DIAMETER RIGIDITY OF SPHERICAL POLYHEDRA

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ABSTRACT. We classify geodesically complete compact 2-dimensional spherical polyhedra X of diameter and injectivity radius π. If X contains a point whose link has diameter $> \pi$ then either (i) X is the spherical join of the finite set P of point whose link has diameter $> \pi$ with the metric graph $E = \{x \in X : d(x,P) = \pi/2\}$ whose diameter is $> \pi$ or (ii) X is a hemispherex, that is, X is obtained by attaching hemispheres to the standard sphere S along great circles so that not all of them pass through the same pair of opposite points in S. If all links of X have diameter π then either (i) X is a thick spherical building of type A_3 or B_3, or (ii) X is the spherical join of a finite set with a graph E of diameter π. In each case the injectivity radius of E is π.

1. INTRODUCTION

The diameter rigidity question considered in this paper is motivated by the rank rigidity problem for spaces of nonpositive curvature. The rank of a complete simply connected space Y of nonpositive curvature is ≥ 2 if every geodesic in Y is contained in an isometrically embedded convex Euclidean plane. The Rank Rigidity Problem asks for a classification of such spaces, at least when the isometry group of Y is large.

By a Euclidean (respectively, spherical) k-simplex we mean the intersection of $k+1$ closed half spaces in $\mathbb{R}^k$ (respectively, hemispheres in S^k) in general position. A polyhedron with a metric is called Euclidean (respectively, spherical) if it admits a triangulation into Euclidean (respectively, spherical) simplices. We are interested in the rank rigidity of Euclidean polyhedra with compact quotients and expect them to be Euclidean buildings or products. The rank rigidity for $\dim Y = 2$ is not difficult and is contained in [BB, Section 6].

For a Euclidean polyhedron Y and $p \in Y$ we denote by S_pY the link of Y at p, that is, the set of directions at p. Clearly S_pY is a spherical polyhedron and, if Y has nonpositive curvature, then the injectivity radius of S_pY is π. Furthermore, if the rank of Y is ≥ 2 then S_pY is geodesically complete and has diameter π. Hence for $\dim Y = 3$ the link $X = S_pY$ is a geodesically complete compact 2-dimensional spherical polyhedron of diameter and injectivity radius π. The aim of this paper is to classify such spaces X. Here are examples of geodesically complete compact spherical polyhedra of diameter and injectivity radius π.

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1.1 Spherical building. If X is a spherical building then X carries a natural metric for which the apartments are unit spheres. For this metric the diameter and injectivity radius of X are π. If Y is a Euclidean building of dimension $n \geq 2$ with the natural metric then every geodesic in Y is contained in a isometrically embedded convex Euclidean n-space. The link of a vertex in Y is a spherical building of dimension $n-1$ which has injectivity radius and diameter π, see [Br, Ch.VI].

1.2 Spherical join. Let Y_i be a Euclidean polyhedron, $y_i \in Y_i$ and $X_i = S_{y_i}Y_i$, $i = 1, 2$. Then the *spherical join* $X_1 * X_2$ is the link of $(y_1, y_2) \in Y_1 \times Y_2$ with angular distance. Clearly $X_1 * X_2$ is a spherical polyhedron of dimension $\dim X_1 + \dim X_2 + 1$ and naturally contains X_1 and X_2 as subpolyhedra. If Y_1 and Y_2 are geodesically complete and of nonpositive curvature then $Y_1 \times Y_2$ has rank ≥ 2 and $X_1 * X_2$ is geodesically complete and has injectivity radius and diameter π.

If $\dim Y_1 = 1$ then X_1 is a finite set and $X_1 * X_2$ admits the following simple description which is sufficient for this paper. For each $p \in X_1$ the *spherical cone* C_p over X_2 with *pole* p is the product $X_2 \times [0, \pi/2]$ in which $X_2 \times \{0\}$ is identified with p and the distance d is given by

$$\cos d((y,s),(z,t)) = \cos s \cos t + \sin s \sin t \cos \min\{\pi, d_2(y,z)\},$$

where d_2 is the distance in X_2. The spherical join $X_1 * X_2$ is the disjoint union of the spherical cones C_p, $p \in X_1$, identified along the *equators* $X_2 \times \{\pi/2\}$.

1.3 Hemispherex. A spherical complex is called a *hemispherex* if it is obtained from the unit sphere by attaching unit hemispheres along great hyperspheres so that no pair of antipodal points on the sphere belongs to all hemispheres. A 1-dimensional hemispherex is called a *semicirclex*. It is not difficult to see that the injectivity radius of a hemispherex is π, the diameter of a semicirclex is $> \pi$ and the diameter of a hemispherex of dimension ≥ 2 is π, see Figure 1.

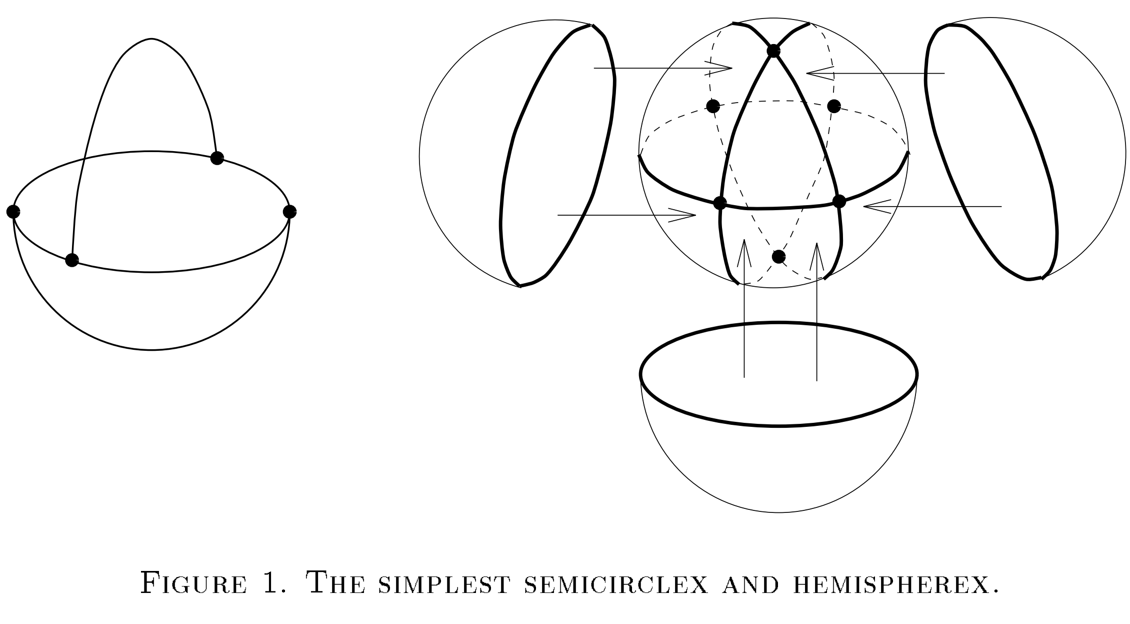

FIGURE 1. THE SIMPLEST SEMICIRCLEX AND HEMISPHEREX.

We refer to the initial sphere (circle) of a hemispherex X as the *central sphere (circle)* of X. The central circle of a semicirclex is distinguished as the set of points for which the maximum of the distance function from x is π. For all other points it is $> \pi$. In dimension $n > 1$, each essential vertex x of a hemispherex X lies on its central sphere S, the link $S_x X$ is an $(n-1)$-dimensional hemispherex, and S is distinguished recursively by the property that S is tangent to the central sphere of $S_x X$.

The Euclidean cone $C(H)$ over a hemispherex H of dimension ≥ 2 is a complete simply connected Euclidean polyhedron of nonpositive curvature and rank ≥ 2. This is the only known to us example of a higher rank space of nonpositive curvature which is not a Euclidean building, a symmetric space or a product. Obviously, every isometry of $C(H)$ must fix the vertex and hence is of finite order. In particular, $C(H)$ does not admit compact factors.

The following theorem is the main result of this paper.

Theorem. *Let X be a geodesically complete compact 2-dimensional spherical polyhedron of diameter and injectivity radius π. Then X is*

(1) *either a thick spherical building of type A_3 or B_3,*
(2) *or a spherical join of a finite set with a metric graph of injectivity radius $\geq \pi$,*
(3) *or a hemispherex.*

The paper is organized as follows. After discussing preliminaries in Section 2 we present the main argument in Section 3. In Section 4 we consider the case when X is not dimensionally homogeneous. In Section 5 we discuss the case when all links of X have diameter π. The main results of Sections 3,4 and 5 are stated as theorems at the beginning of each section. The above theorem is a consequence of these theorems.

2. Preliminaries

Let X be a compact spherical polyhedron. Fix a triangulation of X into spherical simplices. Although the definitions below use this triangulation, they do not depend on the particular choice of triangulation.

Let $x \in X$ and A be a closed k-simplex containing x. View A as a subset of S^k and set S_xA to be the set of unit tangent vectors ξ at x such that a nontrivial initial segment of the geodesic with initial velocity ξ is contained in A. If $B \subset A$ is another closed simplex containing x then naturally $S_xB \subset S_xA$. We define the *link* S_xX of X at x by

$$S_xX = \cup_{A \ni x} S_xA \, ,$$

where the union is taken over all closed simplices containing x. If the maximal dimension of a simplex adjacent to x is n then S_xX has dimension $n-1$. Angles in S_xA induce a natural length metric d_x on S_xX which turns it into a spherical polyhedron. For $\xi, \eta \in S_xX$ define

$$\angle(\xi, \eta) = \min(d_x(\xi, \eta), \pi) \, .$$

For every $x \in X$ there is a neighborhood U of x with polar coordinates (ξ, s), $\xi \in S_xX$, $0 \le s \le \varepsilon$, centered at x and such that

$$\cos d((\xi, s), (\eta, t)) = \cos s \cos t + \sin s \sin t \cos \angle(\xi, \eta) \, .$$

In other words, locally X is isometric to a spherical cone.

Let X be 2-dimensional. We refer to the simplices of X as vertices, edges and faces. A vertex x of X is called *essential* if S_xX is not isometric to the metric graph consisting of two points with $m \ge 0$ edges of length π connecting them. An edge of X is called *essential* if it is not adjacent to exactly two faces. A *maximal* face of X is a connected component of the complement of the union of essential edges and vertices.

A curve $\gamma : I \to X$ is a *geodesic* if it has constant speed and is locally distance minimizing. It it easy to show that a curve $\gamma : [a, b] \to X$ with constant speed is a geodesic if and only if there is a subdivision

$$a = t_0 < t_1 < \dots < t_m = b$$

such that

(1) $\gamma([t_{j-1}, t_j])$ is contained in a closed simplex $A_j \subset X$ and $\gamma : [t_{j-1}, t_j] \to A_j$ is a standard geodesic segment in A_j, $1 \le j \le m$;
(2) $d_{\gamma(t_j)}(-\dot\gamma(t_j), \dot\gamma(t_j)) \ge \pi$, $1 \le j \le m-1$.

