Research Statement
June 2013
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My research work centers around two broad topics: The analysis of nonlinear Partial Differential Equations (PDE) of elliptic and parabolic type (mainly free boundary problems and reaction-diffusion equations) and the study of transport phenomena (particularly the connection between kinetic and hydrodynamic models). My main contributions are described below (preprints are available at `http://www.math.umd.edu/~mellet/research.html`)

1 Nonlinear PDE of elliptic and parabolic type

1.1 Free boundary problems describing contact line phenomena

I have worked on several mathematical aspects of the study of small liquid drops, either lying still on a flat surface, or sliding down an inclined plane. For a drop at equilibrium, we can determine its shape by minimizing an energy. This is a very classical example of calculus of variations, and the corresponding PDE (Euler-Lagrange equation) is a free boundary problem which, in its simplest form, is similar to the classical Bernouilli problem.

$$\begin{cases} -\Delta u = \lambda & \text{in } \{u > 0\} \\ |\nabla u| = \theta_e & \text{on } \partial\{u > 0\} \end{cases} \tag{1}$$

where λ is the Lagrange multiplier for the volume constraint and θ_e is the given equilibrium contact angle.

Some of my earlier works were devoted to the homogenization of such problems. The goal was to understand how small scale perturbations along the free boundary would affect the large scale shape of the droplets. Typically, we consider

$$\begin{cases} -\Delta u^\varepsilon = \lambda & \text{in } \{u^\varepsilon > 0\} \\ |\nabla u^\varepsilon| = \theta(x/\varepsilon) & \text{on } \partial\{u^\varepsilon > 0\} \end{cases} \tag{2}$$

where $x \mapsto \theta(x)$ is a (periodic) function and we want to determine the free boundary condition satisfied by $u^0 = \lim_{\varepsilon\to 0} u^\varepsilon$. This homogenization problem is very different from more classical problems in which the oscillating coefficients appear in the equation rather than in the boundary condition. It is somewhat reminiscent of the homogenization of Neumann boundary conditions (which strongly depends on the geometry of the boundary), but in (2) the boundary $\partial\{u^\varepsilon > 0\}$ is not fixed a priori, leading to more complicated phenomena.

The main feature of my results was to show that some extremal quantities, (rather than some average quantities) play a crucial role in the effective properties of the solution as $\varepsilon \to 0$. I investigated the homogenization of such free boundary problems both in the periodic framework (with L. Caffarelli) and random framework (with J. Nolen), and in both cases we showed that while the homogenized free boundary condition for global energy minimizers is given by $|\nabla u^0| = \langle \theta \rangle$ ($\langle \cdot \rangle$ denotes the mean value or the expectation depending of the framework), local minimizers and solutions of (2) satisfy a homogenized free boundary condition of the form

$$|\nabla u^0| \in [\theta_{\min}, \theta_{\max}],$$

leading to non symmetric solutions and contact angle hysteresis phenomena.

I have also studied the homogenization of other types of free boundary problems. In particular, with I. Kim we studied the homogenization of **Stefan** and **Hele-Shaw** problems. In this case, some averaging process does occur. This fact was actually well known, but we also solved a long standing open problem by proving the uniform convergence of the free boundaries.

Finally, I also studied the homogenization of some **obstacle problems** with highly oscillating obstacles, revisiting an old problem going back to De Giorgi-Dal Maso-Longo (1980) and Cioranescu-Murat (1982). We extended some of these results to the random setting and to boundary obstacle problems (also known as thin obstacle problem).

More recently, I have been interested in models describing the motion of such liquid drops on a solid support. From a modeling point of view, this is a very difficult problem, in part because the motion of the fluid inside the drop should be taken into account, but also because the dynamic of the contact line (or free boundary) is poorly understood. So far, I have focused on two simplified models: the quasi-static approximation and the lubrication approximation.

