Electrified thin films: Global existence of non-negative solutions

C. Imbert[*] and A. Mellet[†]

February 14, 2011

Abstract

We consider an equation modeling the evolution of a viscous liquid thin film wetting a horizontal solid substrate destabilized by an electric field normal to the substrate. The effects of the electric field are modeled by a lower order non-local term. We introduce the good functional analysis framework to study this equation on a bounded domain and prove the existence of weak solutions defined globally in time for general initial data (with finite energy).

Keywords: Higher order equation, Non-local equation, Thin film equation, Non-negative solutions

MSC: 35G25, 35K25, 35A01, 35B09

1 Introduction

In [13], Tseluiko and Papageorgiou study the following equation (see also [11]):

$$u_t + \left(u^3(cu_{xx} - \alpha u - \lambda I(u))_x\right)_x = 0 \qquad x \in [0, L], t > 0 \tag{1}$$

with periodic boundary conditions. This equation models the evolution of a liquid thin film wetting a horizontal solid substrate which is subject to a gravity field and an electric field normal to the substrate. The term $\lambda I(u)$ models the effects of the electric field on the thin film. The operator $I(u)$ is a nonlocal elliptic operator of order 1 which will be defined precisely later on (for now, we can think of it as being the half-Laplace operator: $I(u) = -(-\Delta)^{1/2}u$). When $\lambda > 0$, it has a destabilizing effect (it has the "wrong" sign). The term αu accounts for the effects of gravity, and it is also destabilizing when $\alpha < 0$

[*]CEREMADE, Université Paris-Dauphine, UMR CNRS 7534, place de Lattre de Tassigny, 75775 Paris cedex 16, France

[†]Department of Mathematics. Mathematics Building. University of Maryland. College Park, MD 20742-4015, USA

(”hanging film”). In [13], it is proved that despite these destabilizing terms, positive smooth solutions of (1) do not blow up and remain bounded in H^1 for all time.

The goal of the present paper is to prove the existence of such solutions. This is non trivial because the H^1 a priori estimate established in [13] relies on the conservation of mass which, for non-negative solutions, give a global in time L^1 bound for the solution. However, the usual existence proof for the thin film equation (when $\alpha = \lambda = 0$) relies on the regularization of the mobility coefficient (u^3 is approximated by $|u|^3+\varepsilon$). And for such a regularized equation one cannot show the existence of non-negative solutions (the maximum principle does not hold for fourth order parabolic equations) - It is indeed a remarkable feature of the thin film equation that it is the degeneracy of the diffusion coefficient u^3 that permits the existence of non-negative solutions. Passing to the limit in the regularization process ($\varepsilon \to 0$) present thus an interesting challenge.

Rather than working in the periodic setting, we will consider equation (1) on a bounded domain with Neumann boundary conditions (these Neumann conditions can be interpreted as the usual contact angle conditions and seem physically more relevant - the periodic framework could be treated as well with minor modifications). Further details about the derivation of (1) will be given in Section 2. Since the gravity term is of lower order than the electric field term, it is of limited interest in the mathematical theory developed in this paper. We will thus take

$$\alpha = 0 \quad \text{and} \quad c = \lambda = 1.$$

We thus consider the following problem:

$$\begin{cases} u_t + \big(f(u)(u_{xx} - I(u))_x\big)_x = 0 & \text{for } x \in \Omega, \quad t > 0 \\ u_x = 0,\ f(u)(u_{xx} - I(u))_x = 0 & \text{for } x \in \partial\Omega, \quad t > 0 \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases} \tag{2}$$

The domain Ω is a bounded interval in $\mathbb{R}$; In the sequel, we will always take $\Omega = (0,1)$. The mobility coefficient $f(u)$ is a $\mathcal{C}^1$ function $f : [0,+\infty) \to (0,+\infty)$ satisfying

$$f(u) \sim u^n \quad \text{as } u \to 0 \tag{3}$$

for some $n > 1$. The operator I is a non-local elliptic operator of order 1 which will be defined precisely in Section 3 as the square root of the Laplace operator with Neumann boundary conditions (we have to be very careful with the definition of I in a bounded domain).

Note that the thin film equation

$$u_t + (f(u)u_{xxx})_x = 0, \tag{4}$$

has been extensively studied. The existence of non-negative weak solutions was first established by F. Bernis and A. Friedman [3] for $n > 1$. Further results (existence for $n > 0$ and further regularity results) were later obtained, by similar technics, in particular by E. Beretta, M. Bertsch and R. Dal Passo [2] and A. Bertozzi and M. Pugh [4, 5]. Results in higher dimension were obtained in particular in [9, 8, 7].

A priori estimates. As for the thin film equation (4), we prove the existence of solutions for (2) using a regularization/stability argument. The main tools are integral inequalities which provide the necessary compactness. Besides the conservation of mass, we will see that the solution u of (2) satisfies two important integral inequalities: We define the energy $\mathcal{E}(u)$ and entropy $e(u)$ by

$$\mathcal{E}(u)(t) = \frac{1}{2}\int_\Omega (u_x^2(t) + u(t)Iu(t))dx \quad \text{and} \quad e(u)(t) = \frac{1}{2}\int_\Omega G(u(t))dx$$

where G is a non-negative convex function such that $fG'' = 1$. Classical solutions of (2) then satisfy:

$$\mathcal{E}(u)(t) + \int_0^t \int_\Omega f(u)\big[(u_{xx} - I(u))_x\big]^2 \,dx\,ds \le \mathcal{E}(u_0), \tag{5}$$

$$e(u)(t) + \int_0^t \int (u_{xx})^2 \,dx\,ds + \int_0^t \int u_x I(u)_x \,dx\,ds \le e(u_0). \tag{6}$$

Similar inequalities hold for the thin film equation (4). However, we see here the destabilizing effect of the nonlocal term $I(u)$: First, we note that the energy $\mathcal{E}(u)$ can be written as the difference of two non-negative quantities:

$$\mathcal{E}(u)(t) = \|u(t)\|^2_{\dot H^1} - \|u(t)\|^2_{\dot H^{\frac12}},$$

and may thus take negative values. Similarly, the entropy dissipation can be written as

$$\int_0^t \int (u_{xx})^2 \,dx\,ds + \int_0^t \int u_x I(u)_x \,dx\,ds = \|u(t)\|^2_{\dot H^2} - \|u(t)\|^2_{\dot H^{\frac32}},$$

so the entropy may not be decreasing.

Nevertheless, it is reasonable to expect (2) to have solutions that exist for all times. Indeed, as shown in [13], the conservation of mass, the inequality (5) and the following functional inequality (see Lemma 1)

$$\|u\|^2_{\dot H^1(\Omega)} \le \alpha\mathcal{E}(u) + \beta\|u\|^2_{L^1(\Omega)}, \quad \forall u \in H^1(\Omega),$$

implies that non-negative solution remains bounded in $L^\infty(0,T;H^1(\Omega))$ for all T.

Furthermore, the interpolation inequality

$$\begin{aligned}\|u(t)\|^2_{\dot H^{\frac32}} &\le C\|u(t)\|_{\dot H^1}\|u(t)\|_{\dot H^2} \\ &\le \frac12\|u(t)\|^2_{\dot H^2} + \frac{C}{2}\|u(t)\|^2_{\dot H^1}\end{aligned}$$

yields

$$e(u)(t) + \frac12\int_0^t \|u(r)\|^2_{\dot H^2}\,dr \le e(u_0) + \frac12\int_0^t \|u(r)\|^2_{\dot H^1}\,dr,$$

and so the entropy remains bounded for all time as well.

Main results. We now state the two main results proved in this paper. They should be compared with Theorems 3.1 and 4.2 in [3, pp.185&194]. The first one deals with non-negative initial data whose entropies are finite.