We refer to $-\dot\gamma(t)$ and $\dot\gamma(t)$ as the *incoming* and *outgoing* directions of γ in the link $S_{\gamma(t)}X$. For geodesics γ and σ with $\gamma(s) = \sigma(t) =: x$ we set

$$\angle_x(\gamma, \sigma) = \angle(\dot\gamma(s), \dot\sigma(t)) \, .$$

We say that X is *geodesically complete* if every geodesic $\gamma : I \to X$ can be extended to a geodesic $\tilde\gamma : \mathbb{R} \to X$. Observe that X is geodesically complete if and only if for every $x \in X$ and every $\xi \in S_xX$ there is $\eta \in S_xX$ with $d_x(\xi, \eta) \geq \pi$. If X is geodesically complete then it has no *boundary simplices*, that is, simplices adjacent to exactly one simplex of a higher dimension.

For $x, y, z \in X$ with $0 < d(x, y), d(x, z) < \operatorname{inj} X$ we define $\angle_x(y, z)$ to be the angle at x between the unique minimal connections from x to y and x to z. In general, angles need not depend continuously on the end points of their sides in singular spaces. For example, let X be the *tripod*, that is, the simplicial complex consisting of three edges e, f, g with one common vertex m, the midpoint of the tripod. Let y, z be the other vertices of f and g respectively and let x be a point on e. Then the angle $\angle_x(y, z)$ is zero if $x \neq m$ and π if $x = m$.

2.1 Lemma. *Let σ be a unit speed geodesic with $\sigma(0) = x \in X$. For a sequence (t_n) of positive numbers tending to 0 let γ_n be a unit speed geodesic starting at $\sigma(t_n)$. Assume $\angle_{\sigma(t_n)}(\sigma, \gamma_n) < \pi$ for all n and that γ_n converges to a unit speed geodesic γ starting at x.*

Then $\angle_x(\sigma, \gamma) \leq \liminf_{n \to \infty} \angle_{\sigma(t_n)}(\sigma, \gamma_n)$.

Proof. Since X is a spherical polyhedron, locally near x the space X is isometric to the spherical cone over the space of directions S_xX. Hence there is a uniform $\varepsilon > 0$ such that $\sigma([t_n, t_n + \varepsilon])$ and $\gamma_n([0, \varepsilon])$ are the legs of an immersed isosceles spherical triangle C_n with angle $\angle_{\sigma(t_n)}(\sigma, \gamma_n)$ at the vertex. A subsequence of (C_n) converges to an immersed isosceles spherical triangle with legs $\sigma([0, \varepsilon])$ and $\gamma([0, \varepsilon])$ and the inequality for the angle follows. $\square$

2.2 Berger's Lemma. *Let $y \in X$ and suppose $x \in X$ is a local maximum of the distance function $d(y, .)$.*

Then for every $\xi \in S_xX$ there is $\eta \in S_xX$ such that $d_x(\xi, \eta) \leq \dfrac{\pi}{2}$ and η is tangent to a shortest (of length $d(y, x)$) connection from x to y.

Proof. Let σ be a geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = \xi$. Locally near x the space X is isometric to the spherical cone over the space of directions S_xX. Hence for $t > 0$ small enough the link of X at $\sigma(t)$ is isometric to the spherical join of the set $\{\dot{\sigma}(t), -\dot{\sigma}(t)\}$ with the space of directions at ξ in S_xX. By Lemma 2.1, if η does not exist, then every shortest connection γ_t from $\sigma(t)$ to y satisfies $\angle(\dot{\sigma}(t), \dot{\gamma}_t(0)) > \dfrac{\pi}{2}$. By the structure of the link $S_{\sigma(t)}$, we have $\angle(-\dot{\sigma}(t), \dot{\gamma}_t(0)) < \dfrac{\pi}{2}$. Therefore, $d(y, \sigma(t))$ is strictly increasing in t, and hence x is not a local maximum of $d(y, .)$. This is a contradiction. $\square$

The *injectivity radius* $\operatorname{inj} X$ of X is the supremum of the set of $r \geq 0$ such that any geodesic segment of length $\leq r$ is the unique minimal connection between its ends. We will need the following easy fact.

2.3 Lemma. *Let X be a space of injectivity radius π and σ be a geodesic loop in X of length 2π.*

Then σ is a closed geodesic.

Proof. Let $p = \sigma(\pi/2)$, $q = \sigma(3\pi/2)$. Then p and q are connected by the geodesic $\sigma([\pi/2, 3\pi/2])$ of length π and by the curve $c = \sigma([0, \pi/2]) \cup \sigma([3\pi/2, 2\pi])$, also of

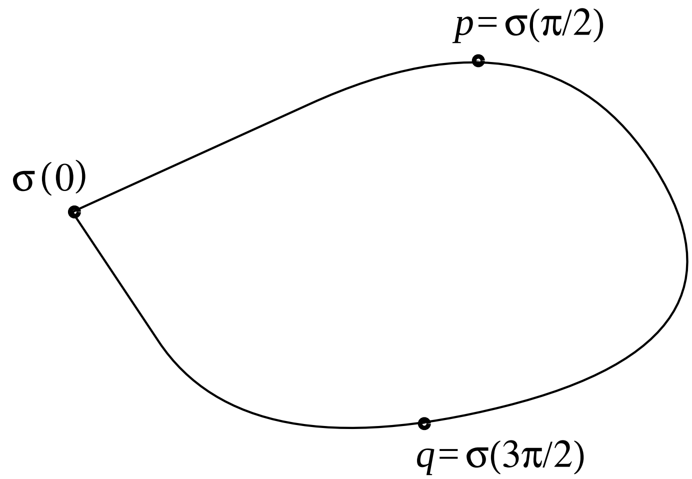

FIGURE 2

length π, see Figure 2. Since the injectivity radius of X is π, c is a curve of minimal length and hence a geodesic. $\square$

2.4 Lemma. *Let* $\dim X = 2$ *and* $\operatorname{inj} X = R > 0$. *Then for every* $x \in X$ *there is a finite subset* F *of* S_xX *such that a unit speed geodesic* $\sigma : [0, R) \to X$ *with initial direction* $\dot\sigma(0) \in S_xX \setminus F$ *never hits an essential vertex of* X, *except possibly* x *at time* 0.

Proof. Since X is compact, there are only finitely many essential vertices in X. Hence there is $\varepsilon > 0$ such that no unit speed geodesic $\sigma : [0, R] \to X$ with $\sigma(0) = x$ hits an essential vertex in time $t \in (R - \varepsilon, R)$. Since the injectivity radius is R, the lemma follows. $\square$

Suppose that for $x \in X$ the injectivity radius of S_xX is $< \pi$. Then there are two directions $\xi, \eta \in S_xX$ connected by two different geodesics of length $\alpha, \beta < \pi$ in S_xX. If γ, σ are unit speed geodesics in X with $\dot\gamma(0) = \xi$, $\dot\sigma(0) = \eta$, then for small enough $s, t > 0$ the points $\gamma(s)$, $\sigma(t)$ are connected in X by two different geodesics of length

$$\arccos(\cos s \cos t + \sin s \sin t \cos \alpha)\,, \quad \arccos(\cos s \cos t + \sin s \sin t \cos \beta)\,.$$

As $s, t \to 0$, both lengths tend to 0, and hence $\operatorname{inj} X = 0$. Therefore $\operatorname{inj} X > 0$ implies that all links of X have injectivity radius $\geq \pi$. This in turn implies that X is of curvature ≤ 1 in the triangle comparison sense of Alexandrov. If $\operatorname{inj} X \geq \pi$ then the triangle comparison holds for triangles in X of perimeter $< 2\pi$ and X is a $CAT(1)$-space in the sense of Gromov, see [Al,Gr,B1,B2].

By a *spherical lune* we mean a subset of the standard sphere S^2 bounded by two meridians making (equal) angles $< \pi$ at the poles.

2.5 Lune Lemma. *Let* $\operatorname{inj} X \geq \pi$ *and let* B *be a geodesic biangle in* X *with vertices* $x_1, x_2 \in X$ *and sides of length* π. *Suppose the angle* α_1 *at* x_1 *is less than* π.

Then the second angle α_2 equals α_1 and B is the boundary of a spherical lune.

Proof. Let y, z be the midpoints of the sides. Then $d(y,z) < \pi$ since otherwise the union of the geodesics from y to x_1 and from x_1 to z would be a minimal geodesic, contradicting the assumption $\alpha_1 < \pi$. Now consider a lune on the standard sphere from the north pole to the south pole such that the distance between the midpoints of the boundary meridians is $d(y,z)$. We obtain in this way simultaneously the comparison triangles for $\Delta(x_i, y, z)$. Note that the perimeter of these triangles is $< 2\pi$. Mark by a prime the objects on the sphere corresponding to objects in X. Let β_i be the angle at y between x_i and z and γ_i the one at z between x_i and y. We have

$$\beta_1 + \beta_2 \geq \pi, \quad \gamma_1 + \gamma_2 \geq \pi$$

since the sides of the biangle are geodesics and

$$\beta_1' = \beta_2' = \pi/2 = \gamma_1' = \gamma_2'$$

since we are dealing with a lune on the sphere and y', z' are the midpoints of the sides. As we pointed out right before the lemma, X is a $CAT(1)$-space and the triangle comparison applies to triangles of perimeter $< 2\pi$. Therefore, $\beta_i \leq \beta_i'$ and $\gamma_i \leq \gamma_i'$. Hence we have equalities for these angles and the equality discussion in the comparison applies, see [B2]. □

A subset M is *r-convex* if every geodesic segment of length $< r$ with ends in M is contained in M.

2.6 Corollary. *Let X be a spherical polyhedron of diameter and injectivity radius π and $x_1, x_2 \in X$ be points at maximal distance π from each other. Denote by $M_i \subset S_{x_i}$, $i = 1, 2$, the set of initial directions to minimal geodesics from x_i to the other point and by $I : M_1 \to M_2$ the map which sends the outgoing direction of such a minimal geodesic to its incoming direction at x_2.*

Then M_i is a π-convex and $\pi/2$-dense subset of S_{x_i} and I is an isometry in the angle metric. □

The following easy statement is proved in [BB, Lemma 6.1].

2.7 Proposition. *Let Γ be a finite graph with all vertices of degree at least 2 and with a length metric of diameter and injectivity radius π. Then*

(1) *either there exist finite subsets $A, B \subset \Gamma$ such that Γ is the bipartite graph $\Gamma(A, B)$ in which each $a \in A$ is connected to each $b \in B$ by an edge of length $\pi/2$,*
(2) *or Γ is a thick spherical building with edges of length π/k, $k \geq 3$, whose apartments are simple loops with $2k$ edges.*

For $|A| = |B| = 2$ the graph $\Gamma(A, B)$ is a unit circle. For $|A| = 2$ and $|B| = m$ the graph $\Gamma(A, B)$ consists of two points connected by m edges of length π.

2.8 Lemma. *Let Γ be a finite graph with all vertices of degree at least 2 and with a length metric of injectivity radius π. Suppose $A \subset \Gamma$ satisfies*

(1) *$d(\eta, \zeta) = \pi$ for all $\eta \neq \zeta \in A$ and*
(2) *$d(\eta, \theta) < \pi$ for all $\eta \in A$ and $\theta \in \Gamma \setminus A$.*

Then Γ *is the bipartite graph* $\Gamma(A, B)$ *with* $B = \{\theta \mid d(\theta, A) = \pi/2\}$.

