Asymptotic analysis for a Vlasov-Fokker-Planck/Compressible Navier-Stokes system of equations

A.Mellet*† and A.Vasseur‡

Department of Mathematics
University of Texas at Austin

Abstract

This article is devoted to the asymptotic analysis of a system of coupled kinetic and fluid equations, namely the Vlasov-Fokker-Planck equation and a compressible Navier-Stokes equation. Such a system is used, for example, to model fluid-particles interactions arising in sprays, aerosols or sedimentation problems. The asymptotic regime corresponding to a strong drag force and a strong Brownian motion is studied and the convergence toward a two phases macroscopic model is proved. The proof relies on a relative entropy method.

1 Introduction

1.1 The model

This paper is devoted to the asymptotic analysis of a system of equations modeling the evolution of dispersed particles in a compressible fluid. This kind of system arises in a lot of industrial applications. One example is the analysis of sedimentation phenomenon, with applications in medicine, chemical engineering or waste water treatment (see Berres, Bürger, Karlsen, and Tory [5], Gidaspow [13], Sartory [21], Spannenberg and Galvin [22]). Such systems are also used in the modeling of aerosols and sprays with

*mellet@math.utexas.edu
†A. Mellet was partially supported by NSF grant DMS-0456647
‡vasseur@math.utexas.edu

applications, for instance, in the study of Diesel engines (see Williams [24], [23]).

At the microscopic scale, the cloud of particles is described by its distribution function $f(x, v, t)$, solution to a Vlasov-Fokker-Planck equation:

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v(F_d f - \nabla_v f) = 0. \tag{1}$$

The fluid, on the other hand, is modeled by macroscopic quantities, namely its density $\rho(x, t) \geq 0$ and its velocity field $u(x, t) \in \mathbb{R}^N$. We assume that the fluid is compressible and isentropic, so that (ρ, u) solves the compressible Euler or Navier-Stokes system of equations:

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x p - \nu \Delta u = F_f \end{cases} \tag{2}$$

(with $\nu = 0$ in the inviscid case and $\nu > 0$ in the viscous case).

The fluid-particles interactions are modeled by a friction (or drag) force exerted by the fluid onto the particles. This force is assumed to be proportional to the relative velocity of the fluid and the particles:

$$F_d = F_0(u(x, t) - v)$$

where F_0 is supposed to be constant equal to 1 (see Remark 1.1 below). The right hand-side in the Euler equation takes into account the action of the cloud of particles on the fluid:

$$F_f = -\int F_d f dv = F_0 \int (v - u(x, t)) f(x, v, t)\, dv.$$

For the sake of simplicity, we assume that the pressure term is given by

$$p = \rho^\gamma,$$

though more general pressure terms could be taken into consideration.

This particular Vlasov-Navier-Stokes system of equations is used, for instance, in the modeling of reaction flows of sprays (see Williams [24],[23]) and is at the basis of the code KIVA-II of the Los Alamos National Laboratory (see O'Rourke et al. [1] and Amsden [2]). We refer to the nice paper of Carrillo and Goudon [11] for a discussion on various modelling issues and stability properties of this system of equations.

A first issue, when dealing with such a kinetic/fluid system of equations is the existence of solutions. Global existence results for the coupling of kinetic equations with incompressible Navier-Stokes equations was proved by Hamdache in [17]. The existence of solutions for short time in the case of the hyperbolic system (i.e. no viscosity in the Navier-Stokes equation ($\nu = 0$) and no Brownian effect in the kinetic equation) is proved by Baranger and Desvillettes in [3]. In [20], we study the coupled system (1)-(2) and prove the existence of weak solutions (f, ρ, u) for any $\gamma > 3/2$ under minimal assumption (finite mass and energy) on the initial data. Our result is recalled in Section 1.5.

In this paper we adress the question on the asymptotic regime corresponding to a strong drag force and strong brownian motion. More precisely, we consider the following system of singular equations:

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon}\mathrm{div}_v((u - v)f_\varepsilon - \nabla_v f_\varepsilon) = 0 \tag{3}$$

$$\partial_t \rho_\varepsilon + \mathrm{div}_x(\rho_\varepsilon u_\varepsilon) = 0 \tag{4}$$

$$\partial_t(\rho_\varepsilon u_\varepsilon) + \mathrm{div}_x(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla_x \rho_\varepsilon^\gamma - \nu\Delta u = \frac{1}{\varepsilon}(j_\varepsilon - n_\varepsilon u_\varepsilon) \tag{5}$$

where $n_\varepsilon = \displaystyle\int f_\varepsilon(x, v, t)\, dv$ and $j_\varepsilon = \displaystyle\int v f_\varepsilon(x, v, t)\, dv$. Note that large time behavior in the case of incompressible fluids has been studied previously by Hamdache [17] and Goudon, Jabin and Vasseur [16, 15].

The main result stated in the present paper is the convergence of weak solutions $(f_\varepsilon, \rho_\varepsilon, u_\varepsilon)$ of (3)-(5) to $(M_{n,u}, \rho, u)$ ($M_{n,u}$ denotes the Maxwellian distribution with density n and velocity u) where (n, ρ, u) is solution of the following system of hydrodynamic equations:

$$\begin{cases} \partial_t n + \mathrm{div}_x(nu) = 0 \\ \partial_t \rho + \mathrm{div}_x(\rho u) = 0 \\ \partial_t((\rho + n)u) + \mathrm{div}_x((\rho + n)u \otimes u) + \nabla_x(n + \rho^\gamma) - \nu\Delta u = 0 \end{cases} \tag{6}$$

This kind of multi-fluid system is also widely used in the modeling of particle/fluid interaction (see for instance Laurent, Massot, and Villedieu [18], or Berthonnaud [7]). Notice that the Brownian effect in the kinetic equation ends up as an additional pressure term in the velocity equation. The derivation of (6) from (3)-(5) was first addressed, formally, by Carrillo and Goudon in [11]. In this paper, we rigorously justify this asymptotic analysis.

Remark 1.1 *The choice of drag force $F_d = F_0(u - v)$ may not be the most relevant one from a physical point of view. In particular, a quadratic dependence in the velocity may be more appropriate in certain regimes. It could* *also seem more relevant from a physical point of view to assume that* $|F_d|$ *depends on the density of the fluid* ρ*, such as*

$$F_d = \rho(u - v). \tag{7}$$

However, the asymptotic analysis is not valid in those situation, unless we can establish a priori lower bounds on the density. Indeed, we cannot expect to have relaxation of the kinetic regime toward an hydrodynamic regime with velocity u *on the vacuum* $\{\rho = 0\}$*.*

The main tool in the proof is a relative entropy method. It relies on the "weak-strong" uniqueness principle established by Dafermos for multidimensional systems of hyperbolic conservation laws admitting a convex entropy functionals [12]. It has been frequently used for system of particles and rarefied gas dynamics, see Yau [25]. It is the main tool also for the asymptotic limit of the Boltzmann equation to the incompressible Navier-Stokes equation, see Bardos, Golse, Levermore [4], Golse, Saint-Raymond [14], Lions, Masmoudi [19]. See also Berthelin, Vasseur [6] and Goudon, Jabin, Vasseur [16] for other kind of Hydrodynamical limit. For different asymptotic problems it is called "modulated energy" method (Brenier [10], [9]).

1.2 Boundary conditions and notion of weak solutions

As usual, the kinetic variable v (the velocity) lies in $\mathbb{R}^N$, while the space variable x lies in a subset Ω of $\mathbb{R}^N$. We will be considering two situations: The case of Ω bounded subset with periodic boundary condition ($\Omega = \mathbb{T}^N$), and the case of Ω bounded subset of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. In the later case, the system (1)-(2) has to be supplemented with boundary condition along $\partial\Omega$. The natural assumption for Navier-Stokes equations is homogeneous Dirichlet condition for the velocity:

$$u(x,t) = 0 \qquad\qquad \forall x \in \partial\Omega$$

To write the boundary condition for the kinetic equation, we introduce γf the trace of f on $\partial\Omega$, and we denote $\gamma^{\pm} f(x,v,t) = f|_{\Sigma^{\pm}}$ where

$$\Sigma^{\pm} = \{(x,v) \in \partial\Omega \times \mathbb{R}^N \mid \pm v \cdot r(x) > 0\}$$

($r(x)$ denotes the outward normal unit vector). In this ppaper, our only requirement is that the boundary condition is such that the following conditions are satisfied:

$$\int_{\mathbb{R}^N} (v \cdot r(x))\, \gamma f(x,v,t)\, dv = 0 \qquad\qquad \forall x \in \partial\Omega \tag{8}$$

and

$$\int_{\mathbb{R}^N} (v \cdot r(x)) \left(\frac{|v|^2}{2} + \log(\gamma f) \right) \gamma f \, dv \geq 0 \quad \forall x \in \partial\Omega. \tag{9}$$

Those conditions are very classical in kinetic theory and they are satisfied if we consider local or diffusive reflection conditions. Typically, we write such conditions in the form

$$\gamma^- f(x,v,t) = \mathcal{B}(\gamma^+ f) \qquad \forall (x,v) \in \Sigma^-,$$

where the operator $\mathcal{B}$ is given by

$$\mathcal{B}(g) = \alpha J(g) + (1-\alpha) D(g) \qquad \alpha \in [0,1]$$

with a specular reflection operator J defined by

$$J(g)(x,v) = g(x, R_x v) \qquad\qquad R_x v = v - 2(v \cdot r(x))\, r(x),$$

and a diffusive operator given by (for example)

$$D(g)(x,v) = M(v) \int_{v \cdot r > 0} g(x,v) v \cdot r \, dv\,,$$

where $M(v)$ is a Maxwellian distribution satisfying $\int_{v\cdot r>0} M(v)\, |v \cdot r|\, dv = 1$ for all $r \in S^2$.