We recall that G is a non-negative convex function such that

$$G''(u) = \frac{1}{f(u)} \qquad \text{for all } u > 0.$$

Theorem 1. *Let $n > 1$ and $u_0 \in H^1(\Omega)$ be such that $u_0 \geq 0$ and*

$$\int_\Omega G(u_0)\, dx < \infty. \tag{7}$$

For all $T > 0$ there exists a function $u(t,x) \geq 0$ with

$$u \in \mathcal{C}(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)), \quad u_x \in L^2(0,T; H^1_0(\Omega))$$

such that, for all $\phi \in \mathcal{D}([0,T) \times \bar{\Omega})$ satisfying $\phi_x = 0$ on $(0,T) \times \partial\Omega$,

$$\iint_Q u\phi_t - f(u)[u_{xx} - I(u)]\phi_{xx} - f'(u)u_x(u_{xx} - I(u))\phi_x \, dt\, dx + \int_\Omega u_0(x)\phi(0,x)\, dx = 0. \tag{8}$$

Moreover, the function u satisfies for every $t \in [0,T]$,

$$\begin{aligned} \int_\Omega u(t,x)\, dx &= \int_\Omega u_0(x)\, dx, \\ \mathcal{E}(u(t)) + \int_0^t \int_\Omega f(u)\big[(u_{xx} - I(u))_x\big]^2 \, ds\, dx &\leq \mathcal{E}(u_0), & (9) \\ \int_\Omega G(u(t))\, dx + \int_0^t \int_\Omega (u_{xx})^2 + u_x I(u)_x \, ds dx &\leq \int_\Omega G(u_0)\, dx. & (10) \end{aligned}$$

We point out that the weak formulation (8) involve two integration by parts. Our second main result is concerned with non-negative initial data whose entropies are possibly infinite (this is the case if u_0 vanishes on an open subset of Ω and $n \geq 2$). In that case, only one integration by parts is possible, and the solutions that we construct are weaker than those constructed in Theorem 1. In particular, the equation is only satisfied on the positivity set of the solution and the boundary conditions are satisfied in a weaker sense.

Theorem 2. *Assume $n > 1$ and let $u_0 \in H^1(\Omega)$ be such that $u_0 \geq 0$. For all $T > 0$ there exists a function $u(t,x) \geq 0$ such that*

$$u \in \mathcal{C}(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \cap \mathcal{C}^{\frac{1}{2},\frac{1}{8}}(\Omega \times (0,T))$$

such that

$$f(u)[u_{xx} - I(u)]_x \in L^2(P)$$

and such that, for all $\phi \in \mathcal{D}([0,T) \times \bar{\Omega})$,

$$\iint_Q u\phi_t \, dt \, dx + \iint_P f(u)[u_{xx} - I(u)]_x \phi_x \, dt \, dx + \int_\Omega u_0(x)\phi(0,x) \, dx = 0 \quad (11)$$

where $P = \{(x,t) \in \bar{Q} : u(x,t) > 0, t > 0\}$. *Moreover, the function* u *satisfies the conservation of mass and the energy inequality* (9).

Finally, u_x *vanishes at all points* (x,t) *of* $\partial\Omega \times (0,T)$ *such that* $u(x,t) \neq 0$.

Comments. These results are comparable to those of [3] when $\lambda = 0$. The reader might be surprised that they are presented in a different order than in [3]. The reason has to do with the proofs; indeed, in contrast with [3], weak solutions given by Theorem 2 are constructed as limits of the solutions given by Theorem 1. This is because the entropy is needed in order to construct the non-negative solutions. See the discussion at the beginning of Section 5 for further details.

Note that the results of [3] have been improved and generalized (see for instance [5]). Many of these results rely on the use of so-called generalized entropies. The generalization of such entropies to this model requires tedious computations (because of the non-local character of I) and is not addressed in this paper.

Organization of the article. In Section 2, we give more details about the physical model leading to (2). We gather, in Section 3, material that will be used throughout the paper. In particular, we detail the functional analysis framework and the definition of the non-local operator I (which is similar to that used in [10]). Section 4, 5 and 6 are devoted to the proofs of the main results. Finally, we give in Appendix a technical result which is more or less classical.

Acknowledgements. The first author was partially supported by the French Ministry of Research (ANR project EVOL). The second author was partially supported by NSF Grant DMS-0901340.

2 Physical model

In this section, we briefly recall the derivation of (2) (see [13] for further details). We consider a viscous liquid film which completely wets a solid horizontal substrate and is constraint between two solid wall (at $x = 0$ and $x = 1$), see Figure 1. The fluid is Newtonian and is assumed to be a perfect conductor. The substrate is a grounded electrode held at zero voltage. Thanks to the presence of another electrode (at infinity), an electric field $\mathbf{E}$ is created which is constant at infinity (in the direction perpendicular to the substrate):

$$\mathbf{E}(x,y) \longrightarrow (0, E_0) \qquad \text{as } y \to +\infty.$$

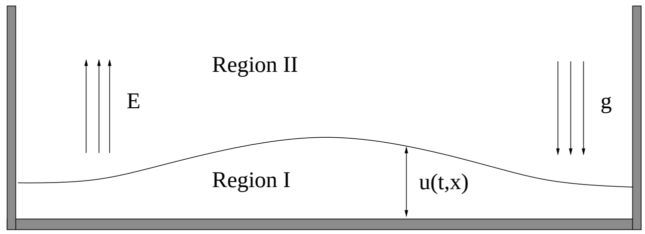

Figure 1: A viscous thin film submitted to an electric field $\mathbf{E}$ and gravity g

The height of the fluid is denoted by $u(t, x)$. Under the assumptions of the lubrication approximation, it is classical that the evolution of u is described by Poiseuille's law:

$$u_t - \partial_x \left(\frac{u^3}{3\mu} \partial_x p \right) = 0 \tag{12}$$

where p is the pressure at the free surface of the fluid $y = u(t, x)$. This pressure is the sum of three terms:

1. The capillary pressure due to surface tension, which can be approximated by
$$p_1 \sim -\sigma u_{xx}$$
(replacing the mean curvature operator by the Laplacian).

2. The effect of gravity, given by
$$p_2 = gu.$$

3. The additional pressure due to the action of the electric field E.

To compute the third term appearing in the pressure, we introduce the potential V such that $\mathbf{E} = -\nabla V$, which satisfies

$$\Delta V = 0 \text{ for } y \geq u(x)$$

and

$$V(x, y) = 0 \quad \text{on } y = u(x).$$

The condition at $y \to \infty$ means that we can write

$$V \sim E_0(Y_0 - y)$$

with (using standard linear approximation)

$$\begin{cases} \Delta Y_0 = 0 & \text{for } y > 0,\ x \in \Omega \\ \nabla Y_0 \to 0 & \text{as } y \to \infty,\ x \in \Omega \\ Y_0(x, 0) = u(x) & \text{for } x \in \Omega. \end{cases} \tag{13}$$

At the boundary of the cylinder, we assume that the electric field has no horizontal component:

$$\partial_x V = 0, \quad \text{for } x \in \partial\Omega,\ y > 0.$$

The pressure exerted by the electric field is then proportional to

$$p_3 = \gamma \mathbf{E}_y = -\gamma \partial_y V(x, 0) = -\gamma E_0(\partial_y Y_0 - 1).$$

The application $u \mapsto \partial_y Y_0(x, 0)$ is a Dirichlet-to-Neumann map for the harmonic extension problem (13). We denote this operator by $I(u)$. We will see in Section 3 that $I(u)$ is in fact the square root of the Laplace operator on the interval Ω with homogeneous Neumann boundary conditions.

We thus have

$$p = p_1 + p_2 + p_3 = -\sigma u_{xx} + gu - \gamma E_0 I(u) + c_0$$

for some constant c_0, and we obtain (1) with $c = \frac{\sigma}{3\mu}$, $\alpha = \frac{g}{3\mu}$ and $\lambda = -\frac{\gamma E_0}{3\mu}$. Note that Poiseuille's law (12) is obtained under the no-slip condition for the fluid along the solid support. Other conditions, such as the Navier slip condition leads to

$$f(u) = u^3 + \Lambda u^s$$

with $s = 1$ or $s = 2$. This explains the interest of the community for general diffusion coefficient $f(u)$.

Boundary conditions. Along the boundary $\partial\Omega$, the fluid is in contact with a solid wall. It is thus natural to consider a contact angle condition at $x = 0$ and $x = 1$: Assuming that the contact angle is equal to $\pi/2$, we then get the boundary condition

$$u_x = 0 \qquad \text{on } \partial\Omega.$$

In [13], the authors derive their analytic results in a periodic setting which is obtained by considering the even extension of u to the interval $(-1, 1)$ (recall that $\Omega = (0, 1)$) and then taking the periodic extension (with period 2) to $\mathbb{R}$.