Proof. Let $\theta \in \Gamma$ be a point with $d(\theta, \eta) = \pi/2$ for some $\eta \in A$. Since the injectivity radius of Γ is π, it suffices to prove that $d(\theta, \zeta) = \pi/2$ for all $\zeta \in A$. Suppose $d(\theta, \zeta) > \pi/2$ for some $\zeta \in A$ and let c be a shortest curve from ζ to θ. By (2), the length l of c satisfies $\pi/2 < l < \pi$. Extend c to a simple unit speed geodesic of length π. Then $d(c(\pi), \eta) < \pi$, and hence by (1), $c(\pi) \notin A$. On the other hand, $d(c(\pi), \zeta) = \pi$, and hence by (2), $c(\pi) \in A$, which is a contradiction. $\square$

3. Spaces with links of diameter $> \pi$

In this section a point $x \in X$ is called a *pole* if the diameter of the link S_xX (with respect to d_x) is $> \pi$. A direction $\xi \in S_xX$ is called a *spreading direction* if there is $\eta \in S_xX$ with $d_x(\xi, \eta) > \pi$. In this section we will prove the following special case of our main theorem.

Theorem 3.1. *Let X be a compact 2-dimensional spherical polyhedron of diameter and injectivity radius π. Assume that every edge of X is adjacent to at least two faces and that X contains a pole.*

Then X is either a spherical join or a hemispherex.

Throughout this section we assume that X is a compact 2-dimensional spherical polyhedron of diameter and injectivity radius π and that every edge of X is adjacent to at least two faces. In particular, X is geodesically complete and for every pole $x \in X$ and spreading direction $\xi \in S_xX$ there is a nontrivial arc of directions $\eta \in S_xX$ such that $d_x(\xi, \eta) > \pi$.

3.2 Lemma. *Let $x \in X$ be a pole, $\xi \in S_xX$ be a spreading direction and $\sigma : [0, \pi] \to X$ be a unit speed geodesic with $\dot\sigma(0) = \xi$.*

Then for $0 < t < \pi/2$ or $\pi/2 < t < \pi$,

(1) *the link $S_{\sigma(t)}X$ is the graph $\Gamma(A, B)$ with $A = \{\dot\sigma(t), -\dot\sigma(t)\}$ and B the set of midpoints of (at least two) edges of length π connecting $\dot\sigma(t)$ and $-\dot\sigma(t)$.*

In particular, $\sigma(t) \notin \mathcal{V}_X$ and

(2) *if $\sigma_1 : [0, \pi/2] \to X$ is a geodesic with $\dot\sigma_1(0) = \dot\sigma(0)$, then $\sigma_1(t) = \sigma(t)$, $0 \le t \le \pi/2$;*

(3) *if $\sigma_2 : [0, \pi/2] \to X$ is a geodesic with $\dot\sigma_2(0) = \dot\sigma(\pi/2)$, then $\sigma_2(\tau) = \sigma(\tau + \pi/2)$, $0 \le \tau \le \pi/2$.*

Proof. By Lemma 2.4, there is a continuous family $\gamma_s : [0, \pi] \to X$, $-\varepsilon \le s \le \varepsilon$, such that

(i) $\gamma_s(0) = x$ for all $s \in (-\varepsilon, \varepsilon)$;
(ii) $\gamma_s(t) \notin \mathcal{V}_X$ for all $s \in (-\varepsilon, \varepsilon)$ and $t \in (0, \pi)$;
(iii) $s \mapsto \eta_s = \dot\gamma_s(0)$ is a unit speed curve which lies in an open edge of S_xX and satisfies $d_x(\xi, \eta_s) > \pi$.

Since no γ_s passes through an essential vertex, the union $\cup\gamma_s(t)$ is isometric to a spherical lune with angle 2ε. Fix $t \in (0, \pi)$, $t \ne \pi/2$. Since essential edges of X are geodesics, by elementary spherical geometry, an open subarc $c : I \to X$ of the curve $s \mapsto \gamma_s(\pi - t)$ is contained in a maximal face F of X. Then c is a smooth curve in F with constant speed and $d(c(\cdot), \sigma(t)) \equiv \pi$. Hence any minimal (of length π) geodesic from c to $\sigma(t)$ is perpendicular to c. By Lemma 2.2, for every s there are exactly two minimal geodesics from $c(s)$ to $\sigma(t)$ with initial directions $\pm \frac{d}{d\tau}\gamma_s(\tau)\Big|_{\pi - t}$. Therefore, their concatenation is a geodesic loop α_s of length 2π at $\sigma(t)$, see Figure 3. By construction, α_s contains the union of $\sigma([0, t])$ and $\gamma_s([0, \pi - t])$ as a subarc of length π. We parameterize α_s by arclength so that $\dot\alpha_s(0) = -\dot\sigma(t)$.By Lemma 2.3, α_s is a closed geodesic. By Lemma 2.2, the balls of

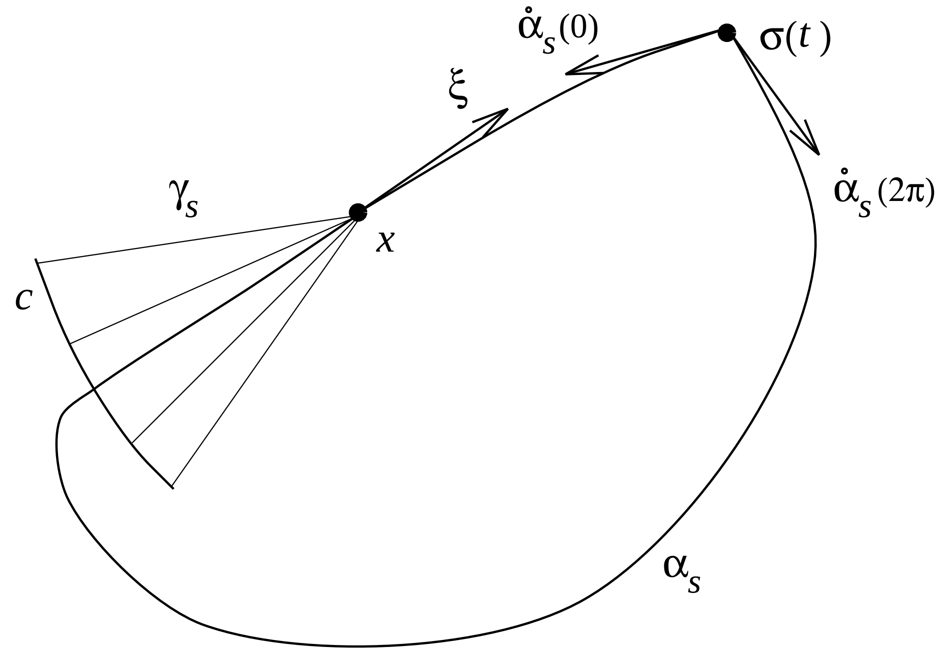

FIGURE 3

radius $\pi/2$ about $\dot\alpha_s(0)$ and $-\dot\alpha_s(2\pi)$ cover the link $S_{\sigma(t)}X$. It follows that $S_{\sigma(t)}X$ is the graph with two vertices, $\dot\alpha_s(0) = -\dot\sigma(t)$ and $-\dot\alpha_s(2\pi)$, connected by edges of length π. In particular, $-\dot\alpha_s(2\pi) = \dot\sigma(t)$. This concludes the proof. □

3.3 Lemma. *Let $x \in X$ be a pole, $\xi \in S_xX$ be a spreading direction and $\sigma : [0, \pi/2] \to X$ be a unit speed geodesic with $\dot\sigma(0) = \xi$. Then*

(1) *the link $S_{\sigma(\pi/2)}X$ is the bipartite graph $\Gamma(A, B)$ with*

$$B = \{\zeta : d_{\sigma(\pi/2)}(-\dot\sigma(\pi/2), \zeta) = \pi/2\} \text{ and}$$

$$A = \{\zeta : d_{\sigma(\pi/2)}(B, \zeta) = \pi/2\} \ni -\dot\sigma(\pi/2);$$

(2) *for any $\eta \in S_xX$ with $d_x(\xi, \eta) > \pi$ and any $\zeta \in A$, $\zeta \neq -\dot\sigma(\pi/2)$, there is a unique continuation of σ to a closed geodesic $\sigma : [0, 2\pi] \to X$ with $\dot\sigma(\pi/2) = \zeta$ and $-\dot\sigma(2\pi) = \eta$; for any such continuation $\sigma(\pi)$ is a pole.*

Proof. **Many of the objects used in this argument are shown in Figure 4.** As in the previous proof, there is a continuous family $\gamma_s : [0, \pi] \to X$, $-\varepsilon \leq s \leq \varepsilon$, such that

(i) $\gamma_s(t) \notin \mathcal{V}_X$ for all $s \in (-\varepsilon, \varepsilon)$ and $t \in (0, \pi)$;
(ii) $s \mapsto \eta_s = \dot\gamma_s(0)$ is a unit speed curve which lies in an open edge of S_xX and satisfies $d_x(\xi, \eta_s) > \pi$.

Since no γ_s passes through an essential vertex, the union $\cup\gamma_s(t)$ is isometric to a spherical lune with angle 2ε.