Other boundary conditions can be considered (such as elastic reflection conditions), we refer the reader to [20] for further considerations on those boundary conditions.

Finally, we recall that (f, ρ, u) is a weak solution of (3)-(5) on $[0,T]$ if

$$\begin{aligned}
& f(x,v,t) \geq 0 \qquad \forall (x,v,t) \in \Omega \times \mathbb{R}^3 \times (0,T), \\
& f \in C([0,T]; L^1(\Omega \times \mathbb{R}^3)) \cap L^\infty(0,T; L^1 \cap L^\infty(\Omega \times \mathbb{R}^3)), \\
& |v|^2 f \in L^\infty(0,T; L^1(\Omega \times \mathbb{R}^3)),
\end{aligned}$$

and

$$\begin{aligned}
&\rho(x,t) \geq 0 \qquad \forall (x,t) \in \Omega \times (0,T),\\
&\rho \in L^\infty(0,T;L^\gamma(\Omega)) \cap C([0,T];L^1(\Omega)),\\
&u \in L^2(0,T;H^1_0(\Omega)), \quad \rho\,|u|^2 \in L^\infty(0,T;L^1(\Omega)),\\
&\rho\, u \in C([0,T];L^{2\gamma/(\gamma+1)}(\Omega)-w).
\end{aligned}$$

Moreover, we ask that (4)-(5) holds in the sense of distribution (Note that the conditions on f yield $n(x,t) \in L^\infty(0,T;L^{6/5}(\Omega))$ which is enough to give a meaning to the product $n\,u$ in $L^1((0,T)\times\Omega)$), and in the case of reflection boundary conditions, we ask that the kinetic equation (3) holds in the following sense:

$$\int_0^T \int_{\Omega\times\mathbb{R}^N} f\big[\partial_t\varphi + v\cdot\nabla_x\varphi + (u-v)\cdot\nabla_v\varphi + \Delta_v\varphi\big]\,dx\,dv\,dt + \int_{\Omega\times\mathbb{R}^N} f_0\varphi(x,v,0)\,dx\,dv = 0 \tag{10}$$

for any $\varphi \in \mathcal{C}^\infty(\overline{\Omega}\times\mathbb{R}^3\times[0,T])$ such that $\varphi(\cdot,T)=0$ and

$$\gamma^+\varphi = \mathcal{B}^*\gamma^-\varphi \quad \text{ on } \Sigma^+\times[0,T] \tag{11}$$

(which holds, in particular, if φ is independent of v).

1.3 Formal derivation of the asymptotic model

The formal derivation of the asymptotic model was first investigated by J. Carrillo and T. Goudon in [11]. We recall here the main steps for the sake of completness. First of all, assuming that $\lim_{\varepsilon\to0} f_\varepsilon = f$ and $\lim_{\varepsilon\to0} u_\varepsilon = u$, (3) formally yields

$$(u-v)f - \nabla_v f = 0,$$

which implies

$$f(x,v,t) = n(x,t)\frac{1}{(2\pi)^{3/2}}e^{-\frac{|u(x,t)-v|^2}{2}},$$

where $n(x,t) = \int f(x,v,t)\,dv$ denotes the density of particles. Furthermore, integrating (3) with respect to v yields

$$\partial_t n + \mathrm{div} j = 0$$

where

$$j(x,t) = \int v\, f(x,v,t)\, dv = n(x,t)u(x,t),$$

is the current of particles. Finally, multiplying (3) by v and integrating with respect to v, we are led to:

$$\partial_t(nu) + \mathrm{div}_x(n\, u \otimes u) + \nabla_x n = -\frac{1}{\varepsilon}\int (u-v) f\, dv, \tag{12}$$

in which we used the equality

$$\int v \otimes v f\, dv = u \otimes u\, n + nI$$

(where I denotes the identity matrix). Adding (12) and (5), it is readily seen that the right hand sides cancel, so that (n, ρ, u) is solution to the following system of equations:

$$\begin{cases} \partial_t n + \mathrm{div}_x(nu) = 0 \\ \partial_t \rho + \mathrm{div}_x(\rho u) = 0 \\ \partial_t((\rho+n)u) + \mathrm{div}_x((\rho+n)u \otimes u) + \nabla_x(n + \rho^\gamma) - \nu \Delta u = 0 \end{cases}$$

with boundary condition

$$u(x,t) = 0 \qquad \forall x \in \partial\Omega.$$

1.4 Entropies

The rigorous proof of the convergence towards the asymptotic equation (6) relies heavily on the use of energy inequalities and relative entropy methods. Before we state our main result, we need to review the classical inequalities satisfied by smooth solutions of (3) and (4)-(5). First of all, setting

$$\mathscr{E}_1(f) = \int_{\mathbb{R}^N} \left(\frac{|v|^2}{2} f + f \log f \right) dv,$$

it is readily seen (multipying (3) by $\frac{|v|^2}{2} + \log f_\varepsilon + 1$ and integrating with respect to x, v) that smooth solutions of (3) satisfy:

$$\begin{aligned} \frac{d}{dt}\int_\Omega \mathscr{E}_1(f_\varepsilon)\, dx + \frac{1}{\varepsilon}\int_\Omega \int_{\mathbb{R}^N} |(u_\varepsilon - v) f_\varepsilon - \nabla_v f_\varepsilon|^2 \frac{1}{f_\varepsilon}\, dv\, dx \\ + \int_{\partial\Omega \times \mathbb{R}^N} (v \cdot r(x)) \left(\frac{|v|^2}{2} + \log \gamma f_\varepsilon + 1 \right) \gamma f_\varepsilon\, d\sigma(x)\, dv \\ = \frac{1}{\varepsilon}\int_\Omega \int_{\mathbb{R}^N} u_\varepsilon (u_\varepsilon - v) f_\varepsilon\, dv\, dx, \end{aligned}$$

where the boundary term is either 0 (periodic boundary condition) or non-negative (using (8) and (9)). In either case, we deduce

$$\frac{d}{dt}\int_\Omega \mathscr{E}_1(f_\varepsilon)\,dx + \frac{1}{\varepsilon}\int_\Omega\int_{\mathbb{R}^N} |(u_\varepsilon - v)f_\varepsilon - \nabla_v f_\varepsilon|^2 \frac{1}{f_\varepsilon}\,dv\,dx \\ \leq \frac{1}{\varepsilon}\int_\Omega\int_{\mathbb{R}^N} u_\varepsilon(u_\varepsilon - v) f_\varepsilon\,dv\,dx.$$

Moreover, defining

$$\mathscr{E}_2(\rho, u) = \rho\frac{|u|^2}{2} + \frac{1}{\gamma - 1}\rho^\gamma\,,$$

it is well-konwn that smooth solutions of (4)-(5) satisfy:

$$\frac{d}{dt}\int_\Omega \mathscr{E}_2(\rho_\varepsilon, u_\varepsilon)\,dx + \nu\int_\Omega |\nabla_x u_\varepsilon|^2\,dx = -\frac{1}{\varepsilon}\int_\Omega\int_{\mathbb{R}^N} u_\varepsilon(u_\varepsilon - v)f_\varepsilon\,dv\,dx.$$

We deduce the following proposition:

Proposition 1.1 *Let $(f_\varepsilon, \rho_\varepsilon, u_\varepsilon)$ be a smooth solution of (3)-(5), then the following energy equality holds:*

$$\frac{d}{dt}\int_\Omega \left[\mathscr{E}_1(f_\varepsilon) + \mathscr{E}_2(\rho_\varepsilon, u_\varepsilon)\right]dx \\ + \frac{1}{\varepsilon}\int_\Omega\int_{\mathbb{R}^N} |(u_\varepsilon - v)f_\varepsilon - \nabla_v f_\varepsilon|^2 \frac{1}{f_\varepsilon}\,dv\,dx + \nu\int_\Omega |\nabla_x u_\varepsilon|^2\,dx \leq 0. \quad (13)$$

In particular

$$\mathscr{E}(f, \rho, u) = \int_{\mathbb{R}^N}\left[\frac{|v|^2}{2}f + f\log f\right]dv + \rho\frac{|u|^2}{2} + \frac{1}{\gamma - 1}\rho^\gamma$$

is an entropy for the system (3)-(5).

From now on, we denote by

$$D(f, u) = \int_\Omega\int_{\mathbb{R}^N} \left|(u - v)f - \nabla_v f\right|^2 \frac{1}{f}\,dv\,dx$$

the kinetic dissipation.