Finally, since the equation is of order 4, we need an additional boundary condition. We thus assume that u satisfies the following null-flux condition

$$u^3(u_{xx} - I(u))_x = 0 \qquad \text{on } \partial\Omega$$

which will guarantee the conservation of mass.

3 Preliminaries

In this section, we recall how the operator I is defined (see [10]) and give the functional analysis results that we will need to prove the main theorem. A very similar operator, with Dirichlet boundary conditions rather than Neumann boundary conditions, was studied by Cabré and Tan [6].

3.1 Functional spaces

The space $H^s_N(\Omega)$. We denote by $\{\lambda_k, \varphi_k\}_{k=0,1,2...}$ the eigenvalues and corresponding eigenfunctions of the Laplace operator in Ω with Neumann boundary conditions on $\partial\Omega$:

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k & \text{in } \Omega \\ \partial_\nu\varphi_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

normalized so that $\int_\Omega \varphi_k^2\,dx = 1$. When $\Omega = (0,1)$, we have

$$\lambda_0 = 0\,, \qquad \varphi_0(x) = 1$$

and

$$\lambda_k = (k\pi)^2\,, \qquad \varphi_k(x) = \sqrt{2}\cos(k\pi x) \qquad k = 1,\, 2,\, 3,\, \ldots$$

The φ_k's clearly form an orthonormal basis of $L^2(\Omega)$. Furthermore, the φ_k's also form an orthogonal basis of the space $H^s_N(\Omega)$ defined by

$$H^s_N(\Omega) = \left\{ u = \sum_{k=0}^{\infty} c_k\varphi_k\,;\ \sum_{k=0}^{\infty} c_k^2(1+\lambda_k^s) < +\infty \right\}$$

equipped with the norm

$$||u||^2_{H^s_N(\Omega)} = \sum_{k=0}^{\infty} c_k^2(1+\lambda_k^s)$$

or equivalently (noting that $c_0 = \int_\Omega u(x)\,dx$ and $\lambda_k \geq 1$ for $k \geq 1$):

$$||u||^2_{H^s_N(\Omega)} = \|u\|^2_{L^1(\Omega)} + \|u\|^2_{\dot{H}^s_N(\Omega)}$$

where the homogeneous norm is given by:

$$\|u\|^2_{\dot{H}^s_N(\Omega)} = \sum_{k=1}^{\infty} c_k^2\lambda_k^s.$$

A characterisation of $H^s_N(\Omega)$. The precise description of the space $H^s_N(\Omega)$ is a classical problem.

Intuitively, for $s < 3/2$, the boundary condition $u_\nu = 0$ does not make sense, and one can show that (see Agranovich and Amosov [1] and references therein):

$$H^s_N(\Omega) = H^s(\Omega) \qquad \text{for all } 0 \leq s < \frac{3}{2}.$$

In particular, we have $H^{\frac{1}{2}}_N(\Omega) = H^{\frac{1}{2}}(\Omega)$ and we will see later that

$$\|u\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)} = \int_\Omega\int_\Omega (u(y)-u(x))^2\nu(x,y)dxdy$$

where $\nu(x,y)$ is a given positive function; see (18) below.

For $s > 3/2$, the Neumann condition has to be taken into account, and we have in particular

$$H^2_N(\Omega) = \{u \in H^2(\Omega)\,;\, u_\nu = 0 \text{ on } \partial\Omega\}$$

which will play a particular role in the sequel. More generally, a similar characterization holds for $3/2 < s < 7/2$. For $s > 7/2$, additional boundary conditions would have to be taken into account, but we will not use such spaces in this paper. In Section 4, we will also work with the space $H^3_N(\Omega)$ which is exactly the set of functions in $H^3(\Omega)$ satisfying $u_\nu = 0$ on $\partial\Omega$.

The case $s = 3/2$ is critical (note that $u_\nu|_{\partial\Omega}$ is not well defined in that space) and one can show that

$$H^{\frac{3}{2}}_N(\Omega) \quad = \quad \left\{ u \in H^{\frac{3}{2}}(\Omega)\,;\, \int_\Omega \frac{u_x^2}{d(x)}\,dx < \infty \right\}$$

where $d(x)$ denotes the distance to $\partial\Omega$. A similar result appears in [6]; more precisely, such a characterization of $H^{\frac{3}{2}}_N(\Omega)$ can be obtained by considering functions u such that $u_x \in \mathcal{V}_0(\Omega)$ where $\mathcal{V}_0(\Omega)$ is defined in [6] as the equivalent of our space $H^{1/2}_N(\Omega)$ with Dirichlet rather than Neumann boundary conditions. We do not dwell on this issue since we will not need this result in this paper.

3.2 The operator I

As it is explained in the Introduction, the operator I is related to the computation of the pressure as a function of the height of the fluid.

Spectral definition. With λ_k and φ_k defined by (14), we define the operator

$$I : \sum_{k=0}^{\infty} c_k \varphi_k \;\longmapsto\; -\sum_{k=0}^{\infty} c_k \lambda_k^{\frac{1}{2}} \varphi_k \tag{15}$$

which clearly maps $H^1(\Omega)$ onto $L^2(\Omega)$ and $H^2_N(\Omega)$ onto $H^1(\Omega)$.

Dirichlet-to-Neuman map. We now check that this definition of the operator I is the same as the one given in Section 2, namely I is the Dirichlet-to-Neumann operator associated with the Laplace operator supplemented with Neumann boundary conditions:

We consider the following extension problem:

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega \times (0, +\infty), \\ v(x,0) = u(x) & \text{on } \Omega, \\ v_\nu = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases} \tag{16}$$

Then, we can show (see [10]):

Proposition 1 ([10])**.** *For all $u \in H^{\frac{1}{2}}_N(\Omega)$, there exists a unique extension $v \in H^1(\Omega \times (0, +\infty))$ solution of (16).*

Furthermore, if $u(x) = \sum_{k=1}^{\infty} c_k \varphi_k(x)$*, then*

$$v(x,y) = \sum_{k=1}^{\infty} c_k \varphi_k(x) \exp(-\lambda_k^{\frac{1}{2}} y). \tag{17}$$

and we have:

Proposition 2 ([10])**.** *For all* $u \in H_N^2(\Omega)$*, we have*

$$I(u)(x) = -\frac{\partial v}{\partial \nu}(x,0) = \partial_y v(x,0) \quad \text{for all } x \in \Omega,$$

where v *is the unique harmonic extension solution of (16).*
Furthermore $I \circ I(u) = -\Delta u$*.*

Integral representation. Finally, the operator I can also be represented as a singular integral operator:

Proposition 3 ([10])**.** *Consider a smooth function* $u : \Omega \to \mathbb{R}$*. Then for all* $x \in \Omega$*,*

$$I(u)(x) = \int_\Omega (u(y) - u(x))\nu(x,y)dy$$

where $\nu(x,y)$ *is defined as follows: for all* $x, y \in \Omega$*,*

$$\nu(x,y) = \frac{\pi}{2}\left(\frac{1}{1-\cos(\pi(x-y))} + \frac{1}{1-\cos(\pi(x+y))}\right). \tag{18}$$

3.3 Functional equalities and inequalities

Equalities. The semi-norms $||\cdot||_{\dot{H}^{\frac{1}{2}}(\Omega)}$, $||\cdot||_{\dot{H}^1(\Omega)}$, $||\cdot||_{\dot{H}^{\frac{3}{2}}(\Omega)}$ and $||\cdot||_{\dot{H}_N^2(\Omega)}$ are related to the operator I by equalities which will be used repeatedly.