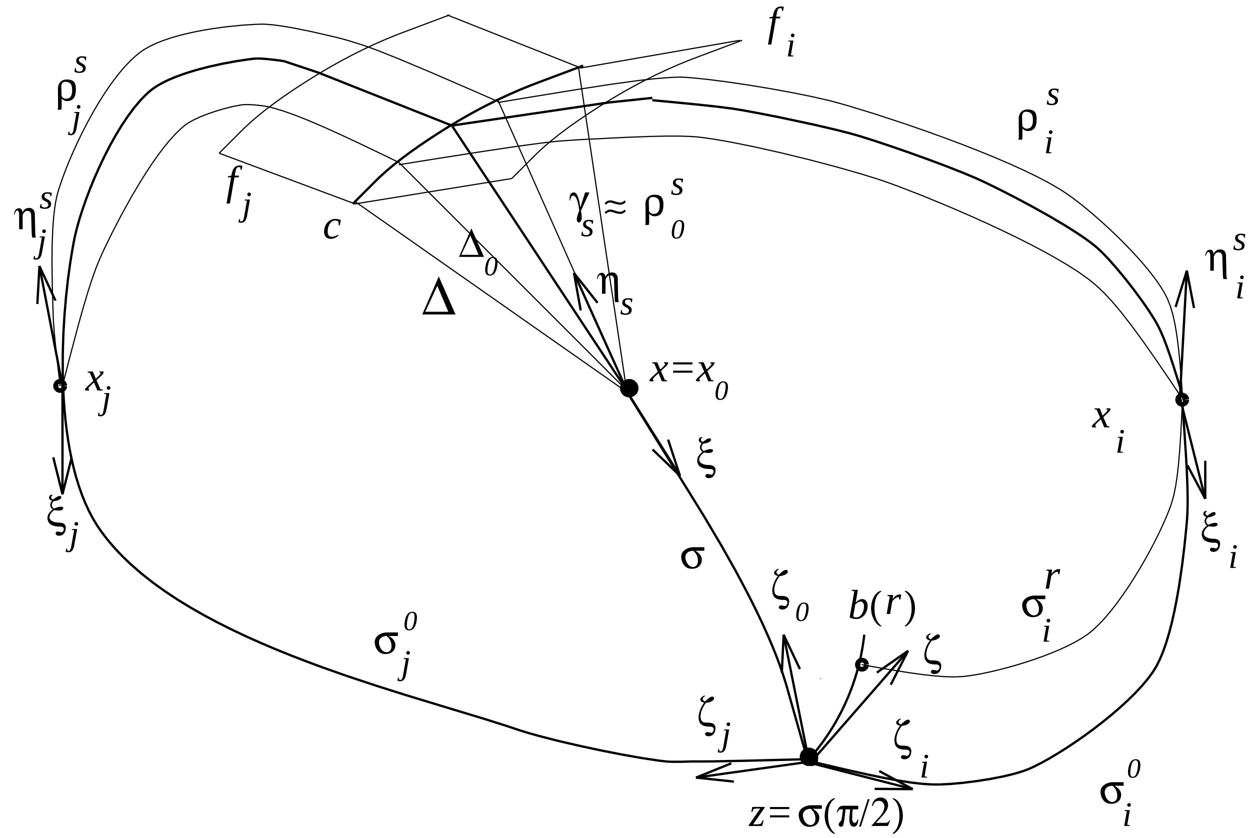

FIGURE 4

The curve $s \mapsto \gamma_s(\pi/2)$ is a unit speed geodesic in X, and hence an open subarc $c : I \to X$ of this curve is either contained in a maximal face of X or is contained in an essential edge of X. A neighborhood of c consists of local faces $f_0, ..., f_k$, $k \geq 1$, adjacent to c with, say, $-\dot{\gamma}_s(\pi/2)$ pointing into f_0. The link $S_{c(s)}X$ is the graph with vertices $\pm\dot{c}(s)$ and $k+1$ edges ε_i^s of length π connecting them and representing the local faces f_i, $0 \leq i \leq k$. The midpoints δ_i^s of these edges lie at distance $\pi/2$ from $\pm\dot{c}(s)$ with $\delta_0^s = -\dot{\gamma}_s(\pi/2)$. Observe that the distance in X from $c(s)$ to $z = \sigma(\pi/2)$ is π and, by elementary geometry, any minimal geodesic from c to z is perpendicular to c. Hence, by Lemma 2.2, there are precisely $k+1$ minimal geodesics $\rho_i^s : [0, \pi] \to X$ from $c(s)$ to z and their initial directions are δ_i^s, $0 \leq i \leq k$. By construction,

$$\rho_0^s(\tau) = \gamma_s(\pi/2 - \tau) \quad 0 \leq \tau \leq \pi/2$$
$$\rho_0^s(\tau) = \sigma(\tau - \pi/2), \quad \pi/2 \leq \tau \leq \pi.$$

Hence the geodesics ρ_0^s focus at $x_0 := x$ and

$$\Delta_0 := \{\rho_0^s(\tau) : 0 \leq \tau \leq \pi/2,\ s \in I\}$$

is an isosceles spherical triangle. Now let $1 \leq i \leq k$. We apply Lemma 3.2 to η_s (instead of ξ) and to the concatenation γ_i^s of γ^s with $\rho_i^s|[0, \pi/2]$ (instead of σ) and conclude that $\cup\gamma_i^s(\tau)$ is a spherical lune, focusing at $x = x_0$ and at another point $x_i \in X$ with $d(x, x_i) = \pi$. Hence

$$\Delta_i := \{\rho_i^s(\tau) : 0 \le \tau \le \pi/2,\ s \in I\}$$

is an isosceles spherical triangle with apex $x_i = \rho_i^s(\pi/2)$ and

$$\rho_i^s(\tau) = \rho_i^r(\tau), \quad \pi/2 \le \tau \le \pi,\ s, r \in I.$$

Let

$$\xi_i = \dot\rho_i^s(\pi/2) \text{ and } \eta_i^s = -\dot\rho_i^s(\pi/2).$$

Then $d_{x_i}(\xi_i, \eta_i^s) \ge \pi$ since ρ_i^s is a geodesic. Now $s \mapsto \eta_i^s$ is a simple unit speed curve in $S_{x_i}X$ and $S_{x_i}X$ is a graph. Hence we have $d_{x_i}(\xi_i, \eta_i^s) > \pi$ for all s by passing to a subinterval of I if necessary. In particular, x_i is a pole.

Let $\zeta_i = -\dot\rho_i^s(\pi) \in S_zX$ be the direction pointing at x_i, $0 \le i \le k$. The union $\rho_i^s \cup \rho_j^s$, $i \ne j$, is a geodesic loop at z of length 2π and hence, by Lemma 2.3, is a closed geodesic through z. Therefore, the angle between ζ_i and ζ_j, $i \ne j$, is π. Let $\zeta \in S_zX$ be a direction such that $d_z(\zeta, \zeta_i) = \pi/2$ for some $i \in \{0, ..., k\}$. To finish the proof of (1), we need to show that $d_z(\zeta, \zeta_j) = \pi/2$ for every $j \in \{0, ..., k\}$, $j \ne i$.

For $\beta \in (0, \pi/2)$ small enough let $b : [0, \beta) \to X$ be a unit speed geodesic with $b(0) = z$, $\dot b(0) = \zeta$ and such that $b((0, \beta))$ is either contained in a maximal face or in an essential edge of X. Let σ_i^r be the minimal unit speed geodesic from x_i to $b(r)$, $0 \le r < \beta$. Then $\sigma_i^0(\tau) = \rho_i^s(\pi/2 + \tau)$.

For a small enough β we have that $d_{x_i}(\dot\sigma_r^i(0), \eta_i^s) > \pi$ for s in an open subinterval $J \subset I$ and $r \in [0, \beta)$. Hence the concatenation of $\rho_i^s|[0, \pi/2]$, $s \in J$, with σ_i^r is a geodesic.

By Lemma 3.2, applied to $\dot\sigma_r^i(0)$ (instead of ξ) and (any extension to $[0, \pi]$ of) $\sigma_i^r(\tau)$, we conclude $\sigma_r^i(\tau) \notin \mathcal{V}_X$, $0 < \tau < \pi/2$. It follows that $\sigma_r^i(\pi/s) = b(r)$, $r \in [0, \beta)$.

The first part of the proof shows that for any $j \in \{0, ..., k\}$ and any $s \in J$, there is a minimal geodesic (of length π) from $c(s)$ to $b(r)$ through x_j and these geodesics are perpendicular to b. It follows that $S_{b(r)}X$, $r > 0$, is the graph with vertices $\pm\dot b(r)$, connected by $k + 1$ edges of length π whose midpoints are the directions of minimal geodesics connecting $b(r)$ to $c(s)$, $s \in J$, through x_j, $0 \le j \le k$. By Lemma (angle lemma in prelim), $d_z(\zeta, \zeta_j) = \pi/2$. Therefore $S_zX = \Gamma(A, B)$ with $A = \{\zeta_i : 0 \le i \le k\}$ and $B = \{\zeta : d_z(\zeta, A) = \pi/2\}$. This concludes the proof of (1) and shows that (2) holds for an open and dense set of η's satisfying $d_x(\xi, \eta) > \pi$. □

3.4 Corollary. *Let $x \in X$ be a pole, $\xi \in S_xX$ be a spreading direction and $\sigma : [0, \pi] \to X$ be a unit speed geodesic with $\dot\sigma(0) = \xi$.*

Then for $0 < t < \pi$ the link $S_{\sigma(t)}X$ has diameter π and $\sigma(\pi)$ is a pole. □

For $k \ge 2$, a *k-pod* is a graph consisting of a central vertex, called the *middle point*, and k edges of length $\pi/2$ adjacent to it.

3.5 Corollary. *Let $z \in X$ be a pole and suppose there is an open local edge β in S_zX adjacent to $\eta \in S_zX$, such that all directions in β are spreading. Suppose $\gamma_s : [0, \pi] \to X$, $0 \le s \le \delta$ for some $\delta > 0$, is a continuous family of unit speed* *geodesics such that* $\dot\gamma_0(0) = \eta$, $\dot\gamma_s(0) \in \beta$ *for* $s > 0$ *and such that* $s \mapsto \dot\gamma_s(0)$ *is a unit speed curve in* S_zX.

Then $L := \cup\gamma_s(t)$ *is a spherical lune with angle* δ. *For* $0 < t < \pi$ *there is a* k*-pod* $\Pi(t) \subset S_{\gamma_0(t)}X$ *whose middle point* ξ *is the direction of the curve* $s \mapsto \gamma_s(t)$, *the directions pointing into* L *belong to* $\Pi(t)$ *and*

(1) *if* $t \neq \pi/2$ *then* $k = 2$; *if* $t = \pi/2$ *then* k *is the number of local faces adjacent to the geodesic segment* $s \mapsto \gamma_s(\pi/2)$;

(2) $\Pi(t)$ *is the ball of radius* $\pi/2$ *centered at* ξ, *that is, no edge of* $S_{\gamma_0(t)}X \setminus \Pi(t)$ *is adjacent to a point in the interior of* $\Pi(t)$.

Proof. It is immediate from Lemmas 3.2 and 3.3 that L is a spherical lune of angle δ. Furthermore, it also follows that there is no essential edge of X passing through the interior of L, except possibly through $s \mapsto \gamma_s(\pi/2)$. □

3.6 Lemma. *Let* $x, y \in X$ *be points with* $d(x, y) = \pi$ *and let* $\gamma : [0, \pi] \to X$ *be a minimal geodesic from* x *to* y.

Then there is $\varepsilon > 0$ *such that for any* $\xi \in S_xX$ *with* $d_x(\xi, \dot\gamma(0)) < \varepsilon$ *we have the alternative*

(1) *either* ξ *is tangent to a minimal geodesic from* x *to* y,

(2) *or else, there are* $t \in (0, \pi)$, *a unit speed geodesic* $\sigma : [0, t] \to X$ *with* $\dot\sigma(0) = \xi$, *an isosceles spherical triangle* Δ *with apex at* x *and legs* $\gamma([0, t])$ *and* $\sigma([0, t])$, *and a nontrivial open arc* $\alpha \subset S_{\gamma(t)}X$ *adjacent to* $-\dot\gamma(t)$ *such that every direction* $\eta \in \alpha$ *points into* Δ *and* $d_{\gamma(t)}(\eta, \dot\gamma(t)) > \pi$.

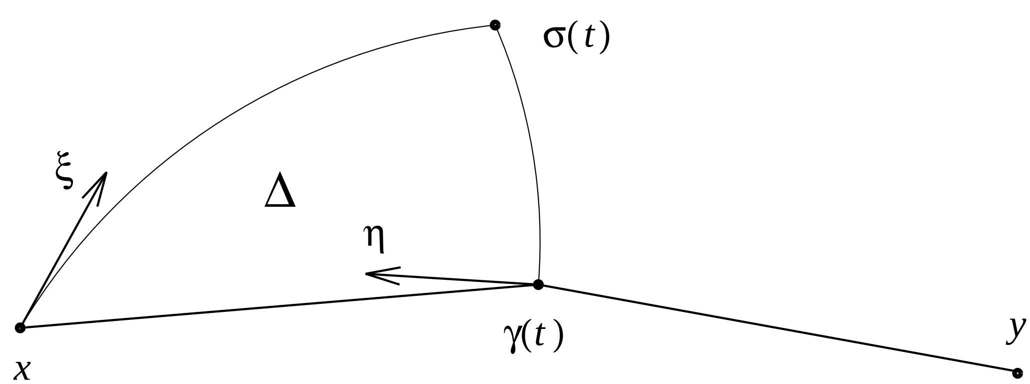

FIGURE 5.