The **quasi-static approximation** assumes that the free surface of the drop is at equilibrium at all time and that the deviation of the "apparent" contact angle $|\nabla u|$ from the equilibrium value θ_e is responsible for the motion of the contact line. In the particular model that I considered, the normal velocity V of the contact line $\partial\{u>0\}$ is given by

$$V = \frac{1}{2}|\nabla u|^2 - \frac{1}{2}\theta_e^2 \qquad \text{on } \partial\{u>0\} \tag{3}$$

(this is only one of many velocity laws that can be found in the literature), where the function u solves

$$-\Delta u = \lambda(t) - f(u,x) \quad \text{in } \{u(t)>0\}, \qquad \int u(t)\,dx = V_0 \tag{4}$$

(here f takes into account body forces such as gravitational forces). This free boundary problem is difficult to study for several reasons: No comparison principle holds, so one cannot develop a viscosity solution type approach. Also topology changes seem unavoidable (splitting and merging of droplets) so any notion of weak solutions should account for this possibility.

With I. Kim, we studied this problem when $f(u,x) = u\cos\alpha + x_1 \sin\alpha$ which models the motion of a drop on a plane inclined at an angle α in the x_1 direction. We first showed that (3) must actually be replaced by

$$\min\left\{-V + \frac{1}{2}|\nabla u|^2 - \frac{1}{2}\theta_e^2\,,\, |\nabla u|\right\} = 0 \quad \text{on } \partial\{u>0\} \tag{5}$$

for the problem to be well posed (that is the degenerate case $|\nabla u| = 0$ must be handled differently). We then fully studied this free boundary problem in one dimension proving in particular the existence, uniqueness and stability of traveling wave type solutions, and we obtained some homogenization results in the case where θ_e depends on x.

The next step will be to derive similar results in two dimensions. Many new issues arise in that case and one of the most interesting question is that of the regularity of the free boundary. Indeed, many experiments show the formation of corner and cusp singularities along the contact line of dropplets sliding with large velocity. Our goal is thus to prove the existence of traveling wave like solutions and determine whether this ”simple” model produces such singularities.

In the **lubrication approximation** the height of the (thin) liquid drop solves a degenerate fourth order parabolic equation, known as the **thin film equation**. If we supplement this equation with the static contact angle condition, we are lead to the following free boundary problem:

$$\begin{cases} \partial_t u + \partial_x(m(u)\partial_{xxx} u) = 0 & \text{in } \{u>0\} \\ m(u)\partial_{xxx} u = 0 & \text{on } \partial\{u>0\} \\ |u_x| = \theta_e & \text{on } \partial\{u>0\} \end{cases} \tag{6}$$

where the mobility coefficient $m(u)$ is degenerate (physical coefficients include in particular $m(u) = u$ and $m(u) = u^3 + \Lambda u^s$ for $s \in [1, 2]$). Note that the first free boundary condition is a null flux condition which is enforced by some integral equality (conservation of mass), while the second condition is our contact angle condition.

This problem has been studied mainly in the case $\theta_e = 0$ (known as complete wetting case). In that case, the free boundary can be ignored and the equation assumed to hold in $\mathbb{R}$. The contact angle condition is then recovered as a consequence of some regularity result ($u \in C^1$). F. Otto (CPDE 1998) proved the existence of weak solutions for (6) in the case $m(u) = u$ using the gradient flow structure of the equation in that particular case. In a recent work, I propose a different approach to prove the existence of weak solutions for (6) for a larger class of coefficients m, including $m(u) = u^3 + \lambda u^s$, $s \in [1, 2)$. The idea is to consider the singular equation

$$\partial_t u + \partial_x \Big(m(u) \partial_x [\partial_{xx} u - P_\varepsilon(u)] \Big) = 0 \tag{7}$$

where P_ε is an approximation of a Dirac mass at $u = 0$. I then prove that the solution u^ε of this equation converges to some weak solutions of (6) as $\varepsilon \to 0$.

There are many problems related to the motion of contact lines that I plan to study in the future. In the framework of the quasi-static approximation, for instance, we have focused so far on a particular choice of velocity law at the contact line which yields a nice gradient flow formulation for the problem. Many other velocity laws have been proposed in the literature, and I would like to investigate the influence that a particular choice of a velocity law can have on general features of the solution, such as the speed of the traveling waves and homogenization behavior. In the lubrication approximation framework it would be interesting to study traveling wave type solutions describing the motion of a thin film down an inclined plane and to study the homogenization of the contact angle condition in that framework.