Next, for a given density distribution n and velocity field u we introduce the maxwellian distribution

$$M_{(n,u)} = \frac{n}{(2\pi)^{3/2}} e^{-\frac{|u-v|^2}{2}}.$$

We know that, for any (f, ρ, u) satisfying $\int f\,dv + \mathscr{E}(f, \rho, u) < \infty$, the following minimization principle holds (see Bouchut [8]):

$$\mathscr{E}(M_{(n,u)}, \rho, u) \le \mathscr{E}(f, \rho, u),$$

where

$$n(x,t) = \int f(x, v, t)\,dv.$$

In particular, introducing

$$\begin{aligned} \mathscr{H}(n, \rho, u) &= \mathscr{E}(M_{(n,u)}, \rho, u) \\ &= (n+\rho)\frac{u^2}{2} + \frac{1}{\gamma - 1}\rho^\gamma + n \log n - \frac{2}{3}\log(2\pi) n, \end{aligned}$$

we have

$$\mathscr{H}(n, \rho, u) \le \mathscr{E}(f, \rho, u), \qquad \text{if} \quad n = \int f\,dv. \tag{14}$$

Moreover, $\mathscr{H}$ is an entropy for the asymptotic system (6). More precisely, we have:

Proposition 1.2 *Let (n, ρ, u) be a solution of (6), and let $U = (n, \rho, P) = (n, \rho, (\rho + n)u)$. We define*

$$\mathscr{H}(U) = \frac{P^2}{2(n+\rho)} + \frac{1}{\gamma-1}\rho^\gamma + n\log n - \frac{2}{3}\log(2\pi)n.$$

Then we have

$$\frac{d}{dt}\mathscr{H}(U) + div_x\left[F(U) - \nu\frac{P}{n+\rho}\nabla\frac{P}{n+\rho}\right] + \nu\left|\nabla\frac{P}{n+\rho}\right|^2 = 0,$$

where $\left|\nabla\frac{P}{n+\rho}\right|^2 = \sum_{i,j}|\partial_j u_i|^2$ *and with*

$$F(U) = \frac{P}{n+\rho}\left[\frac{P^2}{2(n+\rho)} + \frac{\gamma}{\gamma-1}\rho^\gamma + n\log n - \frac{2}{3}\log(2\pi)n\right]$$

In other words, $\mathscr{H}$ is a convex entropy for the system (6).

Finally, we recall that given an entropy $\mathscr{H}(U)$, we can define the relative entropy by

$$\mathscr{H}(V|U) = \mathscr{H}(V) - \mathscr{H}(U) - D\mathscr{H}(U)(V-U)$$

where D stand for the derivation with respect to the conservative variables (n, ρ, P). A simple computation shows that the relative entropy associated with (6) is

$$\mathscr{H}(U|U^*) = (n+\rho)\frac{|u-u^*|^2}{2} + \frac{1}{\gamma-1}p_1(\rho|\rho^*) + p_2(n|n^*)$$

with

$$\begin{aligned} p_1(\rho|\rho^*) &= \rho^\gamma - \rho^{*\gamma} - \gamma\rho^{*\gamma-1}(\rho-\rho^*) \\ p_2(n|n^*) &= n\log n - n^*\log n^* - (\log n^* + 1)(n-n^*) \\ &= n\log\frac{n}{n^*} + (n^*-n). \end{aligned}$$

(Note that the $p_i(\cdot|\cdot)$ are the relative entropies associated to $p_1(\rho) = \rho^\gamma$ and $p_2(n) = n\log n$.)

1.5 Main results

We now have all the notations necessary to state our main results. Throughout the paper, we will assume that the intial data have finite mass and energy. More precisely, we assume that $f_0(x,v)$, $\rho_0(x)$ and $u_0(x)$ are such that

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega \rho_0\,dx < \infty, \qquad \int_\Omega f_0\,dx\,dv < \infty \\ \displaystyle\int_\Omega \mathscr{E}(f_0,\rho_0,u_0)\;dx < +\infty \end{array} \right. \tag{15}$$

We recall that under those hypotheses, we proved in [20] the following result:

Theorem 1.1 ([20]) *Let $f_0(x,v)$, $\rho_0(x)$ and $u_0(x)$ satisfy (15). Assume moreover that $f_0 \in L^\infty(\Omega\times\mathbb{R}^N)$ and that*

$$\nu > 0 \qquad \text{and} \qquad \gamma > 3/2.$$

Then, for any $\varepsilon > 0$ there exists a weak solution $(f_\varepsilon, \rho_\varepsilon, u_\varepsilon)$ of (3)-(5) defined globally in time. Moreover, this solution satisfies the usual entropy inequality:

$$\begin{aligned} \int_\Omega \mathscr{E}(f_\varepsilon(t),\rho_\varepsilon(t),u_\varepsilon(t))\,dx + \frac{1}{\varepsilon}\int_0^t D(f_\varepsilon,u_\varepsilon)\,ds + \nu\int_0^t\int_\Omega |\nabla u_\varepsilon|^2\,dx\,ds \\ \leq \int_\Omega \mathscr{E}(f_0,\rho_0,u_0)\,dx \end{aligned} \tag{16}$$

In this paper, we are concerned with the asynptotic behavior of weak solutions as ε goes to zero. To simplify the analysis, we will assume that the initial data are well prepared, which amounts to assuming that

$$f_0(x,v) = M_{(n_0,u_0)} = \frac{n_0(x)}{(2\pi)^{3/2}} e^{-\frac{|u_0(x)-v|^2}{2}}. \tag{17}$$

Note that (15) can then be rewritten as

$$\begin{cases} \displaystyle\int_\Omega \rho_0 \, dx < \infty, \qquad \int_\Omega n_0 \, dx < \infty \\ \displaystyle\int_\Omega (n_0+\rho_0)\frac{|u_0|^2}{2} + \frac{1}{\gamma-1}\rho_0^\gamma + n_0 \log n_0 \, dx < \infty. \end{cases} \tag{18}$$

Remark 1.2 *Instead of (17), it would actually be enough to assume that the initial data f_0^ε converges to a maxwellian distribution in the following sense:*

$$\int_\Omega \mathscr{E}(f_0^\varepsilon, \rho_0, u_0) \, dx \longrightarrow \int_\Omega \mathscr{H}(n_0, \rho_0, u_0) \, dx$$

Finally, we need to assume that solutions of the asymptotic system of equations associated to initial value (ρ_0, n_0, u_0) stays away from vacuum. This will be satisfied (at least for small time) if the following condition holds:

$$\lambda_0 \le n_0(x) + \rho_0(x) \le \Lambda_0 \qquad \forall x \in \Omega, \tag{19}$$

with λ_0 and Λ_0 positive constants. Indeed, under this assumption, we have the following proposition (see Dafermos [12]):

Proposition 1.3 *Under hypothesis (18) and (19), there exists $T^* > 0$ and n^*, ρ^*, u^* solution of (6) on $[0, T^*)$ such that*

$$0 < \lambda \le n^*(x,t) + \rho^*(x,t) \le \Lambda \qquad \textit{for all } x \in \Omega, t \in [0,T^*)$$

for some positive constants λ and Λ.

The main result of this paper says that any solutions of (3)-(5) satisfying the entropy inequality (and thus in particular the solution constructed in Theorem 1.1), converges, as ε goes to zero to (n^*, ρ^*, u^*):

Theorem 1.2 *Assume that $n_0(x)$, $\rho_0(x)$ and $u_0(x)$ satisfy (15)-(19), and let $(f_\varepsilon, \rho_\varepsilon, u_\varepsilon)$ be a weak solution of (3)-(5) with initial conditions*

$$f_\varepsilon(x,v,0) = f_0(x,v), \quad \rho_\varepsilon(x,0) = \rho_0(x), \quad u_\varepsilon(x,0) = u_0(x),$$

and satisfying the entropy inequality (16). Assume moreover that

$$\nu \geq 0 \qquad \textit{and} \quad \gamma \in (1,2).$$

Then there exists a constant C such that

$$\int_0^T \mathscr{H}(U_\varepsilon|U^*)\,dt + \frac{\nu}{2}\int_0^T \int_\Omega |\nabla(u^\varepsilon - u^*)|^2 \,dx\,dt \leq C\sqrt{\varepsilon} \tag{20}$$

for all $T < T^*$ where $U_\varepsilon = (n_\varepsilon, \rho_\varepsilon, P_\varepsilon)$, $n_\varepsilon = \int f_\varepsilon(x,v,t)\,dv$ and $P_\varepsilon = (n_\varepsilon + \rho_\varepsilon)u_\varepsilon$.

Moreover, any sequence of functions $(f_\varepsilon, \rho_\varepsilon, u_\varepsilon)$ satisfying Inequality (20) satisfies

$$f_\varepsilon \longrightarrow M_{n^*,u^*} \quad \textit{a.e. and } L^1_{loc}(0,T^*;L^1(\Omega\times\mathbb{R}^3))\textit{-strong}$$
$$n_\varepsilon \longrightarrow n^* \quad \textit{a.e. and } L^1_{loc}(0,T^*;L^1(\Omega))\textit{-strong}$$
$$\rho_\varepsilon \longrightarrow \rho^* \quad \textit{a.e. and } L^p_{loc}(0,T^*;L^p(\Omega))\textit{-strong } \forall p<\gamma$$
$$\sqrt{\rho_\varepsilon}u_\varepsilon \longrightarrow \sqrt{\rho^*}u^* \quad L^p_{loc}(0,T^*;L^2(\Omega))\textit{-strong.}$$

When $\nu > 0$, we also have

$$u_\varepsilon \longrightarrow u^* \quad L^2_{loc}(0,T^*;L^2(\Omega))\textit{-strong.}$$

We stress out the fact that it is not necessary to have a positive viscosity coefficient ν to carry out the proof of this Theorem.