Proposition 4 (The operator I and several semi-norms – [10])**.**
For all $u \in H^{\frac{1}{2}}(\Omega)$*, we have*

$$-\int uI(u)\,dx = \frac{1}{2}\int_\Omega\int_\Omega (u(x)-u(y))^2\nu(x,y)dxdy = ||u||^2_{\dot{H}^{\frac{1}{2}}(\Omega)}.$$

For all $u \in H_N^2(\Omega)$*, we have*

$$-\int_\Omega u_x I(u)_x\,dx = ||u||^2_{\dot{H}_N^{\frac{3}{2}}(\Omega)}.$$

For all $k \in \mathbb{N}$ *and* $u \in H_N^{k+1}(\Omega)$*, we have*

$$\int_\Omega (\partial_x^k I(u))^2\,dx = \|u\|^2_{\dot{H}_N^{k+1}(\Omega)}. \tag{19}$$

Inequalities. First, we recall the following Nash inequality:

$$\|u\|_{L^2(\Omega)} \leq C\|u\|_{\dot{H}^1(\Omega)}^{\frac{1}{3}}\|u\|_{L^1(\Omega)}^{\frac{2}{3}}. \tag{20}$$

This inequality will allow us to control the H^1 norm by the energy $\mathcal{E}(u)$ and the L^1 norm. Indeed, we recall that the energy is defined by

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |u_x|^2 + u\,I(u)\,dx = \frac{1}{2}\|u\|^2_{\dot{H}^1(\Omega)} - \frac{1}{2}\|u\|^2_{\dot{H}^{\frac{1}{2}}_N(\Omega)}. \tag{21}$$

We then have:

Lemma 1. *There exist positive constants α, β such that for all $u \in H^1(\Omega)$,*

$$\|u\|^2_{\dot{H}^1(\Omega)} \leq \alpha\mathcal{E}(u) + \beta\|u\|^2_{L^1(\Omega)}.$$

Remark 1. See also Lemma 4.1 in [13].

Proof. We have:

$$\begin{aligned}
\|u\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)} &= -\int u\,I(u)\,dx \\
&\leq \|u\|_{L^2(\Omega)}\|I(u)\|_{L^2(\Omega)} \\
&\leq \|u\|_{L^2(\Omega)}\|u\|_{\dot{H}^1(\Omega)} \\
&\leq \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{\dot{H}^1(\Omega)}.
\end{aligned}$$

Using (20), we deduce

$$\begin{aligned}
\|u\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)} &\leq \frac{1}{2}\|u\|^2_{\dot{H}^1(\Omega)} + \frac{1}{2}\|u\|^{\frac{2}{3}}_{\dot{H}^1(\Omega)}\|u\|^{\frac{4}{3}}_{L^1(\Omega)} \\
&\leq \frac{1}{2}\|u\|^2_{\dot{H}^1(\Omega)} + \frac{1}{3}(\|u\|^2_{\dot{H}^1(\Omega)} + 2\|u\|^2_{L^1(\Omega)}) \\
&\leq \frac{5}{6}\|u\|^2_{\dot{H}^1(\Omega)} + \frac{2}{3}\|u\|^2_{L^1(\Omega)}.
\end{aligned}$$

We thus get the desired result with $\alpha = 12$ and $\beta = 4$. □

4 A regularized problem

We can now turn to the proof of Theorem 1 and 2. As usual, we introduce the following regularized equation:

$$\begin{cases} u_t + (f_\varepsilon(u)(u_{xx} - I(u))_x)_x = 0 & \text{for } x \in \Omega, \quad t > 0 \\ u_x = 0,\ f_\varepsilon(u)(u_{xx} - I(u))_x = 0 & \text{for } x \in \partial\Omega, \quad t > 0 \\ u(x,0) = u_0(x) & \text{for } x \in \Omega \end{cases} \tag{22}$$

where the mobility coefficient $f(u)$ is approximated by

$$f_\varepsilon(u) = \min(\max(\varepsilon, f(|u|)), M)$$

which satisfies

$$\varepsilon \leq f_\varepsilon(u) \leq M \quad \text{ for all } u \in \mathbb{R}.$$

Ultimately, we will show that the solution u satisfies

$$0 \leq u(t,x) \leq M_0$$

for some constant M_0 independent of M, so that we do not have to worry about M (provided we take it large enough). The ε is of course the most important parameter in the regularization since it makes (22) non-degenerate.

The proof of Theorem 1 consists of two parts: First, we have to show that the regularized equation (22) has a solution (which may take negative values). Then we must pass to the limit $\varepsilon \to 0$ and show that we obtain a non-negative solution of (2).

In this section, we prove the first part. Namely, we prove:

Theorem 3. *Let $u_0 \in H^1(\Omega)$. For all $T > 0$ there exists a function $u^\varepsilon(t,x)$ such that*

$$u^\varepsilon \in \mathcal{C}(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^3_N(\Omega))$$

such that, for all $\phi \in \mathcal{D}([0,T) \times \bar{\Omega})$,

$$\iint_Q u^\varepsilon \phi_t + f_\varepsilon(u^\varepsilon)[u^\varepsilon_{xx} - I(u^\varepsilon)]_x \phi_x \, dt\, dx + \int_\Omega u_0(x)\phi(0,x)\, dx = 0. \tag{23}$$

Moreover, the function u^ε satisfies for every $t \in [0,T]$,

$$\begin{aligned} \int_\Omega u^\varepsilon(t,x)\, dx &= \int_\Omega u_0(x)\, dx, \\ \mathcal{E}(u^\varepsilon(t)) + \int_0^t \int_\Omega f_\varepsilon(u^\varepsilon)\big[(u^\varepsilon_{xx} - I(u^\varepsilon))_x\big]^2 \, ds\, dx &\leq \mathcal{E}(u_0), \end{aligned} \tag{24}$$

and

$$\begin{aligned} &\int_\Omega (u^\varepsilon_x)^2\, dx + \int_0^t \int_\Omega f_\varepsilon(u^\varepsilon)(u^\varepsilon_{xxx})^2 \, ds\, dx \\ &\qquad \leq \int_\Omega ((u_0)_x)^2\, dx + \int_0^t \int_\Omega f_\varepsilon(u^\varepsilon) u^\varepsilon_{xxx} (I(u^\varepsilon))_x \, ds\, dx \end{aligned} \tag{25}$$

and

$$\int_\Omega G_\varepsilon(u^\varepsilon(t))\, dx + \int_0^t \int_\Omega (u^\varepsilon_{xx})^2 + u^\varepsilon_x I(u^\varepsilon)_x \, dsdx \leq \int_\Omega G_\varepsilon(u_0)\, dx. \tag{26}$$

where G_ε is a non-negative function such that $f_\varepsilon G''_\varepsilon = 1$.

Finally, u^ε is $\frac{1}{2}$-Hölder continuous with respect to x and $\frac{1}{8}$-Hölder continuous with respect to t; more precisely, there exists a constant C_0 only depending on Ω and $\|u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|f_\varepsilon(u^\varepsilon)[u^\varepsilon_{xx} - I(u^\varepsilon)]_x\|_{L^2(Q)}$ such that

$$\|u^\varepsilon\|_{\mathcal{C}^{\frac{1}{2},\frac{1}{8}}_{t,x}(Q)} \le C_0. \tag{27}$$

Proof of Theorem 3. Theorem 3 follows from a fixed point argument: For some T_*, we denote

$$V = L^2(0, T_*; H^2_N(\Omega))$$

and we define the application $\mathcal{F} : V \to V$ such that for $v \in V$, $\mathcal{F}(v)$ is the solution u of

$$\begin{cases} u_t + (f_\varepsilon(v)(u_{xxx} - I(v)_x))_x = 0 & \text{for } x \in \Omega, \quad t > 0 \\ u_x = 0,\ f_\varepsilon(v)(u_{xx} - I(v))_x = 0 & \text{for } x \in \partial\Omega, \quad t > 0 \\ u(x, 0) = u_0(x) & \text{for } x \in \Omega. \end{cases} \tag{28}$$

The fact that $\mathcal{F}$ is well defined follows from the observation that for $v \in V$, we have

$$a(t, x) = f_\varepsilon(v(t, x)) \in [\varepsilon, M] \quad \text{and} \quad g(t, x) = I(v)_x \in L^2(Q)$$

and the following proposition:

Proposition 5. *Consider $u_0 \in H^1(\Omega)$ and $a(t, x) \in L^\infty(Q)$ such that $\varepsilon \le a(t, x) \le M$ a.e. in Q. If $g \in L^2(Q)$, then there exists a function*

$$u \in \mathcal{C}(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^3_N(\Omega))$$

and

$$\iint_Q \left[u\phi_t + a(u_{xxx} - g)\phi_x\right] dt\, dx + \int_\Omega u_0(x)\phi(0, x)\, dx = 0$$

for all $\phi \in \mathcal{D}([0, T) \times \bar{\Omega})$. Moreover, for every $t \in [0, T]$, u satisfies

$$\int_\Omega u(t, x)\, dx = \int_\Omega u_0(x)\, dx$$

and

$$\int_\Omega (u_x)^2(t)\, dx + \frac{1}{2}\int_0^t \int_\Omega a(u_{xxx})^2\, ds\, dx \le \frac{M}{2}\int_0^t \int_\Omega g^2\, ds\, dx + \int_\Omega (u_0)^2_x\, dx. \tag{29}$$

Furthermore, u is $\frac{1}{2}$-Hölder continuous with respect to x and $\frac{1}{8}$-Hölder continuous with respect to t; more precisely, there exists a constant C_0 only depending on Ω and $\|u\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|u_{xxx} - g\|_{L^2(Q)}$ such that

$$\|u\|_{\mathcal{C}^{\frac{1}{2},\frac{1}{8}}_{t,x}(Q)} \le C_0. \tag{30}$$

This proposition is a very natural existence result for the fourth order linear parabolic equation

$$u_t + (au_{xxx})_x = (ag)_x.$$

Its proof is fairly classical, we give some details in Appendix A for the interested reader.