1.2 A nonlocal parabolic equation arising in Hydraulic fractures modeling

Together with C. Imbert, we initiated a research program devoted to the following equation:

$$\partial_t u - \partial_x (u^n \partial_x I(u)) = 0 \tag{8}$$

where $n > 0$ and I is a non-local elliptic operator of order 1 satisfying $I \circ I = -\partial_{xx}$ (in $\mathbb{R}$, we would have $I = (-\partial_{xx})^{1/2}$). This equation arises in the modeling of hydraulic fractures and there is an extensive literature devoted to formal asymptotic and numerical analysis. However, prior to our work, no rigorous existence results were known.

Equation (8) is similar to the thin film equation mentioned above (which corresponds to $I = -\partial_{xx}$). Like the thin film equation, it lacks a comparison principle, so the existence of a non-negative solutions is non-trivial. It also presents some additional difficulties. In particular, the operator I is non-local and the regularity given by the energy inequality does not give the boundedness and continuity of weak solutions even in dimension 1.

We established the existence of non-negative solution for all $n \geq 1$. This work opens many interesting problems, in particular the existence of self-similar solutions (work in preparation), the regularity of the weak solutions and the finite speed of propagation of the support.

It should be noted that our existence result assumes that the equation is satisfied everywhere rather than in the support of u. As for the thin film equation, this corresponds to very particular behavior of the solution at the tip of the fracture (this is known as the zero toughness regime). Taking the rock toughness into account would lead to a free boundary problem similar to the thin film free boundary problem discussed above, but with new difficulties stemming from the non-local character of the operator I.

1.3 Reaction-diffusion equations in heterogeneous media

Reaction-diffusion equations are a very important and well studied class of parabolic equations. One important feature of these equations is the existence of particular solutions which describe fronts propagation (*traveling waves* in homogeneous medium, *pulsating traveling fronts* in periodic medium, *generalized fronts* in general frameworks).

I have contributed to the study of these solutions when the reaction term is the so-called ignition temperature reaction term. In particular, I have studied fronts propagation in non-homogeneous media (this is related to the homogenization of free boundary problems discussed in the first section). In the periodic case, we proved (with L. Caffarelli and K.-A. Lee) the existence of pulsating fronts and showed that the effective speed of propagation of these fronts depends on the infimum of the combustion rate (we also obtained some results in the random setting in one-dimension). In general heterogeneous media, we established (with J.-M. Roquejoffre, Y. Sire, J. Nolen and L. Ryzhik) the existence, uniqueness and stability of generalized fronts in one-dimension.

More recently, I have studied some problems in which the balance between diffusion and reaction is non-standard. One such problem is a boundary reaction-diffusion equation (with the reaction taking place at the boundary of the domain) for which we proved the existence of traveling wave solutions with L. Caffarelli and Y. Sire. Another problem is the fractional reaction-diffusion equation (in which diffusion mechanisms are modeled by a fractional laplacian $(-\Delta)^s$ instead of the usual laplacian), for which we prove the existence of traveling wave solutions in the super-critical case $s > 1/2$ with J.M. Roquejoffre and Y. Sire (in the subcritical case $s < 1/2$, such solutions do not exist - the case critical $s = 1/2$ is, to my knowledge, still open). An important aspect of my works in both of these problem is the characterization of the asymptotic decay of the solution at infinity (which is faster than the standard exponential decay in the first case, but only algebraic in the second case).

1.4 Existence of multiple solutions for a Mean-Curvature equation

With J. Vovelle, we investigate the existence of multiple solutions to the following equation:

$$-\text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f(\lambda, u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{9}$$

where the function $u \mapsto f(\lambda, u)$ is in particular convex with at least linear growth at infinity. When $f(\lambda, u) = H + \lambda u$, this problem arises in the study of capillary pendent drops.

