb{R}^J$ has no smallest archimedean class, i.e., J has no least element.

Fix $c^* \in M$ such that $L^\circ(c, c^*)$ and $||U(c^*)|| << \gamma$. Since $\beta + \gamma \sim \beta$ whenever $\beta << \gamma$, $||\psi_3|| \geq \gamma$ implies $||U(a)|| \sim ||U(a')||$ whenever $L^\circ(c^*, a)$, $L^\circ(c^*, a')$ and $E^\circ(a, a')$. That is, the archimedean class of $||U(a)||$ depends only on $[a]$. As well, suppose that $L^\circ(c^*, a)$, $L^\circ(a, b)$, and $[b] - [a]$ is a nonstandard element of $\mathbf{M}^\circ$. Let d be any element in the E°-class of $([a] + [b])/2$ (which exists since $\mathbf{M}^\circ \models Q$). Then $[b] - [d]$ and $[d] - [a]$ are nonstandard elements and (5) yields

$$||U(b)|| << ||U(d)|| << ||U(a)|| \tag{6}$$

We will obtain a contradiction by constructing a coinitiality preserving embedding $f : \mathbb{Q} \to J$. Let $D = \{n/2^m : n \in \omega \setminus \{0\}, m \in \omega\}$ denote the positive dyadics. We will construct an embedding $g : D \to M$ such that $L^\circ(c^*, f(d))$ and $||U(d')|| << ||U(d)||$ for all $d < d'$ from D and $\{||U(d)|| : d \in D\}$ is coinitial in $\mathbb{R}^J$. Once we have such a g, then f can be obtained by composing an isomorphism between $(\mathbb{Q}, <)$ and $(D, <)$ with g. Since $\mathbf{M}^\circ$ is nonstandard and $(J, <)$ is countable with no minimal element we can find $\{a_n : n \in \omega\}$ from M such that $L^\circ(c^*, a_0)$, $L^\circ(a_n, a_{n+1})$, $||U(a_{n+1})|| << ||U(a_n)||$ for all n and $\{||U(a_n)|| : n \in \omega\}$ is coinitial in $\mathbb{R}^J$. We begin our construction of g by letting $g(n) = a_n$. Now suppose $\{g(n/2_l) : n \in \omega, l \leq m\}$ have been defined. Fix an odd $n \in \omega$, say $n = 2k - 1$. Let d_k be any element of the E°-class of $([g(k/2^m)] + [g((k+1)/2^m)])/2$. It follows from (6) that $||U(g((k+1)/2^m))|| << ||U(d_k)|| << ||g(k/2^m)||$, so let $g(n/2^{m+1}) = d_k$.

Definition 3.6. *For σ any sentence in τ_a, let σ^* denote the τ_U-sentence $\Psi \to \sigma^\circ$.*

Theorem 3.7. *If $(J, <)$ is countable but not initially dense, then the set of $L(\mathbb{R}^J)$-tautologies in the vocabulary τ_U is not arithmetical.*

Proof. Fix any countable $(J, <)$ that is not initially dense. We argue that for any τ_a-sentence σ, σ^* is an $L(\mathbb{R}^J)$-tautology if and only if $\mathfrak{N} \models \sigma$ (in the usual two-valued sense). The Theorem follows immediately from this by Tarski's Theorem and the recursiveness of the map $\sigma \mapsto \sigma^*$.

First, suppose that σ^* is an $L(\mathbb{R}^J)$-tautology. By Lemma 3.4 we can choose $\mathbf{M}$ such that $||\Psi||_{\mathbf{M}} > 0$. By Proposition 3.5 $\mathbf{M}$ is τ_a-quotientable and $\mathbf{M}^\circ \cong \mathfrak{N}$. Since σ^* is an $L(\mathbb{R}^J)$-tautology and $||\Psi||_{\mathbf{M}} > 0$, $||\sigma^\circ||_{\mathbf{M}} > 0$. But, as noted earlier, this implies $||\sigma^\circ||_{\mathbf{M}} = 1$, hence $\mathbf{M}^\circ \models \sigma^\circ$. Since $\neg\neg\varphi$ is equivalent to φ in the class of two-valued structures and $\mathbf{M}^\circ \cong \mathfrak{N}$, $\mathfrak{N} \models \sigma$.

Conversely, suppose $\mathfrak{N} \models \sigma$. Let $\mathbf{M}$ be any $L(\mathbb{R}^J)$-structure with vocabulary τ_U. We argue that $||\sigma^*||_{\mathbf{M}} = 1$. This is immediate if $||\Psi||_{\mathbf{M}} = 0$, so assume $||\Psi||_{\mathbf{M}} > 0$. Then, again by Proposition 3.5, $\mathbf{M}$ is τ_a-quotientable and $\mathbf{M}^\circ \cong \mathfrak{N}$. Thus, $\mathbf{M}^\circ \models \sigma^\circ$, so $||\sigma^\circ||_{\mathbf{M}} = 1$, which implies $||\sigma^*||_{\mathbf{M}} = 1$.

Remark 3.8. Note that the proofs of both Proposition 3.5 and Theorem 3.7 only require that $(J, <)$ have countable coinitiality (and not initially dense).

Corollary 3.9. *The following are equivalent for a countable linear order $(J, <)$.*

1. *For all countable vocabularies τ, a sentence σ is an $L(\mathbb{R}^J)$-tautology if and only if σ is provable from $\Pi\forall$;*
2. *For all finite vocabularies τ, the set of $L(\mathbb{R}^J)$-tautologies is arithmetic;*
3. *$(J, <)$ is initially dense.*

Proof. Immediate by Theorems 3.2 and 3.7.

Corollary 3.10. *Let σ be any τ_a-sentence such that $\mathfrak{N} \models \sigma$, but $Q \not\vdash \sigma$ (in the usual proof theory of first-order logic). Let $(J, <)$ be a countable linear order. Then σ^* is an $L(\mathbb{R}^J)$-tautology if and only if $(J, <)$ is not initially dense.*

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