Next, we show the following result:

Lemma 2. *There exists a (small) time $T_* > 0$, depending only on ε, M and Ω, such that $\mathcal{F}$ has a fixed point u in $V = L^2(0, T_*; H^2_N(\Omega))$ for any initial data $u_0 \in H^1(\Omega)$. Furthermore, u satisfies*

$$\|u\|_V \le R||u_0||_{\dot{H}^1(\Omega)}$$

and

$$\|u\|_{L^\infty(0,T_*;\dot{H}^1(\Omega))} \le \sqrt{2}||u_0||_{\dot{H}^1(\Omega)}. \tag{31}$$

Before proving this lemma, let us complete the proof of Theorem 3.

Construction of a solution for large times. Lemma 2 gives the existence of a solution u_1^ε of (22) defined for $t \in [0, T_*]$. Since T_* does not depend on the initial condition, we can apply Lemma 2 to construct a solution u_2^ε in $[T_*, 2T_*]$ with initial condition $u_1^\varepsilon(T_*, x)$ which is $H^1(\Omega)$ by (31). This way, we obtain a solution u^ε of (22) on the time interval $[0, 2T_*]$. Note that we also have

$$\|u^\varepsilon\|_{L^\infty(0,2T_*;\dot{H}^1(\Omega))} \le \sqrt{2}^2||u_0||_{\dot{H}^1(\Omega)}.$$

Iterating this argument, we construct a solution u^ε on any interval $[0, T]$ satisfying in particular, for all $k \in \mathbb{N}$ such that $kT_* \le T$,

$$||u^\varepsilon||_{L^\infty(0,kT_*,\dot{H}^1(\Omega))} \le \sqrt{2}^k R||u_0||_{\dot{H}^1(\Omega)}.$$

Energy and entropy estimates. The conservation of mass follows from Proposition 5, but we need to explain how to derive (24), (25) and (26) from (23). Formally, one has to choose successively $\phi = -u^\varepsilon_{xx} + I(u^\varepsilon)$, $\phi = -u^\varepsilon_{xx}$ and $\phi = G'_\varepsilon(u^\varepsilon)$. Making such a formal computation rigourous is quite standard; details are given in Appendix B for the reader's convenience. Finally, (27) follows from (30). □

Proof of Lemma 2. We need to check that the conditions of Leray-Schauder's fixed point theorem are satisfied:

$\mathcal{F}$ is compact. Let $(v_n)_n$ be a bounded sequence in V and let u_n denote $\mathcal{F}(v_n)$. The sequence $(I(v_n))_x$ is bounded in $L^2(Q)$, and so

$$g_n = f_\varepsilon(v_n)\partial_x I(v_n) \text{ is bounded in } L^2(Q).$$

Estimate (29) implies that u_n is bounded in $L^2(0,T_*;H^3_N(\Omega))$. In particular $\partial_{xxx}u_n$ is bounded in $L^2(0,T_*;L^2(\Omega))$ and Equation (28) implies that $\partial_t(u_n)$ is bounded in $L^2(0,T_*,H^{-1}(\Omega))$. Using Aubin's lemma, we deduce that $(u_n)_n$ is pre-compact in $V = L^2(0,T_*;H^2_N(\Omega))$.

$\mathcal{F}$ is continuous. Consider now a sequence $(v_n)_n$ in V such that $v_n \to v$ in V and let $u_n = \mathcal{F}(v_n)$. We have in particular $v_n \to v$ in $L^2(Q)$ and, up to a subsequence, we can assume that $v_n \to v$ almost everywhere in Q. Hence, $f_\varepsilon(v_n) \to f_\varepsilon(v)$ almost everywhere in Q. We also have that $(I(v_n))_x$ converges to $(I(v))_x$ in $L^2(Q)$, and since $|f_\varepsilon(v_n)| \leq M$ a.e., we can show that

$$g_n = f_\varepsilon(v_n)\partial_x I(v_n) \to f_\varepsilon(v)\partial_x I(v) = g \quad \text{ in } L^2(Q).$$

Next, the compacity of $\mathcal{F}$ implies that $(u_n)_n$ is pre-compact in the space $L^2(0,T;H^2_N(\Omega))$, and so u_n converges (up to a subsequence) to U in V. In particular, $u_n \to U$ in $L^2(Q)$ and (up to a another subsequence), $u_n \to U$ almost everywhere in Q. We thus have $f_\varepsilon(u_n) \to f_\varepsilon(U)$ in $L^2(Q)$, and passing to the limit in the equation, we conclude that $U = u = \mathcal{F}(v)$ (by the uniqueness result in Proposition 5). Since this holds for any subsequence of u_n, we deduce that the whole sequence u_n converges to u hence

$$\mathcal{F}(v_n) \to \mathcal{F}(v) \text{ in } V \text{ as } n \to \infty$$

and $\mathcal{F}$ is continuous.

A priori estimates. It only remains to show that there exists a constant $R > 0$ such that for all functions $u \in V$ and $\sigma \in [0,1]$ such that $u = \sigma\mathcal{F}(u)$, we have

$$\|u\|_V \leq R.$$

This is where the smallness of T_* will be needed.

Using (29), we see that

$$\int_\Omega (u_x(t))^2\,dx + \frac{\varepsilon}{2}\int_0^{T_*}\int_\Omega (u_{xxx})^2\,dx\,dt \\ \leq \frac{M}{2}\int_0^{T_*}\int_\Omega ((I(u)_x)^2\,dx\,dt + \int_\Omega (u_0)_x^2\,dx, \quad (32)$$

and using (19) and the interpolation inequality

$$\|u_x\|_{L^2(\Omega)} \leq C\|u\|^{\frac{1}{2}}_{L^2(\Omega)}\|u_{xx}\|^{\frac{1}{2}}_{L^2(\Omega)}, \quad (33)$$

we get:

$$
\begin{aligned}
\int_0^{T_*}\int_\Omega ((I(u)_x)^2\,dx\,dt &\leq 2C_\varepsilon\int_0^{T_*}\|I(u)(t)\|_2^2\,dt+\frac{\varepsilon}{2M}\int_0^{T_*}\|(I(u))_{xx}(t)\|_2^2\,dt\\
&\leq 2C_\varepsilon\int_0^{T_*}\|u_x(t)\|_2^2\,dt+\frac{\varepsilon}{2M}\int_0^{T_*}\|u_{xxx}(t)\|_2^2\,dt\\
&\leq 2C_\varepsilon T_*\sup_{t\in[0,T_*]}\|u_x(t)\|_2^2+\frac{\varepsilon}{2M}\int_0^{T_*}\|u_{xxx}(t)\|_2^2\,dt \quad (34)
\end{aligned}
$$

where C_ε only depends on the constant C in (33) and the parameters ε and M. Combining (32) and (34), we conclude that

$$
(1-MC_\varepsilon T_*)\sup_{t\in[0,T]}\|u_x(t)\|_2^2+\frac{\varepsilon}{4}\int_0^{T_*}\|u_{xxx}(t)\|_2^2\,dt\leq\int_\Omega(u_0)_x^2\,dx.
$$