We now turn to the proof of Theorem 1.2. The next section is devoted to the central argument of the proof, namely the relative entropy inequality and its consequences. The (more technical) results needed to complete the proof (controle of the relative flux and of the kinetics approximation) are then detailed in the following two sections.

2 Relative entropy: Proof of Theorem 1.2

In this section we derive a relative entropy inequality for the asymptotic system (6) which is the corner stone of this paper. This inequality is much more general than the system under consideration and is valid for general system of conservation laws of the form

$$\partial_t U_i + \sum_k \partial_{x_k} A_{ik}(U) = \sum_k \partial_{x_k}\left[B_{ij}(U)\partial_{x_k}\left(D_j\mathscr{H}(U)\right)\right], \tag{21}$$

where $B(U)$ is a positive symmetric matrix and $D\mathscr{H}$ denotes the derivative (with respect to U) of the entropy $\mathscr{H}(U)$ associated with the flux $A(U)$ (the system (6) can be written in this form, see below). The existence of such an entropy is equivalent to the existence of an entropy flux function F such that

$$D_j F_k(U) = \sum_i D_i \mathscr{H}(U) D_j A_{ik}(U) \tag{22}$$

for all U. Then we have

$$\partial t \mathscr{H}(U) + \sum_k \partial_{x_k} \big[F_k(U) - \sum_{i,j} D_j \mathscr{H}(U) B_{ij}(U) \partial_{x_k} D_i \mathscr{H}(U) \big] + \sum_{i,j,k} B_{ij}(U) \big[\partial_{x_k} Di \mathscr{H}(U)\big] \big[\partial_{x_k} D_j \mathscr{H}(U)\big] = 0,$$

which in particular gives

$$\frac{d}{dt} \int_\Omega \mathscr{H}(U)\, dx \leq 0.$$

The asymptotic system (6) can be written in this form, with $U = (n, \rho, P)$ (we recall that $P = (n+\rho)u$) and

$$A(U) = \frac{1}{n+\rho} \begin{pmatrix} nP_1 & nP_2 & nP_3 \\ \rho P_1 & \rho P_2 & \rho P_3 \\ C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}, \qquad B(U) = \nu \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$C_{ij} = P_i P_j + (n + \rho^\gamma)(n+\rho)\delta_{ij}.$$

Note that we have

$$D\mathscr{H}(U) = \Big(-\frac{u^2}{2} + \log n + 1 - \frac{2}{3}\log(2\pi)\ ,\ -\frac{u^2}{2} + \frac{\gamma}{\gamma-1}\rho^{\gamma-1}\ ,\ u \Big)$$

The corner stone of the method is the following very general proposition:

Proposition 2.1 *Consider a system of conservation laws (21) and assume that there exists a smooth convex entropy $\mathscr{H}$ and a corresponding entropy* *flux F defined by (22). Then, for any be a smooth* **solution** U^* *of (21) and for any smooth* **function** V*, the following inequality holds:*

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \mathscr{H}(V|U^*)\,dx \\
&\quad + \int_\Omega B_{ij}(V)\,[\nabla D_i\mathscr{H}(V) - \nabla D_i\mathscr{H}(U^*)]\,[\nabla D_j\mathscr{H}(V) - \nabla D_j\mathscr{H}(U^*)]\;dx \\
&\qquad \le \frac{d}{dt}\int_\Omega \mathscr{H}(V)dx + \int_\Omega B_{ij}(V)\partial_{x_k} Di\mathscr{H}(V)\partial_{x_k} D_j\mathscr{H}(V)\,dx \\
&\qquad - \int_\Omega D\mathscr{H}(U^*)[\partial_t V + div_x A(V) - \partial_{x_k}\left(B_j(V)\partial_{x_k}\left(D_j\mathscr{H}(V)\right)\right)]\,dx \\
&\qquad - \int_\Omega \sum_{j,k} \partial_{x_k}[\partial_j\mathscr{H}(U^*)]A_{jk}(V|U^*)\,dx \\
&\qquad + \int_\Omega div\,(B(U^*)\nabla D\mathscr{H}(U^*))D\mathscr{H}(V|U^*)\,dx \\
&\qquad - \int_\Omega [B(V) - B(U^*)]\,\nabla D\mathscr{H}(U^*)\,[\nabla D\mathscr{H}(V) - \nabla D\mathscr{H}(U^*)]\;dx \quad (23)
\end{aligned}
$$

where

$$
\begin{aligned}
A(V|U) &= A(V) - A(U) - DA(U)\cdot(V-U) \\
D\mathscr{H}(V|U) &= D\mathscr{H}(V) - D\mathscr{H}(U) - D^2\mathscr{H}(U)\cdot(V-U)
\end{aligned}
$$

A similar proposition was first established by Dafermos [12] for general system of hyperbolic conservation laws, without viscosity (see also [6]):. This inequality has been extensively used ever since, and proved to be an important tools in stability/asymptotic analysis of system of conservation laws.

However, it is the first time, to our knowledge, that such an inequality is derived when a viscosity term arise in the equation. We believe that the main interest of this proposition is to show that as long as the viscosity term is of the form

$$\mathrm{div}\big[B(U)\nabla\left(D\mathscr{H}(U)\right)\big],$$

the natural relative entropy inequality (and its consequences) is preserved. Moreover, it is not very hard to see that a viscosity term of the form

$$\nabla\big[E(U)\mathrm{div}\left(D\mathscr{H}(U)\right)\big] = \sum_j \partial_{x_i}\big[E_{ij}\partial_{x_j}(D_j\mathscr{H}(U))\big],$$

could also be added in the system. In particular, the usual viscosity term for Navier-Stokes equation $\mathrm{div}(\nu\nabla u) + \nabla(\lambda\mathrm{div}u)$ can be used, but more general viscosity term are allowed (in particular with diffusion term involving the density).

Finally, we observe that in our model, the viscosity matrix B does not depends on U, so the last term in (23) vanishes.

Proof of Proposition 2.1. First, for any V, $U \in [C^1(\mathbb{R}^n)]^p$, we have

$$\begin{aligned}\partial_t \mathscr{H}(V|U) \;&=\; \partial_t \mathscr{H}(V) - \partial_t \mathscr{H}(U)\\ &\qquad -D^2\mathscr{H}(U)\partial_t U \cdot (V-U)\\ &\qquad -D\mathscr{H}(U)\partial_t V + D\mathscr{H}(U)\partial_t U.\end{aligned}$$

Next, we observe that, using (22), we have, for all $U \in \mathbb{R}^N$:

$$\begin{aligned}&-\sum_{i,j,k} D_{ij}\mathscr{H}(U)\partial_{x_k}(A_{jk}(U))(V_i-U_i)\\ &\qquad +\sum_{i,j,k}\partial_{x_k}\big[D_iF_k(U)(V_i-U_i)\big]\\ &\qquad\qquad -\sum_{i,j,k} D_j\mathscr{H}(U)\partial_{x_k}\big[D_iA_{jk}(U)(V_i-U_i)\big]=0\end{aligned}$$

Using this equality, a carefull computation yields:

$$\begin{aligned}&\partial_t\mathscr{H}(V|U) =\\ &\quad = \partial_t\mathscr{H}(V) - \partial_t\mathscr{H}(U)\\ &\qquad -\mathrm{div}(B(V)\nabla D\mathscr{H}(V))D\mathscr{H}(V) + \mathrm{div}(B(U)\nabla D\mathscr{H}(U))D\mathscr{H}(U)\\ &\qquad -D^2\mathscr{H}(U)[\partial_t U + \mathrm{div}_x A(U) - \mathrm{div}\,(B(U)\nabla\,(D\mathscr{H}(U)))](V-U)\\ &\qquad -D\mathscr{H}(U)[\partial_t V + \mathrm{div}_x A(V) - \mathrm{div}\,(B(V)\nabla\,(D\mathscr{H}(V)))]\\ &\qquad +D\mathscr{H}(U)[\partial_t U + \mathrm{div}_x A(U) - \mathrm{div}\,(B(U)\nabla\,(D\mathscr{H}(U)))]\\ &\qquad +\sum_{i,k}\partial_{x_k}[\partial_i F_k(U)(V_i-U_i)]\\ &\qquad +\sum_{j,k}\partial_j\mathscr{H}(U)\partial_{x_k}[A(V|U)]\\ &\qquad +\mathrm{div}\,[B(V)\nabla D\mathscr{H}(V) - B(U)\nabla D\mathscr{H}(U)]\,[D\mathscr{H}(V) - D\mathscr{H}(U)]\\ &\qquad +\mathrm{div}(B(U)\nabla D\mathscr{H}(U))D\mathscr{H}(V|U).\end{aligned}$$