Proof. Objects relevant to Lemma 3.6 are shown in Figure 5. Choose $\delta > 0$ such that (a) the δ-neighborhood of γ contains no essential vertices of X except for those that belong to γ, (b) the essential vertices on γ lie at least δ apart and (c) each essential edge meeting γ is either tangent to γ or makes an angle $> \delta$ with γ.

Choose $\varepsilon \in (0, \delta/2)$ so that $\angle_{\gamma(t)}(\sigma(s), x) \leq \delta$ for every $t \in [\delta, \pi - \delta]$, every geodesic $\sigma : [0, t] \to X$ with $\sigma(0) = x$ and $d(\sigma(t), \gamma(t)) < \varepsilon$ and every $s \leq t - \delta/2$.

Let $\xi \in S_xX$ satisfy $d_x(\dot\gamma(0), \xi) < \varepsilon$ and let $\tau \leq \pi$ be the maximum of all $t \leq \pi$ for which there is an isosceles spherical triangle $\Delta_t \subset X$ whose apex is at x and whose legs are $\gamma([0,t])$ and a unit speed geodesic $\sigma : [0,t] \to X$ with $\dot\sigma(0) = \xi$. If $\tau = \pi$ then Δ_τ is a spherical lune and σ is a minimal geodesic from x to y.

Suppose $\tau < \pi$. Then we claim that Statement 2 of the lemma holds true for $t = \tau$, a geodesic σ with $\dot\sigma(0) = \xi$ and the corresponding triangle Δ_τ. Clearly $\tau \geq \delta > 0$ since the ball of radius δ at x is a spherical cone over S_xX with apex at x. By our assumption (a) on δ and by the choice of ε, the point $z = \gamma(\tau)$ is an essential vertex of X. Assume by contradiction that 2) does not hold. Then there is an open local edge $\beta \in S_zX$ tangent to Δ_τ and adjacent to $-\dot\gamma(\tau)$ such that there is a curve $c \subset S_zX$ from $-\dot\gamma(\tau)$ to $\dot\gamma(\tau)$ which has length π and contains β.

Since Δ_τ is a spherical triangle, $d(\sigma(\tau), \gamma(\tau)) < \varepsilon$ by spherical geometry. Therefore, by the triangle inequality, $d(\sigma(\tau - \delta/2, \gamma(\tau)) < \delta$. By the choice of ε we have $\angle_{\gamma(\tau}(\sigma(\tau - \delta/2), x) \leq \delta$. Assumption (b) implies that the initial directions of the geodesics from z to $\sigma(t)$, $0 \leq t \leq \tau - \delta/2$, lie in the curve $c \subset S_zX$. We denote the union of these geodesics by Δ'_τ. By our assumption (a) on δ, the ball of radius δ centered at z is a spherical cone over S_zX with apex z. Hence c represents a spherical semidisc D centered at z so that $D \cup \Delta'_\tau$ contains an isosceles spherical triangle along γ whose legs have length $> \tau$. This is a contradiction. □

3.7 Lemma. *Let $x \in X$ be a point whose link is not a bipartite graph and $y \in X$ be a point with $d(x,y) = \pi$. Then*

(1) *for any $\eta \in S_yX$ there is a unique minimal geodesic from y to x with initial direction η;*
(2) *if not every direction in S_xX is tangent to a minimal geodesic from x to y then S_yX is a circle of length 2π and S_xX is a semicirclex; the central circle in S_xX is the set of initial directions of geodesics of length π from x to y.*

Proof. Since X is a spherical polyhedron, any two unit speed geodesics σ_1 and σ_2 in X with a common initial direction, coincide on an interval $[0, \varepsilon]$ with $\varepsilon > 0$. Hence the uniqueness follows from the assumption on the injectivity radius.

The set $\Phi \subset S_yX$ of initial directions tangent to minimal geodesics from y to x is closed and $\pi/2$-dense (by Lemma 2.2). Hence Φ intersects any connected component of S_yX and it suffices to prove that Φ is open in S_yX.

If Φ is not open then, by Lemma 3.6, there exist a minimal geodesic $\gamma : [0,\pi] \to X$ from y to x, a number $t \in (0, \pi)$, and a local open edge $\alpha \subset S_{\gamma(t)}X$ adjacent to $-\dot\gamma(t)$ such that $d_{\gamma(t)}(\eta, \dot\gamma(t)) > \pi$ for all $\eta \in \alpha$. We apply Lemmas 3.2 and 3.3 with $\xi = \dot\gamma(t)$ and $\sigma(\cdot) = \gamma(t + \cdot)$ to conclude that S_xX is a bipartite graph. This is a contradiction. Hence Φ is open and (1) follows.

To prove (2) observe that the set $\Psi \subset S_xX$ of initial directions of minimal geodesics from x to y is a proper closed subset of S_xX with the following properties:

(i) Ψ is π-convex in S_xX and is isometric to S_yX;
(ii) $d_x(\xi, \Psi) \leq \pi/2$ for any $\xi \in S_xX$;

Property (i) follows from Corollary 2.6 and (1). Property (ii) follows from Lemma 2.2 since $d(x,y) = \pi = \operatorname{diam} X$.

Let α be a local edge in $S_xX \setminus \Psi$ adjacent to $\xi \in \Psi$ and let $\sigma : [0,\pi] \to X$ be a minimal geodesic from x to y with $\dot\sigma(0) = \xi$. By Lemma 2.4, we can choose α

so small that geodesics of length π with initial directions in α never hit essential vertices.

Observe that $d_{\sigma(t)}(-\dot\sigma(t), \dot\sigma(t)) = \pi$ for all $t \in (0, \pi)$ since otherwise S_xX would be a bipartite graph by Lemmas 3.2 and 3.3, applied to $-\dot\sigma(t)$. Since $\xi \in \Psi$ but $\alpha \subset S_xX \setminus \Psi$, Lemma 3.6 implies that there is $t \in (0, \pi)$ and an isosceles spherical triangle Δ such that one of its legs is $\sigma([0, t])$, it is tangent to α at x and any direction ζ in the local edge $\beta \subset S_{\sigma(t)}X$ adjacent to $-\dot\sigma(t)$ and tangent to Δ lies at distance more than π from $\dot\sigma(t)$. In particular, $\sigma(t)$ is a pole. We apply Lemmas 3.2 and 3.3 to $\dot\sigma(t)$ and recall that $\Psi \cong S_yX$ to obtain

(iii) $\Psi = \Gamma(A, B)$ with $|A| \geq 2$ and $|B| \geq 2$.

By Lemmas 3.2 and 3.3 and since Δ is convex, there is a continuous family of unit speed geodesics $\gamma_s : [0, \pi] \to X$, $0 \leq s \leq \delta$ for some $\delta > 0$, such that

$$\begin{aligned} \gamma_0(\tau) &= \sigma(t - \tau), \quad 0 \leq \tau \leq t \\ \dot\gamma_s(0) &\in \beta, \quad s > 0 \end{aligned}$$

and α is tangent to $\cup\gamma_s$ at x. By Corollary 3.5, applied to $-\dot\sigma(t)$ and β, this family of geodesics gives rise to a k-pod Π in S_xX, $k \geq 2$, such that no edge of $S_xX \setminus \Pi$ is adjacent to the interior of Π. Clearly Π contains α. Note that $\dot\sigma(0)$ is an end point of Π. If an interior point ζ of Π lies in Ψ, then so does the arc $a \subset \Pi$ between ζ and $\dot\sigma(0)$ (because length$(a) < \pi$ and because of (i)). But $a \supset \alpha$, which is a contradiction. Hence no interior point of Π belongs to Ψ.

On the other hand, any end point of Π belongs to Ψ. Assume, by contradiction that there is an end point ζ of Π which does not belong to Ψ. Then an ε-neighborhood of ζ does not intersect Ψ and the point $\eta \in \Pi$ lying between the middle point ν and ζ with $d_x(\eta, \nu) = \varepsilon/2$ is $> \pi/2$ away from Ψ. This contradicts Lemma 2.2. Hence we get:

(iv) S_xX is obtained by attaching k-pods (with $k \geq 2$) by their end points to $\Psi \cong S_yX$.

It follows that S_xX is not a 1-dimensional thick spherical building with edges of length π/k, $k \geq 3$. We will need one more property of S_xX:

(v) $d_x(\xi, \eta) \leq \pi$ for any $\xi \in \Psi$ and $\eta \in S_xX$; in particular, S_xX and Ψ are connected.

If (v) did not hold, ξ would be a spreading direction and Corollary 3.4 would imply that y is a pole in contradiction with (iii) and (i).

Suppose that $\Psi = \Gamma(A, B)$ with $|A|+|B| \geq 5$. Because S_xX has injectivity radius π, the end points of a k-pod have to be attached to points with pairwise distances π in Ψ. Property (v) implies that if $\theta \in \Psi$ is an attaching point then all points $\zeta \in \Psi$ with $d_x(\theta, \zeta) = \pi$ are attaching points for the same k-pod. Assume WLOG that $|A| \geq 3$. Let $\theta \in \Psi$ be an attaching point which is not a vertex of $\Psi = \Gamma(A, B)$ which lies on the edge connecting $a_1 \in A$ to $b_1 \in B$ and set $d_x(a_1, \theta) = \tau$, $0 < \tau < \pi/2$. Let ζ_i be the point on the edge connecting a_i to b_2 with $d_x(\zeta_i, a_i) = \tau$, $i = 2, 3$, see Figure 6. Clearly $d_x(\zeta_i, \theta) = \pi$ and $d_x(\zeta_2, \zeta_3) = \pi - 2\tau < \pi$. This is a contradiction. Hence any attaching point $\theta \in \Psi$ is a vertex and S_xX is a bipartite graph with edges of length $\pi/2$. This contradicts the assumption on S_xX. Hence $|A| = |B| = 2$ and $\Psi \cong S_yX$ is a circle of length 2π.

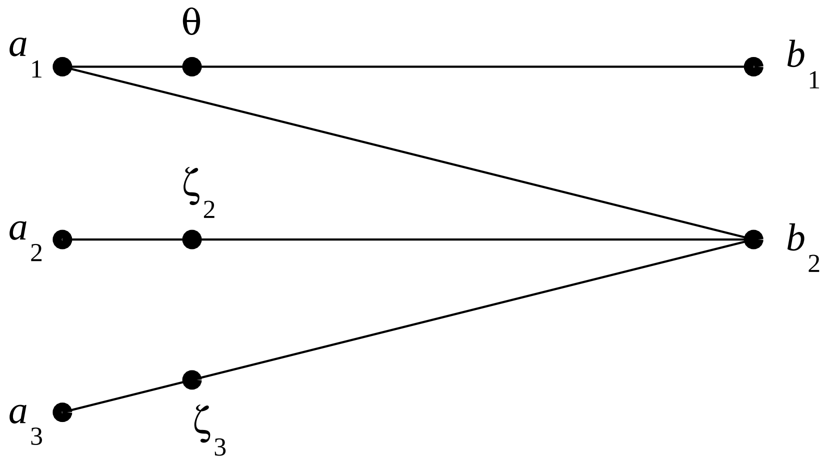

FIGURE 6.