Therefore, choosing $T_*:=\frac{1}{2MC_\varepsilon}$, we get the following estimates

$$
\|u\|_{L^\infty(0,T_*;\dot H^1(\Omega))}\leq\sqrt{2}||u_0||_{\dot H^1(\Omega)}\qquad\text{and } \|u\|_{L^2(0,T_*;\dot H^3_N(\Omega))}\leq\frac{2}{\sqrt{\varepsilon}}||u_0||_{\dot H^1(\Omega)}
$$

Since we also have

$$
\int_\Omega u(t)\,dx=\int_\Omega u_0\,dx,
$$

we deduce that $\|u\|_V\leq R$ for some constant R depending on ε, which completes the proof. □

5 Proof of Theorem 1

As pointed out in the introduction, one of the main difficulties in the proof of Theorem 1 is that the natural energy estimate (29) does not give any information by itself, since $\mathcal{E}(u)$ may be negative. Even if Lemma 1 implies that the quantity

$$
\alpha\mathcal{E}(u^\varepsilon)+\beta||u^\varepsilon||_{L^1(\Omega)}
$$

is bounded below by the H^1 norm of u, the mass conservation only allows us to control the L^1 norm of u^ε if we know that u^ε is non-negative. Unfortunately, it is well known that equation (22) does not satisfy the maximum principle, and that the existence of non-negative solutions of (2) is precisely a consequence of the degeneracy of the diffusion coefficient, so that while we can hope (and we will prove) to have $\lim_{\varepsilon\to 0}u^\varepsilon\geq 0$ we do not have, in general, that $u^\varepsilon\geq 0$. Lemma 3 below will show that it is nevertheless possible to derive some a priori estimates that are enough to pass to the limit, provided the initial entropy is finite (7).

Proof of Theorem 1. Consider the solution u^ε of (22) given by Theorem 3. In order to prove Theorem 1, we need to show that $\lim_{\varepsilon\to 0}u^\varepsilon$ exists and solves (8).

Since we cannot use the energy inequality to get the necessary estimates on u^ε, we will need the following lemma:

Lemma 3. *Let H_ε denote the following functional:*

$$H_\varepsilon(v) = \int_\Omega [v_x^2 + 2MG_\varepsilon(v)]\,dx.$$

Then the solution u^ε given by Theorem 3 satisfies

$$H_\varepsilon(u^\varepsilon(t)) + \frac{M}{2}\int_0^t\int_\Omega (u^\varepsilon_{xx})^2\,dx\,ds + \frac{1}{2}\int_0^t\int_\Omega f_\varepsilon(u^\varepsilon)(u^\varepsilon_{xxx})^2\,dx\,ds \le H_\varepsilon(u_0)e^{t/2}$$

for every $t \in [0,T]$.

Proof of Lemma 3. Using (25) and (26) we see that

$$\int_\Omega G_\varepsilon(u^\varepsilon(t))\,dx + \frac{1}{2}\int_0^t\int_\Omega (u^\varepsilon_{xx})^2\,dx\,ds \le \int_\Omega G_\varepsilon(u_0)\,dx + \frac{1}{2}\int_0^t\int_\Omega (u^\varepsilon_x)^2\,dx\,ds$$

and

$$\int_\Omega (u^\varepsilon_x)^2\,dx + \frac{1}{2}\int_0^t\int_\Omega f_\varepsilon(u^\varepsilon)[u^\varepsilon_{xxx}]^2\,dx\,ds \le \int_\Omega ((u_0)_x)^2\,dx + \frac{M}{2}\int_0^t\int_\Omega (u^\varepsilon_{xx})^2\,dx\,ds.$$

This implies

$$H_\varepsilon(u^\varepsilon(t)) \le H_\varepsilon(u_0) + \int_0^t H_\varepsilon(u^\varepsilon(s))ds,$$

and Gronwall's lemma yields the desired result. □

Sobolev and Hölder bounds. We now gather all the a priori estimates: Using the conservation of mass, Lemma 3 and inequality (7), we see that there exists a constant C independent of ε such that:

$$\sup_{t\in[0,T]}\int_\Omega (u^\varepsilon_x(t))^2\,dx \le C, \quad (35)$$

$$\sup_{t\in[0,T]}\int_\Omega G_\varepsilon(u^\varepsilon(t))\,dx \le C, \quad (36)$$

$$\int_0^T\int_\Omega f_\varepsilon(u^\varepsilon)[u^\varepsilon_{xxx}]^2\,dx\,dt \le C, \quad (37)$$

$$\int_0^T\int_\Omega (u^\varepsilon_{xx})^2\,dx\,dt \le C. \quad (38)$$

Next, we note that (35) yields

$$\mathcal{E}(u^\varepsilon) \ge -||u^\varepsilon||_{L^\infty(0,T;\dot H^{1/2}(\Omega))} \ge -C||u^\varepsilon||_{L^\infty(0,T;H^1(\Omega))} \ge -C$$

and so (24) gives

$$\int_0^T \int_\Omega f_\varepsilon(u^\varepsilon)\big[u^\varepsilon_{xxx} - I(u^\varepsilon)_{xx}\big]^2\, ds\, dx \le C. \tag{39}$$

Finally, estimates (27), (35) and (39) yield that u^ε is bounded in $\mathcal{C}^{1/2,1/8}_{x,t}(Q)$.

Limit $\varepsilon \to 0$. The previous Hölder estimate implies that there exists a function $u(x,t)$ such that u^ε converges uniformly to u as ε goes to zero (up to a subsequence). Inequality (38) also implies that

$$u^\varepsilon \rightharpoonup u \text{ in } L^2(0,T;H^2(\Omega))\text{-weak}$$

and Aubin's lemma gives

$$u^\varepsilon \longrightarrow u \text{ in } L^2(0,T;H^1(\Omega))\text{-strong}.$$

After integration by parts, (23) can be written as

$$\iint_Q u^\varepsilon \phi_t - f_\varepsilon(u^\varepsilon)[u^\varepsilon_{xx} - I(u^\varepsilon)]\phi_{xx} - f'_\varepsilon(u^\varepsilon)u^\varepsilon_x(u^\varepsilon_{xx} - I(u^\varepsilon))\phi_x\, dt\, dx$$
$$= \int_\Omega u_0(x)\phi(x)\, dx$$

and passing to the limit $\varepsilon \to 0$ gives (8).

Non-negative solution. It only remains to show that u is non-negative. This can be done as in [3], using (36) (and the fact that f satisfies (3) with $n > 1$).

L^∞ a priori estimate. Finally, (35) and Sobolev's embedding implies that there exits a constant M_0 depending only on $||u_0||_{H^1(\Omega)}$ such that

$$0 \le u(t,x) \le M_0. \tag{40}$$

Choosing $M > M_0$, we deduce that u solves (2). □

6 Proof of Theorem 2

In order to get Theorem 2, we need to derive the following corollary from Theorem 1.

Corollary 1. *The solution u constructed in Theorem 1 satisfies for all $\phi \in \mathcal{D}((0,T)\times\bar{\Omega})$,*

$$\iint_Q u\phi_t\, dt\, dx + \iint_P f(u)[u_{xx} - I(u)]_x \phi_x\, dt\, dx = 0 \tag{41}$$

where $P = \{(x,t) \in \bar{Q} : u(x,t) > 0, t > 0\}$.

Proof. In view of the proof of Theorem 1, u is the uniform limit of a subsequence of $(u^\varepsilon)_{\varepsilon>0}$ where u^ε is given by Theorem 3. Since u^ε satisfies (23), it is thus enough to pass to the limit in this weak formulation as $\varepsilon \to 0$ in order to get the desired result. Let h^ε denote $f_\varepsilon(u^\varepsilon)[u^\varepsilon_{xx} - I(u)]_x$. Estimates (39) and (40) imply

$$\iint_Q h_\varepsilon^2 \, dx\, dt \leq C. \tag{42}$$

In other words, $(h^\varepsilon)_\varepsilon$ is bounded in $L^2(Q)$. Hence, up to a subsequence,

$$h^\varepsilon \rightharpoonup h \qquad \text{in } L^2(Q)-\text{weak}.$$

Furthermore, we recall that there exists a continuous function $u(x,t)$ such that u^ε converges uniformly to u as ε goes to zero (up to a subsequence).