Thus, if $U = U^*$ is a smooth solution of (21), we get

$$\begin{aligned}
&\partial_t \mathscr{H}(V|U^*) \\
&\quad = \partial_t \mathscr{H}(V) - \mathrm{div}(B(V)\nabla D\mathscr{H}(V))D\mathscr{H}(V) + \mathrm{div}_x F(U^*) \\
&\qquad - D\mathscr{H}(U^*) \cdot [\partial_t V + \mathrm{div}_x A(V) - \mathrm{div}\,(B(V)\nabla\,(D\mathscr{H}(V)))] \\
&\qquad + \sum_{i,k} \partial_{x_k}[\partial_i F_k(U^*)(V_i - U_i^*)] \\
&\qquad + \sum_{j,k} \partial_j \mathscr{H}(U^*)\partial_{x_k}[A(V|U^*)] \\
&\qquad + \mathrm{div}\,[B(V)(\nabla D\mathscr{H}(V) - \nabla D\mathscr{H}(U^*))]\,[D\mathscr{H}(V) - D\mathscr{H}(U^*)] \\
&\qquad + \mathrm{div}\,[(B(V) - B(U^*))\nabla D\mathscr{H}(U^*)]\,[D\mathscr{H}(V) - D\mathscr{H}(U^*)] \\
&\qquad + \mathrm{div}(B(U^*)\nabla D\mathscr{H}(U^*))D\mathscr{H}(V|U^*).
\end{aligned}$$

Integrating with respect to x (using the boundary condition $u = 0$ on $\partial\Omega$ when Ω is a bounded subset of $\mathbb{R}^N$), it is relatively easy to deduce (23). □

We can now write the main proposition of this section:

Proposition 2.2 *Let $(f^\varepsilon, \rho^\varepsilon, u^\varepsilon)$ be a weak solutions of (3)-(5) with initial data satisfying (15)-(19) and verifying the entropy decay inequality (13), and let $U^* = (n^*, \rho^*, (n^* + \rho^*)u^*)$ be a smooth solution of (6). Denote by U^ε the macroscopic quantities corresponding to $(f^\varepsilon, \rho^\varepsilon, u^\varepsilon)$:*

$$U^\varepsilon = (n^\varepsilon, \rho^\varepsilon, (n^\varepsilon + \rho^\varepsilon)u^\varepsilon) \quad \text{with} \quad n^\varepsilon = \int f^\varepsilon \, dv.$$

Then U^ε satisfies the following inequality:

$$\begin{aligned}
&\int_\Omega \mathscr{H}(U^\varepsilon|U^*)(t)\, dx + \nu \int_0^t \int_\Omega |\nabla(u^\varepsilon - u^*)|^2 \, dx\, ds \\
&\leq - \int_0^t \int_\Omega \sum_{j,k} \partial_{x_k}[\partial_j \mathscr{H}(U^*)]A_{jk}(U^\varepsilon|U^*)\, dx\, ds \\
&\quad + \int_0^t \int_\Omega div(B\nabla D\mathscr{H}(U^*))D\mathscr{H}(U^\varepsilon|U^*)\, dx\, ds \\
&\quad - \int_0^t \int_\Omega D\mathscr{H}(U^*)\,[\partial_t U^\varepsilon + div_x(A(U^\varepsilon)) - div_x(B\nabla D\mathscr{H}(U^\varepsilon))]\, dx\, ds
\end{aligned}$$

(where the last integral has to be understood in the distributional sense).

Proof of Proposition 2.2. First, we note that since U^* is a smooth solution of (6) and using the expression of $B(U)$, Inequality (23) gives:

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \mathscr{H}(U^\varepsilon|U^*)\,dx + \nu\int_\Omega [\nabla u^\varepsilon - \nabla u^*]^2 \; dx \\
&\quad = \frac{d}{dt}\int_\Omega \mathscr{H}(U^\varepsilon)\,dx + \nu\int_\Omega |\nabla u^\varepsilon|\,dx \\
&\quad - \int_\Omega \sum_{j,k} \partial_{x_k}[\partial_j \mathscr{H}(U^*)] A_{jk}(U^\varepsilon|U^*)\,dx \\
&\quad - \int_\Omega D\mathscr{H}(U^*)(\partial_t U^\varepsilon + \mathrm{div}_x(A(U^\varepsilon)) - \mathrm{div}(B\nabla D\mathscr{H}(U^\varepsilon)))\,dx. \\
&\quad + \int_\Omega \mathrm{div}\,(B\nabla D\mathscr{H}(U^*))\cdot D\mathscr{H}(U^\varepsilon|U^*)\,dx
\end{aligned}$$

Integrating this equality with respect to t, we get:

$$\begin{aligned}
&\int \mathscr{H}(U^\varepsilon|U^*)(t)\,dx + \nu\int_0^t\int_\Omega [\nabla u^\varepsilon - \nabla u^*]^2 \; dx\,ds \\
&\le \int_\Omega \mathscr{H}(U^\varepsilon|U^*)(0)\,dx + \int_\Omega [\mathscr{H}(U^\varepsilon)(t) - \mathscr{H}(U_0)] \; dx \\
&\quad - \int_0^t\int_\Omega \sum_{j,k} \partial_{x_k}[\partial_j \mathscr{H}(U^*)] A_{jk}(U^\varepsilon|U^*)\,dx\,ds \\
&\quad - \int_0^t\int_\Omega D\mathscr{H}(U^*)(\partial_t U^\varepsilon + \mathrm{div}_x(A(U^\varepsilon)) - \mathrm{div}_x(B\nabla D\mathscr{H}(U^\varepsilon)))\,dx\,ds \\
&\quad + \int_0^t\int_\Omega \mathrm{div}\,(B\nabla D\mathscr{H}(U^*))\cdot D\mathscr{H}(U^\varepsilon|U^*)\,dx\,ds.
\end{aligned}$$

Next, we recall that $\mathscr{E}$ denotes the entropy associated to the initial system (3-5) (see Proposition 1.1) and we write:

$$\begin{aligned}
\mathscr{H}(U^\varepsilon)(t) - \mathscr{H}(U_0) \;=\;& \mathscr{H}(U^\varepsilon)(t) - \mathscr{E}(f^\varepsilon,\rho^\varepsilon,u^\varepsilon)(t) \\
&+ \mathscr{E}(f^\varepsilon,\rho^\varepsilon,u^\varepsilon)(t) - \mathscr{E}(f_0,\rho_0,u_0) \\
&+ \mathscr{E}(f_0,\rho_0,u_0) - \mathscr{H}(U_0).
\end{aligned}$$

The well-preparedness of the initial data (17) yields

$$\mathscr{E}(f_0,\rho_0,u_0) - \mathscr{H}(U_0) = 0,$$

and (14) gives:

$$\mathscr{H}(U^\varepsilon)(t) - \mathscr{E}(f^\varepsilon,\rho^\varepsilon,u^\varepsilon)(t) \le 0.$$

Moreover, the entropy inequality (13) implies that

$$\int_\Omega \mathscr{E}(f^\varepsilon,\rho^\varepsilon,u^\varepsilon)(t)\,dx - \int_\Omega \mathscr{E}(f^\varepsilon,\rho^\varepsilon,u^\varepsilon)(0)\,dx \le 0,$$

and so

$$\int_\Omega \left[\mathscr{H}(U^\varepsilon)(t) - \mathscr{H}(U_0)\right]\,dx \le 0.$$

Finally, hypothesis (17) yields

$$\mathscr{H}(U^\varepsilon|U^*)(0) = 0.$$

We deduce

$$\begin{aligned}
&\int \mathscr{H}(U^\varepsilon|U^*)(t)\,dx + \nu\int_0^t\int_\Omega \left[\nabla u^\varepsilon - \nabla u^*\right]^2\,dx\,ds\\
&\le -\int_0^t\int_\Omega \sum_{j,k} \partial_{x_k}[\partial_j\mathscr{H}(U^*)]A_{jk}(U^\varepsilon|U^*)\,dx\,ds\\
&\quad -\int_0^t\int_\Omega D\mathscr{H}(U^*)(\partial_t U^\varepsilon + \mathrm{div}_x(A(U^\varepsilon)) - \mathrm{div}(B\nabla D\mathscr{H}(U^\varepsilon)))\,dx\,ds\\
&\quad +\int_0^t\int_\Omega \mathrm{div}\,(B(U^*)\nabla D\mathscr{H}(U^*))\cdot D\mathscr{H}(U^\varepsilon|U^*)\,dx\,ds.
\end{aligned}$$

which gives Proposition 2.2. □

It is readily seen that Inequality (20) and Theorem 1.2 follow from Proposition (2.2) if we can prove that the following facts hold (under the hypotheses of Theorem 1.2):

(i) the relative flux is controled by the relative entropy:

$$\int_\Omega |A_{j,k}(U|U^*)|\,dx \le C\int_\Omega \mathscr{H}(U|U^*)\,dx \qquad \text{for all function } U,$$

where the constant C depends on $\lambda = \inf(n^*+\rho^*)$ and $\Lambda = \sup(n^*+\rho^*)$ (see Lemma 3.1).

(ii) the term due to the viscosity can be controled by the relative entropy and the viscosity:

$$\begin{aligned}
&\int_0^t\int_\Omega \mathrm{div}(B_j(U^*)\nabla\partial_j\mathscr{H}(U^*))D\mathscr{H}(U|U^*)\,dx\,ds\\
&\qquad \le C\left\|\frac{\Delta u^*}{n^*+\rho^*}\right\|_{L^\infty}\int_\Omega \mathscr{H}(U|U^*)\,dx + \frac{\nu}{2}\int_\Omega |\nabla(u-u^*)|^2\,dx\,ds
\end{aligned}$$

for all $U = (n,\rho,(n+\rho)u)$ (see Lemma 3.3).