Since the points at which a k-pod Π is attached to Ψ must be π apart, we conclude that $k = 2$ and Π is an edge of length π. By assumption, S_xX is not a bipartite graph and hence is a semicirclex. □

3.8 Corollary. *If $x, y \in X$ are poles and $d(x, y) = \pi$ then for any $\xi \in S_xX$ there is a unique minimal geodesic γ with $\dot\gamma(0) = \xi$ and $\gamma(\pi) = y$. In particular, the map $\dot\gamma(0) \mapsto -\dot\gamma(\pi)$ is an isometry between the links S_xX and S_yX.* □

3.9 Proposition. *Let $x \in X$ be a pole. Assume that any point at distance π from x is a pole and that any point at distance π from such a point is also a pole.*
Then X is a spherical join.

Proof. For $\xi \in S_xX$ and $y \in X$ with $d(x, y) = \pi$ denote by $\sigma_\xi^y : [0, \pi] \to X$ the unique unit speed geodesic from x to y with initial velocity $\dot\sigma_\xi^y(0) = \xi$. By Corollary 3.8, $\xi \mapsto -\dot\sigma_\xi^y(\pi)$ is an isometry $S_xX \to S_yX$. Therefore, if $\xi \in S_xX$ is a spreading direction, then, by Lemmas 3.2 and 3.3,

(i) for any $y \in X$ with $d(x, y) = \pi$ and for any $t \in (0, \pi)$, $t \neq \pi/2$, the link S_pX, where $p = \sigma_\xi^y(t)$, is the graph with vertices $\dot\sigma_\xi^y(t)$, $-\dot\sigma_\xi^y(t)$ and edges of length π connecting them;

(ii) for any $y \in X$ with $d(x, y) = \pi$ the link S_pX, where $p = \sigma_\xi^y(\pi/2)$, is the graph $\Gamma(A, B)$ with

$$B = \{\zeta : d_p(-\dot\sigma_\xi^y(\pi/2), \zeta) = \pi/2\} \text{ and}$$

$$A = \{\zeta : d_p(B, \zeta) = \pi/2\} \ni -\dot\sigma_\xi^y(\pi/2).$$

Hence

(iii) for any $z \in X$ with $d(x, z) = \pi$, $z \neq y$, the geodesics σ_ξ^y and σ_ξ^z coincide on $[0, \pi/2]$, branch at time $\pi/2$ and the union of $\sigma_\xi^y([\pi/2, \pi])$ with $\sigma_\xi^z([\pi/2, \pi])$ is a geodesic of length π from y to z.

In particular, we have that $d(y,z)=\pi$ and that the set $P_x \subset X$ of points at distance π from x is finite. Since all points in P_x are poles and all points at distance π from them are poles, it also follows that for any $y \in P_x$, the set P_y of points at distance π from y is $P_x \cup \{x\} \setminus \{y\}$. Now let $G \subset S_xX$ be the subset of directions satisfying (i), (ii) and (iii) for all $y,z \in P_x$. The proposition follows if $G = S_xX$.

We know that all spreading directions belong to G. Hence $G = S_xX$ if S_xX is not connected. Since all spreading directions lie in G, it is not empty. Note that Conditions (i) and (ii) describe explicitly the link, and hence a small neighborhood, of $\sigma_\xi^y(t)$. Therefore (i), (ii) and (iii) are satisfied in an open neighborhood of $\xi \in G$ and G is open. It remains to show that G is closed. To that end, let $\xi \in S_xX$ be a point in the closure of G. Then the geodesics σ_ξ^y, $y \in P_x$, satisfy (iii). Since P_x is the set of *all* points at distance π from x and the injectivity radius of X is π, any unit speed geodesic γ of length π and with $\dot\gamma(0)=\xi$ coincides with one of the geodesics σ_ξ^y. In particular, if $p=\sigma_\xi^y(t)$, $0<t<\pi$, $t\neq\pi/2$, then $\eta_y=\dot\sigma_\xi^y(t)$ is the only point in S_pX at distance $\geq\pi$ from $\eta_x=-\dot\sigma_\xi^y(t)$. By symmetry, we can apply the same argument to the pole y and obtain that η_x is the only point in S_pX at distance $\geq\pi$ from η_y. Similarly, for $t=\pi/2$ the set $A=\{\eta_y \in S_pX \mid y \in P_x \cup \{x\}\}$ satisfies

$$d_p(\eta,\zeta)=\pi \ \text{ for all } \ \eta,\zeta\in A \ \text{ and } \ d_p(\eta,\theta)<\pi \ \text{ for all } \ \eta\in A \text{ and } \theta\in S_pX\setminus A.$$

Lemma 2.8 shows now that ξ satisfies (i) and (ii). Hence G is closed and therefore $G=S_xX$. □

Proposition 3.9 and Lemma 3.7 imply the following corollary.

3.10 Corollary. *Let X contain a pole x whose link S_xX is not a semicirclex. Then X is a spherical join.* □

By Corollary 3.10 and Proposition 2.7, to prove Theorem 3.1 it is sufficient to consider spaces X whose links belong to the following list:

(1) semicirclexes;
(2) bipartite graphs $\Gamma(A,B)$ (this includes unit circles and graphs consisting of 2 vertices and edges of length π connecting them);
(3) thick 1-dimensional spherical buildings with edges of length π/k, $k>2$.

From now and until the end of this section we assume that the links of points in X are of types 1,2,3 only and that X has at least one pole.

Let $x\in X$ be a pole and recall that the links of types 2 and 3 have diameter π. Hence the link S_xX is a semicirclex and it consists of its central circle c and semicircles of length π attached to c. Points in the interior of these semicircles are spreading directions. If we start along such a spreading direction and go up to distance π to a point $y\in X$, then y is also a pole (Corollary 3.4). Each direction at x is tangent to a geodesic of length π from x to y, and the map which sends the direction to the incoming direction at y is an isometry $S_xX\to S_yX$. The union of the geodesics corresponding to the central circle in S_xX is a unit sphere $S=S(x,y)$. By Lemma 2.5, S is π-convex.

The end points of the semicircles are vertices of S_xX. The geodesics of length π from x starting in the direction of these vertices and ending at poles will be called *special geodesics*. Special geodesics are (unions of) essential edges.

3.11 Lemma. *Assume that the links of points in X are only of types 1,2,3. Let $x, y \in X$ be poles at distance π from each other.*

Then no point in $S = S(x, y)$ has a link of type 3.

Proof. If $\sigma : [0, \pi] \to X$ is a special geodesic from x to y, then the geodesics of length π from x to y with initial direction in a semicircle $\subset S_x X$ adjacent to $\dot\sigma(0)$ give rise to a family as required in Corollary 3.5. In particular, $S_{\sigma(t)}X$ contains edges of length $\geq \pi/2$ and therefore is not of type 3 for $0 < t < \pi$.

Assume that a point $z \in S$ does not lie on a special geodesic and that its link is of type 3. Let b be a semicircle in $S_x X$ attached to the central circle. The ends of b lie at distance π from each other in $S_x X$ and determine a great circle $C \subset S$ which is the boundary of a hemisphere H_1 made of geodesics of length π from x to y with initial velocity in b. The great circle C divides S into two hemispheres, one of which, call it H_2, contains z. The union $H_1 \cup H_2$ is a round sphere and contains, in particular, a point q at distance π from z. Now necessarily $q \in H_1$. By Lemma 3.7 and Corollary 2.6, the link $S_q X$ is of type 3. On the other hand, q lies on a geodesic from x starting in a spreading direction. Hence the link has to be of type 2, a contradiction. □

3.12 Proposition. *Assume that the links of points in X are only of types 1,2,3. Let $x \in X$ be a pole whose link $S_x X$ is a semicirclex. Suppose that there are at least two poles in X at distance π from x.*

Then X is a spherical join.

Proof. We will show that all points at distance π from x are poles lying at distance π from each other and will apply Proposition 3.9 to conclude that X is a spherical join.

By assumption, the link of each pole is a semicirclex. Let $y, z \in X$ be two poles at distance π from x. By Corollary 3.8, the links $S_y X$ and $S_z X$ are isometric to $S_x X$. Let $\xi \in S_x X$ be a spreading direction and let $\sigma_y, \sigma_z : [0, \pi] \to X$ be the minimal geodesics from x to y,z with initial velocity ξ. By Lemmas 3.2 and 3.3, σ_y and σ_z branch at $\pi/2$ and $\sigma_y([\pi/2, \pi]) \cup \sigma_z([\pi/2, \pi])$ is a geodesic of length π from y to z. Hence

(i) $d(y, z) = \pi$ for any two poles y, z at distance π from x.

The central circles c_x, c_y and c_z in the links of x, y and z give rise to the unit spheres $S(x, y), S(x, z), S(y, z) \subset X$ made of the unit speed geodesics $\gamma : [0, \pi] \to X$ from x to y, x to z and y to z, respectively, whose initial velocities lie in the central circles of the corresponding links. By Lemma 2.5, these spheres are π-convex. Hence their pairwise intersections are also π-convex. The boundary of $S(x, y) \cap S(x, z)$ is exactly the set of points where the geodesics from x with initial velocities in c_x branch to y and z. The branching must occur not further than $\pi/2$ from x because otherwise the union of the two continuations would be a curve of length $< \pi$ from y to z. Hence $S(x, y) \cap S(x, z)$ is contained in the ball of radius $\pi/2$ about x.

By Lemmas 3.2 and 3.3, the branching along geodesics, whose initial velocities are spreading directions, occurs precisely at time $\pi/2$. Recall that every point in the interior of one of the semicircles in $S_x X$ attached to c_x, is a spreading direction. By taking limits we conclude that every geodesic of length π from x whose initial velocity is an end point of one of the attached semicircles, branches at time $\pi/2$

to y and z. View x as the south pole of $S(x,y)$ and $S(x,z)$. Since the link S_xX contains at least two semicircles with different ends attached to c_x, the intersection $S(x,y) \cap S(x,z)$ contains at least two different pairs of antipodal points on the equator of $S(x,y)$ and $S(x,z)$. Since the intersection $S(x,y) \cap S(x,z)$ is π-convex, it is the southern hemisphere in $S(x,y)$ and in $S(x,z)$ (with x being the south pole). We conclude that

(ii) for any two poles x,y at distance π from x all geodesics of length π from x to y and z branch exactly at distance $\pi/2$ from x. In particular, the equator in $S(x,y)$ is a union of essential edges.

Recall that geodesics of length π from x to y whose initial velocities are the ends of one of the semicircles attached to c_x are called *special*. Special geodesics are (unions of) essential edges. If $\sigma : [0,\pi] \to X$ is a special geodesic, it is the limit of geodesics from x to y whose initial velocities are spreading directions. This excludes type 3 as a possible link of X at $\sigma(t)$, $0 < t < \pi$. Consider the link of $\sigma(\pi/2)$. By the above argument, each of the directions in $S_{\sigma(\pi/2)}X$ pointing to x, y, z lies at distance $\pi/2$ from both the directions tangent to the equator of $S(x,y)$ and the directions perpendicular to $S(x,y)$ and pointing into the lunes corresponding to the semicircles in S_xX attached to c_x at $\dot{\sigma}(0)$. This excludes type 1 as a possible link of $\sigma(\pi/2)$. Hence

(iii) for any special geodesic $\sigma : [0,\pi] \to X$ from x, the link $S_{\sigma(\pi/2)}X$ is a bipartite graph.