If we replace the mean curvature operator by the Laplace operator, we recover a very classical semilinear problem. We show that there exists a critical value $\lambda^* \in (0, \infty)$ for the parameter λ such that one (or more) solution exists for $\lambda < \lambda^*$, a unique weak solution u^* exists for $\lambda = \lambda^*$ and no solution exist for $\lambda > \lambda^*$. The most challenging problem is the regularity of the extremal solution u^*. We obtain some partial regularity results (L^∞ bound) for the extremal solution and full regularity in the radially symmetric case for power like nonlinearities (in any dimension). Recently, various authors have investigated this problem in one-dimension and obtained complete descriptions of the number of solutions as a function of λ, for various f. Our result is a first step in studying these questions in higher dimension.

2 Transport phenomena

My other topic of interest concerns the study of transport phenomena which are of fundamental importance in many disciplines (physics, biology, engineering etc.). Transport phenomena can be modeled by various PDE and I am mainly interested in fluid type equations (e.g. Navier-Stokes and Euler systems of equations) and kinetic models (e.g. Vlasov and Boltzmann equation) and the connection between these two classes of models.

2.1 Navier-Stokes equations with density dependent viscosity coefficients

The mathematical analysis of the isentropic compressible Navier-Stokes system of equations is famously challenging. When the viscosity coefficients are constant, the existence of weak solutions was first proved by P.L. Lions in the early 90's for isentropic gas. I am primarily interested in models featuring coefficients that depend on the density ρ (or on the temperature).

With A. Vasseur, we studied such models, when the viscosity coefficients are degenerate (i.e. they vanish when the density vanishes). We proved the stability of a class of weak solutions without restriction on the size of the initial data and in presence of vacuum in any dimension. In one-dimension and for smooth initial density bounded away from zero, we prove global existence of strong solutions even when the viscosity coefficient μ is degenerate.

Also with A. Vasseur, we introduced new methods, based on some classical methods from the Calculus of Variations, to derive estimates for the solutions of Navier-Stokes equations (and with similar techniques we derive lower bounds on the temperature in the full compressible Navier-Stokes system of equations).

2.2 From Kinetic models to fluid equations

Kinetic models provide an accurate description of many transport phenomena. They are simple to write but can be costly to simulate numerically (large number of variables). This is one motivation for the study of asymptotic regimes that yield hydrodynamic type equations modeling the evolution of macroscopic quantities such as the density of particles (these can be diffusion type equations or fluid equations such as Euler of Navier-Stokes systems of equations).

Electrons transport in semiconductor. My earlier work on this topic was concerned with the study of diffusion regimes for some kinetic models arising in the modeling of electrons transport in semiconductors. My most interesting results concerned models in which collisions with the boundaries were the dominant mechanism in the relaxation toward an equilibrium.

Flocking models. More recently, I have been studying kinetic models arising in the study of flocking (or swarming) phenomena. These models describe basic "social" interactions between individuals (birds or fishes for instance) such as long range attraction, short range repulsion, alignment, noise, etc. My main interest is the derivation of fluid type equations from these simple kinetic models. With T. Karper and K. Trivisa, we consider a model which includes a short range strong alignment term which we obtain as singular limit of a correction to the Cucker-Smale model first introduced by Motsch and Tadmor. We prove the existence of weak solutions for this model and rigorously derive an Euler system of equations.

I have many projects related to this work, in particular involving the addition of attraction-repulsion phenomena to these equations. Many issues then arise, in particular due to the lack of convexity (at least in the classical sense) of the corresponding entropy.

Kinetic models and anomalous diffusion. With C. Mouhot and S. Mischler, I have initiated the investigation of anomalous diffusion regimes for kinetic equations. Such regimes arise when the mean squared displacement of the particles is not a linear function of time. The corresponding macroscopic equations are fractional diffusion equations. With several co-authors, I have developed methods which can be used to deal with a wide range of models (linear Boltzmann operator, Lévy-Fokker-Planck operator for instance).

This work has a lot of potential applications, such as the derivation of fractional fluid equation (these methods have been used recently to derive fractional Stokes equation from Boltzmann type equations), the study of energy transport in anharmonic chains (in which anomalous heat diffusion appears to be the rule rather than the exception) or the study of weak turbulence.