(iii) the asymptotic system (6) is consistent with the original system, in the sense that

$$\left\| \int_0^t \int_\Omega \Phi \left[\partial_t U^\varepsilon + \mathrm{div}(A(U^\varepsilon)) - \mathrm{div}(B\nabla \mathscr{H}(U^\varepsilon)) \right] dx\, dt \right\|_{L^1(0,T)} \leq C(T)\sqrt{\varepsilon}$$

for any smooth function $\Phi(x,t)$ (see Proposition 4.1).

Note that the first two points are concerned with the stability of the asymptotic system and will result from an algebraic computation (see Section 3), while the last point says that (6) is indeed the correct asymptotic system for (3)-(5) and relies on the dissipative properties of the entropy (see Section 4).

When those three conditions are fullfilled, we can deduce (20). More precisely, we have (using Proposition 2.2, and Lemma Lemma 3.1, 3.3 and Proposition 4.1):

Proposition 2.3 *Let $(f^\varepsilon, \rho^\varepsilon, u^\varepsilon)$ be a weak solutions of (3)-(5) with initial data satisfying (15)-(19), and verifying the entropy decay inequality (13). Let $U^* = (n^*, \rho^*, (n^* + \rho^*)u^*)$ be a smooth solution of (6) such that*

$$\lambda \leq \rho^* + n^* \leq \Lambda.$$

Then, there exists a constant C depending on λ^{-1}, Λ, $||\nabla u||_{L^\infty}$, $||\partial_t u||_{L^\infty}$, $||\nabla u^2||_{L^\infty}$, $||\Delta u||_{L^\infty}$ and $||\nabla \log n||_{L^\infty}$ such that

$$\int_\Omega \mathscr{H}(U^\varepsilon|U^*)(t)\, dx + \frac{\nu}{2}\int_0^t \int_\Omega |\nabla(u^\varepsilon - u^*)|^2\, dx\, ds \leq C\int_0^t \int_\Omega \mathscr{H}(U^\varepsilon|U^*)\, dx\, ds + R(t)$$

with $||R||_{L^1(0,T^*)} \leq C\sqrt{\varepsilon}$.

A Gronwall argument then leads to

$$\int_0^T \int_\Omega \mathscr{H}(U^\varepsilon|U)(t)\, dx\, dt + \frac{\nu}{2}\int_0^T \int_\Omega |\nabla(u^\varepsilon - u^*)|^2\, dx\, dt \leq C(T)\sqrt{\varepsilon},$$

which conclude the proof of Theorem 1.2.

3 Relative flux

In this section, we study the structure of the asymptotic system (6). We denote respectively by $A(\cdot|\cdot)$ and $\mathscr{H}(\cdot|\cdot)$ the relative flux and the relative entropy associated with the system (6) (see Section 1.4 for details). We establish the following lemma:

Lemma 3.1 *Assume $1 < \gamma < 2$, then for any positive constants λ and Λ, there exists $C(\lambda^{-1}, \Lambda)$ such that*

$$\int_\Omega |A(U|U^*)|\, dx \le C \int_\Omega \mathscr{H}(U|U^*)\, dx$$

for any $U = (n, \rho, (n+\rho)u)$, $U^* = (n^*, \rho^*, (n^*+\rho^*)u^*)$ smooth functions satisfying

$$\lambda \le \rho^* + n^* \le \Lambda.$$

Proof of Lemma 3.1:
First, we check that the relative flux is given by

$$A(U|U^*) = \begin{pmatrix} (\alpha-\alpha^*)(\rho+n)(u_i-u_i^*) \\ (\beta-\beta^*)(\rho+n)(u_i-u_i^*) \\ (\rho+n)(u_i-u_i^*)(u_j-u_j^*) + p_1(\rho|\rho^*)\delta_{ij} \end{pmatrix}$$

with $\alpha = \frac{n}{n+\rho}$, $\alpha^* = \frac{n^*}{n^*+\rho^*}$, $\beta = \frac{\rho}{n+\rho}$ and $\beta^* = \frac{\rho^*}{n^*+\rho^*}$, and we recall that the relative entropy satisfies

$$\mathscr{H}(U|U^*) = (n+\rho)\frac{|u-u^*|^2}{2} + \frac{1}{\gamma-1}p_1(\rho|\rho^*) + p_2(n|n^*).$$

with

$$\begin{aligned} p_1(\rho|\rho^*) &= \rho^\gamma - \rho^{*\gamma} - \gamma\rho^{*\gamma-1}(\rho-\rho^*) = \frac{\gamma}{2}\xi_1^{\gamma-2}(\rho-\rho^*)^2 \\ p_2(n|n^*) &= n\log n - n^*\log n^* - (\log n^* + 1)(n-n^*) = \frac{1}{2}\frac{1}{\xi_2}(n-n^*)^2 \end{aligned}$$

where ξ_1 between ρ and ρ^* and ξ_2 between n and n^*. Those computations are very closed to the ones developed in [6], and we refer the reader to [6] for more details.

The L^1 norm of $A(U|U^*)$ involves the following terms:

$$\int |(\alpha-\alpha^*)(\rho+n)(u_i-u_i^*)|\, dx\,, \qquad \int |(\beta-\beta^*)(\rho+n)(u_i-u_i^*)|\, dx \quad (24)$$

and

$$\int (\rho+n)(u_i-u_i^*)(u_j-u_j^*)\,dx\,, \qquad \int p_1(\rho|\rho^*)\,dx, \tag{25}$$

and it is readily seen that the last two terms (25) are bounded above by $\int \mathscr{H}(U|U^*)\,dx$. Moreover, the terms in (24) are alike, so we only need to treat in detail one of them (the first one).

We note that Cauchy-Schwartz's inequality gives

$$\begin{aligned}\int &|(\alpha-\alpha^*)(\rho+n)(u_i-u_i^*)|\,dx\\ &\leq \left(\int(\alpha-\alpha^*)^2(\rho+n)\,dx\right)^{1/2}\left(\int(\rho+n)|(u_i-u_i^*)|^2\,dx\right)^{1/2},\end{aligned}$$

where the second term is bounded above by $\left(\int \mathscr{H}(U|U^*)dx\right)^{1/2}$. So we are left with the task of showing that the quantity

$$I=(\alpha-\alpha^*)^2(n+\rho)$$

is bounded above by $\mathscr{H}(U|U^*)$.

To that purpose, we need to distinguish the case where $n+\rho$ is larger than Λ and the case where $n+\rho$ is smaller than Λ. In each case, we will use one of the following expression for $\alpha-\alpha^*$:

$$\alpha-\alpha^*=\frac{\rho(n-n^*)+n(\rho^*-\rho)}{(n+\rho)(n^*+\rho^*)} \tag{26}$$

or

$$\alpha-\alpha^*=\frac{\rho^*(n-n^*)+n^*(\rho^*-\rho)}{(n+\rho)(n^*+\rho^*)} \tag{27}$$

1 - When $n+\rho<\Lambda$, using the fact that $\rho<\Lambda$ and $\rho^*\leq\Lambda$, we get $\xi_1<\Lambda$. Since $\gamma-2<0$ we deduce

$$p_1(\rho|\rho^*)\geq C(\Lambda)(\rho-\rho^*)^2.$$

Similarly, using the fact that $n<\Lambda$ and $n^*\leq\Lambda$, we have $\xi_2<\Lambda$ which yields

$$p_2(n|n^*)\geq C(\Lambda)(n-n^*)^2.$$

Finally, using (26) together with the fact that $n/(n+\rho)\leq 1$ and $\rho/(n+\rho)\leq 1$, we get

$$\begin{aligned}I &\leq (n+\rho)\left(\frac{|n-n^*|}{(n^*+\rho^*)}+\frac{|\rho^*-\rho|}{(n^*+\rho^*)}\right)^2\\ &\leq \frac{\Lambda}{\lambda}((n-n^*)^2+(\rho^*-\rho)^2)\end{aligned}$$

and therefore

$$I \leq C(\Lambda, \lambda)\big[p_1(\rho|\rho^*) + p_2(n|n^*)\big]$$

2 - When $n + \rho > \Lambda$, we first note that using (27) and the fact that $n^*/(n^* + \rho^*) \leq 1$, we have

$$I \leq \frac{(|\rho^* - \rho| + |n - n^*|)^2}{(n + \rho)} \leq \frac{(n - n^*)^2}{n + \rho} + \frac{(\rho - \rho^*)^2}{n + \rho}$$

In order to control the first term, we again distinguish two situations:

- When $n \geq \Lambda$, then $n^* < n$, and so $\xi_2 < n$. Therefore

$$p_2(n|n^*) > \frac{1}{n}(n - n^*)^2 > \frac{(n - n^*)^2}{n + \rho}.$$

- When $n < \Lambda$, then $1/\xi_2 > 1/\max(n, n^*) > 1/\Lambda$, and since $\frac{(n-n^*)^2}{\rho+n} < \frac{1}{\Lambda}(n - n^*)^2$, we get

$$p_2(n|n^*) > C(\Lambda)(n - n^*)^2 > C(\Lambda)\Lambda\frac{(n - n^*)^2}{\rho + n}$$

  where we used the fact that $n + \rho \geq \Lambda$.