Passing to the limit in (23), we deduce that the function u satisfies

$$\iint_Q u\phi_t \, dt\, dx + \iint_\Omega h\phi_x \, dt\, dx = 0.$$

We now have to show that

$$h = \begin{cases} 0 & \text{in } \{u=0\}, \\ f(u)\big[u_{xx} - I(u)\big]_x & \text{in } P = \{u>0\}. \end{cases}$$

First we note that for any test function ϕ and $\eta > 0$, we have

$$\begin{aligned} &\left| \int_0^T \int_{\{u\leq\eta\}} f_\varepsilon(u^\varepsilon)\big[u^\varepsilon_{xxx} - I(u^\varepsilon)_x\big]\phi \, dx\, dt \right| \\ &\qquad \leq C(\phi)\bigg(f_\varepsilon(3\eta/2)\bigg)^{1/2} \left(\int_0^T \int_{\{u\leq\eta\}} f_\varepsilon(u^\varepsilon)\big[u_{xxx} - I(u)_x\big]^2 \, dx\, dt\right)^{1/2} \end{aligned}$$

for ε small enough (so that $|u^\varepsilon - u| \leq \eta/2$). Inequality (39) thus implies

$$\limsup_{\varepsilon\to 0} \left| \int_0^T \int_{\{u\leq 2\eta\}} f_\varepsilon(u^\varepsilon)\big[u^\varepsilon_{xxx} - I(u^\varepsilon)_x\big]\phi \, dx\, dt \right| \leq C(\phi) f(\eta/2)^{1/2}.$$

We deduce (since $f(0) = 0$)

$$h = 0 \qquad \text{on } \{u=0\}. \tag{43}$$

Next, (42) yields (for $\varepsilon > 0$ and η small enough)

$$\iint_{\{u>2\eta\}} |u^\varepsilon_{xxx} - I(u^\varepsilon)_x|^2 \, dx\, ds \leq C(\eta).$$

This implies that, if Q_η denotes $\{u > 2\eta\}$, $(u^\varepsilon_{xxx} - I(u^\varepsilon)_x)$ is bounded in the space $L^2(Q_\eta)$. Hence, we can extract from $(u^\varepsilon_{xxx} - I(u^\varepsilon)_x)_{\varepsilon>0}$ a subsequence converging weakly in $L^2(Q_\eta)$. Moreover, remark that Q_η is an open subset of Q (recall that u is Hölder continuous) and $u^\varepsilon_{xxx} - I(u^\varepsilon)_x$ converges in the sense of distributions to $u_{xxx} - I(u)_x$ (use the integral representation for $I(\cdot)$). We thus conclude that,

$$u^\varepsilon_{xxx} - I(u^\varepsilon)_x \rightharpoonup u_{xxx} - I(u)_x \quad \text{ in } L^2(Q_\eta).$$

This yields

$$h = f(u)\big[u_{xxx} - I(u)_{xx}\big] \quad \text{ in } \{u > 0\}$$

which concludes the proof of Corollary 1. □

We now turn to the proof of Theorem 2.

Proof of Theorem 2. When u_0 does not satisfy (7), we lose the $L^2(0,T;H^2(\Omega))$ bound on u^ε, and the previous analysis fails. However, we can introduce

$$u_0^\delta = u_0 + \delta$$

which satisfies (7). Theorem 1 then provides the existence of a *non-negative* solution u^δ of (8). In view of Corollary 1, u^δ satisfies:

$$\iint_Q u^\delta \phi_t \, dt\, dx + \iint_P f(u^\delta)[u^\delta_{xx} - I(u^\delta)]_x \phi_x \, dt\, dx = 0. \tag{44}$$

Since u^δ is non-negative, the conservation of mass gives a bound in the space $L^\infty(0,T;L^1(\Omega))$ and allows us to make use of the energy inequality: Indeed, using (9) and Lemma 1 we see that there exists a constant C independent of δ such that

$$||u^\delta||_{L^\infty(0,T;H^1(\Omega))} \leq C$$

and

$$\int_0^T \int_\Omega f(u^\delta)\big[u^\delta_{xxx} - I(u^\delta)_x\big]^2 \, ds\, dx \leq C. \tag{45}$$

We now define the flux

$$h^\delta = f(u^\delta)\big[u^\delta_{xxx} - I(u^\delta)_x\big].$$

Inequality (45) implies that h^δ is bounded in $L^2(Q)$, and that there exists a function $h \in L^2(Q)$ such that

$$h^\delta \rightharpoonup h \qquad \text{ in } L^2(Q)-\text{weak}.$$

Proceeding as in the proof of Theorem 1, we deduce that u^δ is bounded in $\mathcal{C}^{1/2,1/8}(\Omega \times (0,T))$ and that there exists a function $u(x,t)$ such that u^δ converges uniformly to u as δ goes to zero (up to a subsequence). We can now argue (with minor changes) as in the proof of Corollary 1 and conclude. □

A Proof of Proposition 5

Our goal here is to prove the existence of a weak solution of

$$u_t + (au_{xxx})_x = (ag)_x.$$

We first prove the following proposition.

Proposition 6. *For all $h \in H^1(\Omega)$, there exists $v \in V_0 := H^1 \cap H^3_N$ such that for all $\phi \in \mathcal{D}(\bar{\Omega})$,*

$$-\int_\Omega \frac{v-h}{\tau}\phi\,dx + \int av_{xxx}\phi_x\,dx = \int_\Omega ag\phi_x\,dx. \tag{46}$$

In particular,

$$\begin{aligned} \int_\Omega v\,dx &= \int_\Omega h\,dx, \\ \frac{1}{2}\int_\Omega v_x^2 + \tau\int_\Omega av_{xxx}^2 &\leq \frac{1}{2}\int_\Omega h_x^2 + \tau\int_\Omega agv_{xxx}. \end{aligned} \tag{47}$$

Proof. In order to prove this proposition, we have to reformulate the equation. More precisely, instead of choosing test functions $\phi \in \mathcal{D}(\bar{\Omega})$, we choose $\phi = -\psi_{xx} + \int_\Omega \psi\,dx$ where ψ is given by the following lemma

Lemma 4. *For all $\phi \in \mathcal{D}(\bar{\Omega})$, there exists $\psi \in \mathcal{D}(\bar{\Omega})$ such that*

$$-\psi_{xx} + \int_\Omega \psi\,dx = \phi.$$

Hence, we consider $V_0 := H^1 \cap H^3_N$ equipped with the norm $\|v\|^2_{V_0} = \|v_{xxx}\|^2_{L^2} + (\int v\,dx)^2$ and we look for $v \in V_0$ such that for all $\psi \in \mathcal{D}(\bar{\Omega})$,

$$\begin{aligned} &\int_\Omega v_x\psi_x\,dx + \tau\int av_{xxx}\psi_{xxx}\,dx + \left(\int_\Omega v\,dx\right)\left(\int_\Omega \psi\,dx\right) \\ &\qquad = \int_\Omega h_x\psi_x\,dx + \left(\int_\Omega h\,dx\right)\left(\int_\Omega \psi\,dx\right) + \tau\int_\Omega ag\psi_{xxx}\,dx. \end{aligned} \tag{48}$$

We thus consider the bilinear form A in V_0 defined as follows: for all $v, w \in V_0$,

$$A(v,w) = \int_\Omega v_x w_x\,dx + \tau\int av_{xxx}w_{xxx}\,dx + \left(\int_\Omega v\,dx\right)\left(\int_\Omega w\,dx\right).$$

We check that it is continuous and coercive:

$$\begin{aligned} |A(v,w)| &\leq \|v_x\|_2\|w_x\|_2 + M\tau\|v_{xxx}\|_2\|w_{xxx}\|_2 + \|v\|_1\|w\|_1 \\ &\leq C\|v\|_{V_0}\|w\|_{V_0} \\ A(v,v) &\geq \int_\Omega [(v_x)^2 + \varepsilon\tau(v_{xxx})^2]\,dx + \left(\int_\Omega v\,dx\right)^2 \geq \|v\|^2_{V_0}. \end{aligned}$$

We now consider the following linear form L in V_0: for all $w \in V_0$,

$$L(w) = \int_\Omega h_x w_x \, dx + \left(\int_\Omega h \, dx \right) \left(\int_\Omega w \, dx \right) + \tau \int_\Omega a g w_{xxx} \, dx$$

Since $0 \leq a \leq M$ and $g \in L^2$, L is continuous as soon as $h \in H^1(\Omega)$. Lax-Milgram theorem thus implies that there exists $v \in V_0$ such that (48) holds true for all $w \in V_0$.