Assume that $p \in S(x,y)$ is not the midpoint of a special geodesic and that S_pX is a semicirclex. We claim that its central circle is tangent to $S(x,y)$. Suppose not. Then there is a semicircle s of spreading directions at p tangent to our central sphere. If one of these directions points to x or y, we get a contradiction with Lemmas 3.2 and 3.3. If not, the ends of s point to x and y. Recall that there are at least 2 different distinguished great circles on $S(x,y)$ made of special geodesics. Hence a geodesic, whose initial velocity is an appropriate spreading direction in s , crosses one of the distinguished great circles at time $0 < t < \pi$, $t \neq \pi/2$, since p is not the midpoint of a special geodesic. This contradicts Lemma 3.2 because special geodesics are unions of essential edges. Hence

(iv) if y is a pole at distance π to x and the link of $p \in S(x,y)$ isa semicirclex, then the central circle of S_pX is tangent to $S(x,y)$.

It follows that the union of essential edges of X in $S(x,y)$ consists of great circles. Now suppose that there is a great circle C in $S(x,y)$ which consists of essential edges, does not pass through x and is not the equator. Then C intersects the equator at a point q with an angle different from 0 and $\pi/2$. Hence S_qX is not of type 2. If it is of type 1 then its central circle represents $S = S(x,y)$ and $S(x,z)$ which is impossible. Type 3 cannot occur by Lemma 3.11. Therefore,

(v) for any pole y at distance π from x the essential edges of X in $S(x,y)$ are the equator and the special geodesics from x to y.

Now suppose there is a point p at distance π from x which is not a pole. By Lemma 3.7, the link S_pX is a unit circle and the set of initial directions of the geodesics of length π from x to p is precisely the central circle c_x of S_xX. Moreover, by (v), these geodesics branch off exactly at time $\pi/2$ from the geodesics from x to any other pole at distance π from x. It follows that p lies at distance π from these poles too. More precisely, if $\sigma : [0,\pi] \to X$ is a geodesic from x to any other pole and $\dot\sigma(0) \in c_x$, then

$$\text{(vi)} \qquad d(p,\sigma(t)) = \pi/2 + |t - \pi/2|.$$

Let s be a semicircle in S_xX attached to c_x, let ξ be its midpoint and $\sigma_\xi : [0,\pi/2] \to X$ be the unit speed geodesic with initial velocity ξ. We want to show that the distance from p to $q = \sigma_\xi(\pi/2)$ is π.

Since the directions of s are spreading, Lemmas 3.2 and 3.3 imply that the union D of all geodesics of length π from x with initial velocity in s consists of hemispheres that branch along a common half equator consisting of points at distance $\pi/2$ from x. The boundary ∂D of D consists of special geodesics from x to other poles. It follows from Lemmas 3.2 and 3.3 that any curve from a point in X outside of D to a point inside D has to intersect ∂D. Since q lies at distance $\pi/2$ from any point on ∂D, we conclude that $d(p,q) = \pi$ and that a minimal geodesic from p to q has to pass through the midpoint m of a special geodesic (see (vi)). This leads to a contradiction since the link at m is a bipartite graph. On the one hand, the outgoing direction ζ at m of the unit speed geodesic $\gamma : [0,\pi] \to X$ from x to p lies at distance π from its incoming direction η. On the other hand, since

$$d(p,q) = d(p,m) + d(m,q),$$

ζ must also lie at distance π from the direction θ at m pointing to q. Now η is a vertex of the bipartite graph S_mX and (by the choice of q) $d_m(\eta,\theta) = \pi/2$. This contradicts $d_m(\eta,\zeta) = \pi$, and hence

(vii) all points at distance π from x are poles.

By (i), if y is any pole with $d(x,y) = \pi$ then there are at least two poles x,z at distance π from y. Hence, by (vii), all points at distance π from y are poles. Now Proposition 3.9 implies that X is a spherical join. $\square$

To finish the proof of Theorem 3.1 it remains to consider the following case.

3.13 Proposition. *Suppose that X has poles, the link of each pole is a semicirclex and for any pole $x \in X$ there is exactly one pole in X at distance π from x.*

Then either X has at least four poles and is a hemispherex or X is a spherical join obtained from the unit sphere by attaching hemispheres passing through the north and south poles; in the latter case X has exactly two poles – the north and south poles.

Proof. Fix a pole $x \in X$ and let y be the pole at distance π from x. Let S be the unit sphere consisting of all geodesics of length π from x to y with initial velocity in the central circle of S_xX.

Assume that the link of $z \in S$ is a semicirclex. Then z is a pole. We claim that its central circle must be tangent to S. This is so for $z = x$ or y. Suppose the claim does not hold at $z \in S \setminus \{x,y\}$. Then there is a semicircle s of spreading directions at z tangent to S. None of the spreading directions can point to x or y since this would contradict Lemmas 3.2 and 3.3. Hence, the ends of s point to

x and y. Since S_xX is a semicirclex, here are at least two different great circles on S which are unions of special geodesics consisting of essential edges. Hence a geodesic from z with an appropriate spreading direction crosses an essential edge at time $0 < t < \pi$. This implies that there are at least two poles at distance π from the pole z which contradicts our assumption. Therefore, the central circle at z is tangent to S. Hence, by the structure of possible links, the set of essential edges of X in S is a union of great circles. Moreover,

(∗) if $g \subset S$ is a great circle consisting of essential edges, $p \in g$ and $\alpha \subset S_pX$ is a local edge which is adjacent to one of the two directions of g and is not tangent to S then there is a spherical strip attached to S along the whole length of g which passes through p and is tangent to α.

By assumption, any pole has only one pole at distance π from it. Hence, the geodesics in spreading directions cannot branch at time $\pi/2$ and the hemispheres in the direction of the semicircles in the semicirclex links of S are maximal faces since the links of their interior points are unit circles. Thus, it suffices to show that points in S cannot have links isometric to a type 2 link $\Gamma(A, B)$, where $|A|, |B| > 2$ (recall that, by Lemma 3.11, points in S do not have links of type 3).

Suppose that $z \in S$ be a point with such a link. Assume first that $z = \sigma(t)$ for some special geodesic $\sigma : [0, \pi] \to S$ from x to y, where $0 < t < \pi$. Since special geodesics are unions of essential edges, $\dot\sigma(t)$ and $-\dot\sigma(t)$ are vertices of S_zX. Therefore, $S_zX = \Gamma(A, B)$ with

$$\begin{aligned} B =&\{\xi \,:\, d_z(\xi, \dot\sigma(t)) = \pi/2\}, \\ A =&\{\eta \,:\, d_z(\zeta, B) = \pi/2\} \ni \pm\dot\sigma(t) \end{aligned}$$

and $|A| > 2$ by our assumption. Since σ is a special geodesic, there is a local edge $\alpha \in S_xX$ adjacent to $\dot\sigma(0)$ such that α consists of spreading directions. By (∗), there is a spherical strip f adjacent to $\sigma([0, t])$ such that α is tangent to f at x. Hence for every $\xi \in S_zX$ with $d_z(-\dot\sigma(t), \xi) = \pi$, there is a continuous family $\gamma_s : [0, \pi] \to X$, $0 \le s \le \delta$ for some $\delta > 0$, such that

$$\begin{aligned} &\gamma_0(\tau) = \sigma(\tau), \quad 0 \le \tau \le t \\ &\dot\gamma_0(t) = \xi \quad \text{and} \quad \dot\gamma_s(0) \in \alpha \quad \text{for} \quad s > 0. \end{aligned}$$

Any such family γ_s has a common end point and such an end point is a pole. Since $|A| > 2$, there are at least two choices for the end point of γ_s, $s > 0$. But these end points are poles, contradicting our assumption that there is only one pole at distance π from x.

Suppose now that $z \in S$ is a point whose link is of type $\Gamma(A, B)$ with $|A|, |B| > 2$ and that z does not lie on a special geodesic from x to y. In a bipartite graph, a point at distance π from a vertex is also a vertex. Hence for an essential edge adjacent to z, its geodesic continuation through z is also essential. Since there are at least two different great circles in S consisting of special geodesics, there is an essential edge e through z tangent to S such that the unit speed geodesic ρ in S starting from z in the direction of e crosses a special geodesic σ at a time $0 < t < \pi/2$. Set $\rho(t) = p$. By the previous paragraph, S_pX is not a bipartite graph, and hence is a semicirclex (see Lemma 3.11). By $(*)$, there is a spherical strip along ρ and not tangent to S. Hence the incoming direction $-\dot{\rho}(t)$ at p satisfies the assumption of Corollary 3.5. Since $t < \pi/2$, it follows that S_zX contains a simple arc of length π such that none of its interior points is a vertex. This contradicts the assumption that $|A|, |B| > 2$. $\square$

This completes the proof of Theorem 3.1.

4. Dimensionally non-homogeneous polyhedra.

In this section we discuss spherical polyhedra that contain edges not adjacent to any face.

4.1 Theorem. *Let X be a geodesically complete compact 2-dimensional spherical polyhedron of diameter and injectivity radius π. Assume that X contains an edge that is not adjacent to any face.*

Then X is a spherical join whose equator has isolated points.

A *k-pod in X* consists of k closed edges $e_1, ..., e_k$ of length $\pi/2$ with a common point m, called the *middle point* of the k-pod, such that $B(m, \pi/2) = e_1 \cup ... \cup e_k$. A 2-pod is an edge of length π not adjacent to any face in X. For $k \geq 3$, the middle point is an essential vertex of X.

The following lemma replaces lemmas 3.2 and 3.3. For the convenience of the reader we repeat in the proof some of the arguments from those lemmas.

4.2 Lemma. *Under the assumptions of Theorem 4.1, X contains a k-pod with $k \geq 2$.*

Proof. Since X is 2-dimensional, there is a point $x \in X$ adjacent to a face f and an edge e such that e is not adjacent to any face of X. Let $\sigma : [0, \pi] \to X$ be a unit speed geodesic with $\sigma(0) = x$ which starts along e.

Since $\xi = \dot{\sigma}(0)$ is an isolated point of $S_x X$, its distance from any direction pointing inside f is infinite. By Lemma 2.4, there is a continuous family γ_s : $[0, \pi] \to X$, $-\varepsilon \leq s \leq \varepsilon$, such that

- (i) $\gamma_s(0) = x$ for all $s \in (-\varepsilon, \varepsilon)$;
- (ii) $\gamma_s(t) \notin \mathcal{V}_X$ for all $s \in (-\varepsilon, \varepsilon)$ and $t \in (0, \pi)$;
- (iii) $s \mapsto \eta_s = \dot{\gamma}_s(0)$ is a unit speed curve which lies in an open edge of $S_x X$ and satisfies $d_x(\xi, \eta_s) = \infty$.

Since no γ_s passes through an essential vertex, the union $\cup \gamma_s(t)$ is isometric to a spherical lune with angle 2ε.