In either case, we have

$$\frac{(n - n^*)^2}{n + \rho} \leq C(\Lambda)p_2(n|n^*).$$

Finally, we proceed similarly to show that the term $\frac{(\rho-\rho^*)^2}{n+\rho}$ is controled by $p_1(\rho|\rho^*)$:

- When $\rho > \Lambda$, then $\xi_1 \leq \rho$ and so $\xi_1^{\gamma-2} > \rho^{\gamma-2} > C(\Lambda)/\rho$ (using the fact that $\gamma > 1$). Since

$$\frac{(\rho^* - \rho)^2}{n + \rho} \leq \frac{(\rho - \rho^*)^2}{\rho},$$

  we deduce

$$\frac{(\rho^* - \rho)^2}{n + \rho} < C\xi_1^{\gamma-2}(\rho^* - \rho)^2 \leq p_1(\rho|\rho^*).$$

– When $\rho < \Lambda$, then $\xi_1^{\gamma-2} > (\max(n, n^*))^{\gamma-2} > \Lambda^{\gamma-2}$, and since $\frac{(\rho-\rho^*)^2}{\rho+n} < \frac{1}{\Lambda}(\rho - \rho^*)^2$ (we recall that we still have $n + \rho > \Lambda$), we get

$$p_1(n|n^*) > C(\Lambda)(\rho - \rho^*)^2 > C(\Lambda)\frac{(\rho - \rho^*)^2}{\rho + n}.$$

The proof of Lemma 3.1 is now complete. □

Note that we actually proved the following fact:

Lemma 3.2 *Assume $1 < \gamma < 2$, then for any positive constants λ and Λ, there exists $C(\lambda, \Lambda)$ such that for any non-negative functions (n, ρ) and (n^*, ρ^*) satisfying*

$$\lambda \leq \rho^* + n^* \leq \Lambda,$$

we have

$$\begin{aligned} p_1(\rho|\rho^*) &\geq C \min\left(\frac{1}{n+\rho}, \frac{1}{\Lambda}\right) |\rho - \rho^*|^2 \\ p_2(n|n^*) &\geq C \min\left(\frac{1}{n+\rho}, \frac{1}{\Lambda}\right) |n - n^*|^2 \end{aligned}$$

We deduce the following result:

Lemma 3.3 *Assume $1 < \gamma < 2$, then for any positive constants λ and Λ, there exists $C(\lambda, \Lambda)$ such that for every $t > 0$:*

$$\begin{aligned} &\int_0^t \int_\Omega div(B\nabla D\mathscr{H}(U^*))D\mathscr{H}(U|U^*)\,dx\,ds \\ &\quad \leq C\left\|\frac{\Delta u^*}{n^* + \rho^*}\right\|_{L^\infty} \int_0^t \int_\Omega \mathscr{H}(U|U^*)\,dx\,ds + \frac{\nu}{2}\int_0^t \int_\Omega |\nabla(u - u^*)|^2\,dx\,ds \end{aligned}$$

for any $U = (n, \rho, (n + \rho)u)$, $U^* = (n^*, \rho^*, (n^* + \rho^*)u^*)$ smooth functions satisfying

$$\lambda \leq \rho^* + n^* \leq \Lambda,$$

(and $u = u^* = 0$ on $\partial\Omega$ when Ω is a bounded domain).

Proof. We first check that we have

$$BD\mathscr{H}(U|U^*) = (BD\mathscr{H})(U|U^*)$$

where $BD\mathscr{H}(U) = \frac{P}{n+\rho}$. Thus we have

$$BD\mathscr{H}(U|U^*) = \frac{1}{n^* + \rho^*}(n - n^* + \rho - \rho^*)(u^* - u).$$

It follows that

$$\mathrm{div}(B\nabla D\mathscr{H}(U^*))D\mathscr{H}(U|U^*) = \frac{\Delta u^*}{n^*+\rho^*}\,(n-n^*+\rho-\rho^*)\,(u^*-u)$$

and so

$$\begin{aligned}
&\int_0^t\int_\Omega \mathrm{div}(B\nabla D\mathscr{H}(U^*))D\mathscr{H}(U|U^*)\,dx\,ds\\
&\leq \left\|\frac{\Delta u^*}{n^*+\rho^*}\right\|_{L^\infty}\left(\int_0^t\int_\Omega \min\left(\frac{1}{n+\rho},\frac{1}{\Lambda}\right)\left[|\rho-\rho^*|^2+|n-n^*|^2\right]dx\,ds\right)^{1/2}\\
&\qquad\qquad\left(\int_0^t\int_\Omega \max(\rho+n,\Lambda)(u-u^*)^2\,dx\,ds\right)^{1/2}\\
&\leq \left\|\frac{\Delta u^*}{n^*+\rho^*}\right\|_{L^\infty}\left(\int_0^t\int_\Omega p_1(n|n^*)+p_2(\rho|\rho^*)\,dx\,ds\right)^{1/2}\\
&\qquad\qquad\left(\int_0^t\int_\Omega \max(\rho+n,\Lambda)(u-u^*)^2\,dx\,ds\right)^{1/2}.
\end{aligned}$$

Moreover, we can write

$$\begin{aligned}
&\int_0^t\int_\Omega \max(\rho+n,\Lambda)(u-u^*)^2\,dx\,ds\\
&\qquad\leq \int_0^t\int_\Omega (\rho+n)(u-u^*)^2\,dx\,ds+\Lambda\int_0^t\int_\Omega (u-u^*)^2\,dx\,ds\\
&\qquad\leq \int_0^t\int_\Omega \mathscr{H}(U|U^*)\,dx\,ds+C\int_0^t\int_\Omega \nabla(u-u^*)^2\,dx\,ds
\end{aligned}$$

thanks to Poincaré inequality (and the fact that $u=u^*$ on $\partial\Omega$). The Lemma follows easily using Young's inequality. □

4 Control of the kinetic approximation

In this section, we prove the consistency of the asymptotic system with the kinetic model. More precisely, we prove:

Proposition 4.1 *Let U_ε be a weak solution of (3)-(5) satisfying the entropy inequality (13), and let $U=(n,\rho,(n+\rho)u)$ be a smooth function. Then, there exists a constant C depending only on $||\nabla u||_{L^\infty}$, $||\partial_t u||_{L^\infty}$, $||\nabla u^2||_{L^\infty}$ and $||\nabla\log n||_{L^\infty}$ such that*

$$\left\|\int_0^t\int_\Omega D\mathscr{H}(U)\left[\partial_t U_\varepsilon+div(A(U_\varepsilon))-div(B\nabla(D\mathscr{H}(U_\varepsilon)))\right]dxds\right\|_{L^1(0,T)}\leq C\sqrt{\varepsilon}.$$

The proof relies on inequality (13). However in order to controle the dissipation term, we first need to show that the negative part of the entropy can be controled by its positive part. More precisely, we need to show:

Lemma 4.1 *There exists a constant C such that for every $T > 0$ and ε:*

$$\int_\Omega \int_{\mathbb{R}^N} \frac{|v|^2}{4} f_\varepsilon + |f_\varepsilon \log f_\varepsilon| \, dv \, dx + \frac{1}{\varepsilon} \int_0^T D(f_\varepsilon) \, dt \leq C.$$

Proof of Lemma 4.1. This is a fairly classical result, and its proof can be found in particular in [15]. We recall it here for the sake of completness: We write $|s \ln s| = s \ln s - 2s \ln s \chi_{0 \leq s \leq 1}$, and for $\omega > 0$, we note that

$$-s \ln s \chi_{0 \leq s \leq 1} \quad \leq \quad s\omega + C\sqrt{s}\chi_{e^{-w} \geq s} \leq s\omega + Ce^{-\omega/2}$$

Using these relations with $s = f_\varepsilon$ and $\omega = v^2/8$, we deduce:

$$\begin{aligned} \iint |f_\varepsilon \ln(f_\varepsilon)| \, dx \, dv \leq & \iint f_\varepsilon \ln(f_\varepsilon) \, dx \, dv \\ & + \frac{1}{4} \iint v^2 f_\varepsilon \, dx \, dv + 2C \iint e^{-v^2/16} \, dx \, dv. \end{aligned}$$

Since Ω is bounded, it follows that

$$\begin{aligned} & \iint f_\varepsilon (1 + |\ln(f_\varepsilon)|) \, dx \, dv + \frac{1}{4} \iint v^2 f_\varepsilon \, dv \, dx \\ & \quad + \int \left[\rho_\varepsilon \frac{u_\varepsilon^2}{2} + \frac{1}{\gamma - 1} \rho^\gamma \right] dx + \frac{1}{\varepsilon} \int_0^t D(f_\varepsilon) \, ds \\ & \leq \mathscr{E}(f_\varepsilon, \rho_\varepsilon, u_\varepsilon)(t) + C \\ & \leq \mathscr{E}(f_\varepsilon, \rho_\varepsilon, u_\varepsilon)(0) + C. \end{aligned}$$

□

Proof of Proposition 4.1. Formally, we have, integrating (3) with respect to v:

$$\partial_t n_\varepsilon = -\mathrm{div}_x \int v f_\varepsilon \, dv$$

and thus

$$\partial_t n_\varepsilon + \mathrm{div}_x (n_\varepsilon u_\varepsilon) \quad = \quad \mathrm{div}_x \int (u_\varepsilon - v) f_\varepsilon \, dv.$$