Eventually, remark that conservation of mass and (47) are direct consequences of (46). The proof of Proposition 6 is now complete. □

We can now prove Proposition 5.

Proof of Proposition 5. For any $\tau > 0$, we consider $N_\tau = \lceil \frac{T}{\tau} \rceil$. We then define inductively a sequence $(u^n)_{n=0,\ldots,N_\tau}$ of V_0 as follows: $u^0 = u_0$ and u^{n+1} is obtained by applying Proposition 6 to $h = u^n$. We then define $u^\tau : [0, N_\tau \tau) \times \Omega$ as follows:

$$u^\tau(t,x) = u^n(x) \quad \text{ for } t \in [n\tau, (n+1)\tau).$$

We have $\int_\Omega u^\tau(t,x)\,dx = \int_\Omega u_0(x)\,dx$ for all t. We also derive from (47) that we have

$$\int_\Omega (u^\tau_x)^2(T,x)\,dx + \int_0^T \int_\Omega a(u^\tau_{xxx})^2(t,x)\,dtdx \leq \int_\Omega ((u_0)_x)^2\,dx + \int_0^T \int_\Omega (agu_{xxx})(t,x)\,dtdx$$

In particular, $(u^\tau)_\tau$ is bounded in $L^\infty(0,T;H^1(\Omega))$ and $(S_\tau u^\tau - u^\tau)_\tau$ is bounded in $L^2(0,T-\tau;H^{-1}(\Omega))$ where $S_\tau v(t,x) = v(t+\tau,x)$. We derive from [12, Theorem 5] that $(u^\tau)_\tau$ is relatively compact in $\mathcal{C}(0,T;L^2(\Omega))$.

We now have to pass to the limit in (46). Since $(u^\tau_{xxx})_\tau$ is bounded in $L^2(Q)$ and we can find a sequence $\tau_n \to 0$ such that $u^{\tau_n} \to u$ in $\mathcal{C}(0,T,L^2(\Omega))$ and $u^{\tau_n}_{xxx} \to u_{xxx}$ in $L^2(Q)$. This is enough to conclude.

We next explain how to get (30). Sobolev's embedding imply that there exists a constant K (depending on $\|u\|_{L^\infty(0,T;H^1(\Omega))}$) such that

$$|u(x_1,t) - u(x_2,t)| \leq K|x_1 - x_2|^{1/2}$$

for all x_1, $x_2 \in \Omega$ and for all $t \in (0,T)$. Since u satisfies

$$u_t = h_x$$

with $h \in L^2(Q)$, it is a fairly classical result that Hölder regularity in space implies Hölder regularity in time. More precisely, we have (see [3], Lemma 2.1 for details):

Lemma 5. *There exists a constant C such that for all x_1, x_2 in Ω and all t_1, $t_2 > 0$,*

$$|u(x_1,t_1) - u(x_2,t_2)| \leq C|x_1 - x_2|^{1/2} + C|t_1 - t_2|^{1/8}.$$

The proof of Proposition 5 is thus complete. □

B Proof of (24), (25) and (26)

We have to derive (24), (25) and (26) from (23). Using the fact that u^ε lies in $\mathcal{C}([0,T],H^1(\Omega))$, we first state the following lemma

Lemma 6. *For all $\phi \in \mathcal{D}(\bar{Q})$,*

$$\iint_Q u^\varepsilon \phi_t + f_\varepsilon(u^\varepsilon)[u^\varepsilon_{xx} - I(u^\varepsilon)]_x \phi_x \, dt\, dx = \int_\Omega u^\varepsilon(x,T)\phi(x,T) - \int_\Omega u_0(x)\phi(x,0)\, dx. \quad (49)$$

The proof of such a lemma is fairly classical. It relies on mollifiers that are decentered in the time variable. More precisely, one considers a smooth even function $\rho : \mathbb{R} \to [0,1]$ compactly supported in $[-1,1]$ and such that $\int \rho = 1$. Then for $\alpha > 0$ and $\delta \in \mathbb{R}$, one can define

$$\rho_{\alpha,\delta}(t) = \tau_\delta \rho_\alpha(t) = \alpha\rho\left(\frac{t-\delta}{\alpha}\right).$$

If now a function f is defined in $[0,T]$, it can be extended by 0 to $\mathbb{R}$; in other words, it can be replaced with $f\mathbf{1}_\Omega$ where $\mathbf{1}_\Omega(x) = 1$ if $x \in \Omega$ and $\mathbf{1}_\Omega(x) = 0$ if not; then the convolution product in $\mathbb{R}$: $(f\mathbf{1}_\Omega) \star \rho_{\alpha,\delta}$ is a smooth function in $\mathbb{R}$ which vanishes near $t = 0$ (resp. $t = T$) if $\delta > 0$ (resp. $\delta < 0$).

Consider a smooth function ρ such as in the proof of Lemma 6. Consider $\alpha > 0$ and define $\rho_\alpha(x) = \alpha\rho(\frac{x}{\alpha})$. Then consider $\theta(x,t) = \rho_\alpha(x)\rho_\alpha(t)$.

Lemma 7. *Recall that $u^\varepsilon \in L^\infty(Q)$ and $h^\varepsilon = f_\varepsilon(u^\varepsilon)[u^\varepsilon_{xx} - I(u^\varepsilon)]_x \in L^2(Q)$. Then for all $v \in L^1(Q)$,*

$$\iint_Q u^\varepsilon (v \star \theta)_t = \iint_Q (u^\varepsilon \star \theta)_t v \quad (50)$$

where $f \star g$ means $(f\mathbf{1}_Q) \star (g\mathbf{1}_Q)$.

We next apply Lemma 6 with $\phi = v \star \theta$ where v is chosen to be successively $u^\varepsilon_{xx} \star \theta$, $I(u^\varepsilon \star \theta)$ and $G'_\varepsilon(u^\varepsilon \star \theta)$. After direct computations, we can let $\alpha \to 0$ and get the desired estimates.

References

[1] M. S. Agranovich and B. A. Amosov, *On Fourier series in eigenfunctions of elliptic boundary value problems*, Georgian Math. J., 10 (2003), pp. 401–410. Dedicated to the 100th birthday anniversary of Professor Victor Kupradze.

[2] E. Beretta, M. Bertsch, and R. Dal Passo, *Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation*, Arch. Rational Mech. Anal., 129 (1995), pp. 175–200.

[3] F. Bernis and A. Friedman, *Higher order nonlinear degenerate parabolic equations*, J. Differential Equations, 83 (1990), pp. 179–206.

[4] A. L. Bertozzi and M. Pugh, *The lubrication approximation for thin viscous films: the moving contact line with a "porous media" cut-off of van der Waals interactions*, Nonlinearity, 7 (1994), pp. 1535–1564.

[5] ———, *The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions*, Comm. Pure Appl. Math., 49 (1996), pp. 85–123.

[6] X. Cabré and J. Tan, *Positive solutions of nonlinear problems involving the square root of the laplacian.* Preprint, 2009.

[7] R. Dal Passo, H. Garcke, and G. Grün, *On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions*, SIAM J. Math. Anal., 29 (1998), pp. 321–342 (electronic).

[8] G. Grün, *Degenerate parabolic differential equations of fourth order and a plasticity model with non-local hardening*, Z. Anal. Anwendungen, 14 (1995), pp. 541–574.

[9] ———, *On Bernis' interpolation inequalities in multiple space dimensions*, Z. Anal. Anwendungen, 20 (2001), pp. 987–998.

[10] C. Imbert and A. Mellet, *Existence of solutiosn for a higher order non-local equation appearing in crack dynamics*, 2010. arXiv:1001.5105.

[11] D. T. Papageorgiou, P. G. Petropoulos, and J.-M. Vanden-Broeck, *Gravity capillary waves in fluid layers under normal electric fields*, Phys. Rev. E (3), 72 (2005), pp. 051601, 9.

[12] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Math. Pura Appl., 146 (1987), pp. 65–96.

[13] D. Tseluiko and D. T. Papageorgiou, *Nonlinear dynamics of electrified thin liquid films*, SIAM J. Appl. Math., 67 (2007), pp. 1310–1329 (electronic).