Fix $t \in (0, \pi)$, $t \neq \pi/2$. Since essential edges of X are geodesics, by elementary spherical geometry, an open subarc $c : I \to X$ of the curve $s \mapsto \gamma_s(\pi - t)$ is contained in a maximal face F of X. Then c is a smooth curve in F with constant speed and $d(c(\cdot), \sigma(t)) \equiv \pi$. Hence any minimal (of length π) geodesic from c to $\sigma(t)$ is perpendicular to c. By Lemma 2.2, for every s there are exactly two minimal geodesics from $c(s)$ to $\sigma(t)$ with initial directions $\pm \frac{d}{d\tau} \gamma_s(\tau) \Big|_{\pi - t}$. Therefore, their concatenation is a geodesic loop α_s of length 2π at $\sigma(t)$. By construction, α_s contains the union of $\sigma([0, t])$ and $\gamma_s([0, \pi - t])$ as a subarc of length π. We parameterize α_s by arclength so that $\dot{\alpha}_s(0) = -\dot{\sigma}(t)$. By Lemma 2.3, α_s is a closed geodesic. By Lemma 2.2, the balls of radius $\pi/2$ about $\dot{\alpha}_s(0)$ and $-\dot{\alpha}_s(2\pi)$ cover the link $S_{\sigma(t)} X$. It follows that $S_{\sigma(t)} X$ is the graph with two vertices, $\dot{\alpha}_s(0) = -\dot{\sigma}(t)$ and $-\dot{\alpha}_s(2\pi)$, connected by edges of length π. In particular, $-\dot{\alpha}_s(2\pi) = \dot{\sigma}(t)$. Hence

- (iv) for $t \in (0, \pi/2)$ or $t \in (\pi/2, \pi)$, the link $S_{\sigma(t)}$ is the graph with vertices $\{\dot{\sigma}(t), -\dot{\sigma}(t)\}$ connected by $l \neq 1$ edges of length π.

If $t > 0$ is small then $\sigma(t) \in e$ and $S_{\sigma(t)}X = \{\dot\sigma(t), -\dot\sigma(t)\}$. The set of such t is open; by (iv), it is closed in $(0, \pi/2)$. Therefore

(v) $S_{\sigma(t)}X = \{\dot\sigma(t), -\dot\sigma(t)\}$ for all $t \in (0, \pi/2)$; in particular, $-\dot\sigma(\pi/2)$ is an isolated point in $S_{\sigma(\pi/2)}X$.

We now consider the case $t = \pi/2$. The curve $s \mapsto \gamma_s(\pi/2)$ is a unit speed geodesic in X, and hence an open subarc $c : I \to X$ of this curve is either contained in a maximal face of X or is contained in an essential edge of X. A neighborhood of c consists of local faces $f_1, ..., f_k$, $k \geq 2$, adjacent to c with, say, $-\dot\gamma_s(\pi/2)$ pointing into f_1. The link $S_{c(s)}X$ is the graph with vertices $\pm\dot c(s)$ and k edges ε_i^s of length π connecting them and representing the local faces f_i, $1 \leq i \leq k$. The midpoints δ_i^s of these edges lie at distance $\pi/2$ from $\pm\dot c(s)$ with $\delta_1^s = -\dot\gamma_s(\pi/2)$. Observe that the distance in X from $c(s)$ to $m = \sigma(\pi/2)$ is π and, by elementary geometry, any minimal geodesic from c to m is perpendicular to c. Hence, by Lemma 2.2, there are precisely k minimal geodesics from $c(s)$ to m and their initial directions are δ_i^s, $1 \leq i \leq k$.

Assume that the link S_mX contains a non-trivial arc. Since $-\dot\sigma(\pi/2)$ is an isolated point in S_mX, there is a continuous family of geodesics $\sigma_r : [0, \pi/2+\varepsilon] \to X$ extending $\sigma|[0, \pi/2]$ such that $r \mapsto \dot\sigma_r(\pi/2)$ is a unit speed curve in S_mX and such that $b_\delta(r) := \sigma_r(\pi/2 + \delta)$ is in an open face for all r and all $\delta \in (0, \varepsilon)$. Applying Lemma 2.2 as above, we get a family of geodesics ρ_δ^r of length π from $b_\delta(r)$ to $\gamma_s(\pi/2 - \delta)$, where s is fixed, and such that $\dot\rho_\delta^r(0) = \dot\sigma_r(\pi/2 + \delta)$. By passing to the limit as $\delta \to 0$, we get for every r a minimal geodesic from m to $c(s)$ with initial direction $\dot\sigma_r(\pi/2)$. This contradicts the fact that there are only finitely many minimal geodesics from m to $c(s)$. Hence

(vi) $S_{\sigma(\pi/2)}X$ is a finite set.

It follows that for $t > \pi/2$ and sufficiently close to $\pi/2$ the point $\sigma(t)$ lies in an edge of X not adjacent to any face. For such t the link at $\sigma(t)$ is $\{-\dot\sigma(t), \dot\sigma(t)\}$. The set of such t is clearly open; by (iv), it is closed in $(\pi/2, \pi)$. Hence $S_{\sigma(t)}X = \{-\dot\sigma(t), \dot\sigma(t)\}$ for all $t \in (0, \pi)$, $t \neq \pi/2$. The same holds true for any continuation of $\sigma|[0, \pi/2]$ beyond m. Hence the ball of radius $\pi/2$ about m is a k-pod for some $k \geq 2$. $\square$

Proof of Theorem 4.1. By Lemma 4.2, X contains a k-pod P with $k \geq 2$. Denote its middle point by m, its legs by $e_1, ..., e_k$ and the end points by $x_1, ..., x_k$. Let $\xi_i \in S_{x_i}X$ be the direction of e_i. Clearly ξ_i is an isolated point in $S_{x_i}X$. Fix $i \in \{1, 2, ..., k\}$. Let $\gamma : [0, \pi/2] \to X$ be a unit speed geodesic such that $\gamma(0) = x_i$ and $\dot\gamma(0) \neq \xi_i$. Since $d_{x_i}(\xi_i, \dot\gamma(0)) = \infty$, the union $e_i \cup \gamma$ is a geodesic in X and so is $e_j \cup e_i \cup \gamma$ for $j \neq i$.

Fix $t \in (0, \pi/2)$ and set $z = \gamma(t)$. Choose $j \neq i$ and let y be the point on e_j at distance $\pi/2 - t$ to m. Then $d(z, y) = \pi = \operatorname{diam} X$. Since the distance function $d(z, \cdot)$ decreases in both directions from y, there are exactly two geodesics of length π from y to z. By Lemma 2.2 applied to $d(y, \cdot)$, the balls of radius $\pi/2$ in S_zX about the incoming directions of these two geodesics cover S_zX. One of the incoming directions is $-\dot\gamma(t)$. It follows that S_zX consists of two points $\pm\dot\gamma(t)$ connected by $l \neq 1$ edges of length π.

It follows that each ball $B(x_i, \pi/2) \subset X$ is the spherical cone with center x_i and equator $\{z \in X : d(z, x_i) = \pi/2\} \cong S_{x_i}X$. It remains to show that $d(z, x_j) = \pi/2$

for any $z \in X$ with $d(z, x_i) = \pi/2$, $j \neq i$. For $z = m$ this is clear. For $z \neq m$ we have $d(z, m) = \pi = \operatorname{diam} X$. Since the distance function $d(z, \cdot)$ decreases in all k directions from m, there are exactly k different geodesics of length π from m to z which go along $e_1, ..., e_k$. Obviously these geodesics pass through $x_1, ..., x_k$, and hence $d(z, x_j) = \pi/2$ for all j. $\square$

5. Characterization of spherical buildings.

In this section we prove the following theorem.

5.1 Theorem. *Let X be a geodesically complete compact 2-dimensional spherical polyhedron of diameter and injectivity radius π. Assume that all links of X have diameter π.*

Then either X is a spherical join over a graph of injectivity radius and diameter π or else X is a thick spherical building of type A_3 or B_3.

5.2 Lemma. *Under the assumptions of Theorem 5.1 suppose that $x, y \in X$ satisfy $d(x, y) = \pi$.*

Then for every $\xi \in S_x X$, there is a unique geodesic of length π from x to y with initial velocity ξ.

Proof. Let Φ be the set of initial directions of geodesics of length π from x to y. Then Φ is closed and not empty. By Lemma 3.6, Φ is open. Since $S_x X$ is connected, $\Phi = S_x X$. □

Proof of Theorem 5.1. If X is not a sphere, then X contains an essential edge. If X does not contain essential vertices, essential edges cannot branch and have to be closed geodesics of length 2π by Lemma 5.2. The Gauß-Bonnet formula implies that maximal faces are hemispheres and that X is a spherical join over a graph $\Gamma(A, B)$.

Suppose now that X contains essential vertices and edges. For any essential vertex x in X, all (maximal) edges in $S_x X$ have the same length π/n, where $n = n(x) \geq 2$ (see Proposition 2.7). We claim that every maximal face of X is either a spherical triangle or a spherical lune whose sides in both cases are (by definition) essential edges and whose vertices are essential vertices of X. If $x_1, \ldots, x_k$ are the vertices on the boundary of a maximal face F with interior angles $\pi/n(x_1), \ldots, \pi/n(x_k)$ then, by the Gauß-Bonnet formula,

$$(k - 2\chi(F))\pi + \int_F K = \frac{\pi}{n(x_1)} + \ldots + \frac{\pi}{n(x_k)} \leq k \cdot \frac{\pi}{2}.$$

Because F is a compact surface with boundary, $\chi(F) \leq 1$. Since $K \equiv 1$, F is a topological disc and $k < 4$. The sum of the interior angles in a spherical triangle is $> \pi$. Hence we have the following possibilities for the interior angles of F if $k = 3$:

$$\frac{\pi}{2}, \quad \frac{\pi}{2}, \quad \frac{\pi}{m}, \quad \text{where} \quad m \geq 2$$
$$\frac{\pi}{2}, \quad \frac{\pi}{3}, \quad \frac{\pi}{m}, \quad \text{where} \quad m = 3, 4, 5.$$

If $k = 2$ then F is a spherical lune, the angles at its vertices are equal to each other and equal to π/n for some $n \geq 2$.

If all maximal faces of X are spherical lunes then X is obviously a spherical join over the set of vertices of these lunes.

Suppose X contains a maximal face F which is a spherical triangle. Let e be a side of F and let x, y be the vertices adjacent to e. The length $|e|$ of e is $< \pi$, and the isometry type of F is determined by $|e|$ and the angles of F at x and y. If F' is any other maximal face adjacent to e then it shares a common side with F and its angles at x and y are equal to the corresponding angles of F (see Proposition 2.7). Therefore F' is isometric to F. It can actually be obtained by *reflecting* F with respect to e. Each of the types of triangles listed above is a simplex in a spherical Coxeter complex. It follows easily that X is a thick spherical building. In the case

$$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{m}, \quad m \geq 2,$$

we see that X is a spherical join over the vertices with angles π/m. In the case

$$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}, \quad m = 3, 4, 5,$$

X is a thick spherical building of type A_3, B_3 and H_3, respectively. This completes the proof since there are no thick spherical buildings of type H_3 (see [Ti, p.275]). $\square$

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