Moreover, multiplying (3) by v and integrating with respect to v, we get (still formally):

$$\partial_t \int v f_\varepsilon \, dv = -\mathrm{div}_x \int v \otimes v f_\varepsilon \, dv + \frac{1}{\varepsilon} \int (u_\varepsilon - v) f_\varepsilon \, dv$$

and thus

$$\begin{aligned}
&\partial_t((n_\varepsilon + \rho_\varepsilon)u_\varepsilon) + \mathrm{div}_x(u_\varepsilon \otimes u_\varepsilon(n_\varepsilon + \rho_\varepsilon)) + \nabla_x(\rho_\varepsilon^\gamma + n_\varepsilon) - \nu \Delta u_\varepsilon \\
&\qquad = \partial_t \int (u_\varepsilon - v) f_\varepsilon \, dv + \mathrm{div}_x \left(\int (u_\varepsilon \otimes u_\varepsilon - v \otimes v + I) f_\varepsilon \, dv \right)
\end{aligned}$$

Hence, we can write

$$\begin{aligned}
&\int_0^t \int_\Omega D\mathscr{H}(U) \big[\partial_t U^\varepsilon + \mathrm{div}_x(A(U^\varepsilon)) - \mathrm{div}_x(B \nabla D\mathscr{H}(U^\varepsilon)) \big] \, dx \, ds \\
&= - \int_0^t \int_\Omega [\nabla_x D_1 \mathscr{H}(U) + \partial_t D_3 \mathscr{H}(U)] \cdot \left(\int (u_\varepsilon - v) f_\varepsilon \, dv \right) dx \, ds \\
&\quad + \int_\Omega D_3 \mathscr{H}(U(t)) \left(\int (u_\varepsilon - v) f_\varepsilon(t) \, dv \right) dx \\
&\quad - \int_0^t \int_\Omega \left(\int (u_\varepsilon \otimes u_\varepsilon - v \otimes v + I) f_\varepsilon \, dv \right) : \nabla_x D_3 \mathscr{H}(U) \, dx \, ds
\end{aligned}$$

where we recall that

$$D_1 \mathscr{H}(U) = -\frac{u^2}{2} + \log n + 1 - \frac{2}{3} \log(2\pi), \qquad \text{and} \qquad D_3 \mathscr{H}(U) = u.$$

This equality can be made rigorous by taking $D_1\mathscr{H}(U) + v \cdot D_3\mathscr{H}(U)$ as a test function in (10) (note that this function satisfies the compatibilty condition (11) along $\partial\Omega$ in the case of reflection boundary conditions) and $D_3\mathscr{H}(U)$ as a test function in (5).

We deduce that there exists a constant C depending only on $||\nabla u||_{L^\infty}$, $||\partial_t u||_{L^\infty}$, $||\nabla u^2||_{L^\infty}$ and $||\nabla \log n||_{L^\infty}$ such that

$$\begin{aligned}
&\int_0^t \int_\Omega D\mathscr{H}(U)(\partial_t U^\varepsilon + \mathrm{div}_x(A(U^\varepsilon)) - \mathrm{div}_x(B \nabla D\mathscr{H}(U^\varepsilon))) \, dx \, ds \\
&\leq C \int_0^t \int_\Omega \left| \int_{\mathbb{R}^N} (u_\varepsilon - v) f_\varepsilon \, dv \right| dx \, ds && (28) \\
&\quad + C \int_\Omega \left| \int_{\mathbb{R}^N} (u_\varepsilon - v) f_\varepsilon(t) \, dv \right| dx \\
&\quad + C \int_0^t \int_\Omega \left| \int_{\mathbb{R}^N} (u_\varepsilon \otimes u_\varepsilon - v \otimes v + I) f_\varepsilon \, dv \right| dx \, ds. && (29)
\end{aligned}$$

We are thus left with the task of showing that

$$\int_\Omega \left| \int_{\mathbb{R}^N} (u_\varepsilon - v) f_\varepsilon \, dv \right| \, dx \quad \text{and} \quad \int_\Omega \left| \int_{\mathbb{R}^N} (u_\varepsilon \otimes u_\varepsilon - v \otimes v + I) f_\varepsilon \, dv \right| \, dx$$

can be controled by the dissipation.

1 - We have

$$\begin{aligned} \left| \int (u_\varepsilon - v) f_\varepsilon \, dv \right| &= \left| \int (u_\varepsilon - v) f_\varepsilon - \nabla_v f_\varepsilon \, dv \right| \\ &\leq \left(\int f_\varepsilon dv \right)^{1/2} \left(\int |(u_\varepsilon - v) f_\varepsilon - \nabla_v f_\varepsilon|^2 \frac{1}{f_\varepsilon} dv \right)^{1/2} \end{aligned}$$

and therefore

$$\int_0^t \int_\Omega \left| \int (u_\varepsilon - v) f_\varepsilon \, dv \right| \, dx \, ds \leq C \sqrt{\int_0^t D(f_\varepsilon) \, ds} \leq C\sqrt{\varepsilon},$$

which gives a bound for the first two terms in (29) (note that only the $L^1(0,t)$-norm of the second one is bounded).

2 - Next, we write

$$\begin{aligned} &\int (u_\varepsilon \otimes u_\varepsilon - v \otimes v + I) f_\varepsilon \, dv \\ &= \int \left[u_\varepsilon \otimes (u_\varepsilon - v) + (u_\varepsilon - v) \otimes v + I \right] f_\varepsilon \, dv \\ &= \int u_\varepsilon \sqrt{f_\varepsilon} \otimes \left[(u_\varepsilon - v)\sqrt{f}_\varepsilon - 2\nabla_v \sqrt{f}_\varepsilon \right] + u_\varepsilon \otimes 2\sqrt{f}_\varepsilon \nabla_v \sqrt{f}_\varepsilon \, dv \\ &\quad + \int \left[(u_\varepsilon - v)\sqrt{f}_\varepsilon - 2\nabla_v \sqrt{f}_\varepsilon \right] \otimes v\sqrt{f}_\varepsilon + 2\sqrt{f}_\varepsilon \nabla_v \sqrt{f}_\varepsilon \otimes v + I f_\varepsilon \, dv \\ &= \int u_\varepsilon \sqrt{f_\varepsilon} \otimes \left[(u_\varepsilon - v)\sqrt{f}_\varepsilon - 2\nabla_v \sqrt{f}_\varepsilon \right] + u_\varepsilon \otimes \nabla_v f_\varepsilon \, dv \\ &\quad + \int \left[(u_\varepsilon - v)\sqrt{f}_\varepsilon - 2\nabla_v \sqrt{f}_\varepsilon \right] \otimes v\sqrt{f}_\varepsilon + \nabla_v f_\varepsilon \otimes v + I f_\varepsilon \, dv \\ &= \int u_\varepsilon \sqrt{f_\varepsilon} \otimes \left[(u_\varepsilon - v)\sqrt{f}_\varepsilon - 2\nabla_v \sqrt{f}_\varepsilon \right] \, dv \\ &\quad + \int \left[(u_\varepsilon - v)\sqrt{f}_\varepsilon - 2\nabla_v \sqrt{f}_\varepsilon \right] \otimes v\sqrt{f}_\varepsilon \, dv \end{aligned}$$

and so

$$\int_0^t \int_\Omega \left| \int (u_\varepsilon \otimes u_\varepsilon - v \otimes v + I) f_\varepsilon \, dv \right| dx \, ds$$

$$\leq \left(\int_0^t \int_\Omega \int (|u_\varepsilon|^2 + |v|^2) f_\varepsilon \, dv \, dx \, ds \right)^{1/2} \sqrt{\int_0^t D(f_\varepsilon) \, ds}$$

So it only remains to see that $\int\int (|u_\varepsilon|^2 + |v|^2) f_\varepsilon \, dv \, dx$ is bounded uniformly by a constant. Using the entropy inequality, we already know that $\int\int |v|^2 f_\varepsilon \, dv \, dx$ is bounded, so it is enough to check that $\int\int (u_\varepsilon - v)^2 f_\varepsilon \, dv \, dx$ is bounded. To that purpose, we write

$$\begin{aligned} \int (u_\varepsilon - v)^2 f_\varepsilon \, dv \, dx &= \int\int (u_\varepsilon - v)\sqrt{f_\varepsilon} \left[(u_\varepsilon - v)\sqrt{f}_\varepsilon - 2\nabla_v \sqrt{f}_\varepsilon \right] dv \, dx \\ &\quad + \int\int (u_\varepsilon - v) \nabla_v f_\varepsilon \, dv \, dx \\ &\leq \left(\int (u_\varepsilon - v)^2 f_\varepsilon \, dv \, dx \right)^{1/2} \sqrt{D(f_\varepsilon)} + \int f_\varepsilon \, dv \, dx \end{aligned}$$

which gives

$$\int_0^t \int (u_\varepsilon - v)^2 f_\varepsilon \, dv \, dx \, ds \leq \int_0^t D(f_\varepsilon) \, ds + 2 \int_0^t \int f_\varepsilon \, dv \, dx \, ds.$$

and yields the result

Putting all the pieces together, (29) gives

$$\left\| \int_0^t \int D\mathscr{H}(U) \left[\partial_t U^\varepsilon + \mathrm{div}_x(A(U^\varepsilon)) - \mathrm{div}_x(B \nabla D\mathscr{H}(U^\varepsilon)) \right] dx \, ds \right\|_{L^1(0,T)}$$

$$\leq (1+T) \left(\sup_x \int_0^T |\nabla_x D_1 \mathscr{H}(U)| + |\partial_t D_3 \mathscr{H}(U)| + |\nabla_x D_3 \mathscr{H}(U)| ds \right) \sqrt{\varepsilon}$$

which is the desired result. □

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