HODGE TYPE THEOREMS FOR ARITHMETIC MANIFOLDS ASSOCIATED TO ORTHOGONAL GROUPS

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Abstract. We show that special cycles generate a large part of the cohomology of locally symmetric spaces associated to orthogonal groups. We prove in particular that classes of totally geodesic submanifolds generate the cohomology groups of degree n of compact congruence p-dimensional hyperbolic manifolds "of simple type" as long as n is strictly smaller than $\frac{p}{3}$. We also prove that for connected Shimura varieties associated to $\mathrm{O}(p,2)$ the Hodge conjecture is true for classes of degree $< \frac{p+1}{3}$. The proof of our general theorem makes use of the recent endoscopic classification of automorphic representations of orthogonal groups by [6]. As such our results are conditional on the hypothesis made in this book, whose proofs have only appeared in preprint form so far; see the second paragraph of subsection 1.20 below.

Contents

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1. Introduction

1.1. Let D be the p-dimensional hyperbolic space and let $Y_\Gamma = \Gamma\backslash D$ be a compact hyperbolic manifold.

Thirty-five years ago one of us (J.M.) proved, see [60], that if Y_Γ was a compact hyperbolic manifold of simple arithmetic type (see subsection 1.3 below) then there was a congruence covering $Y'_\Gamma \to Y_\Gamma$ such that Y'_Γ contained a nonseparating embedded totally geodesic hypersurface F'. Hence the associated homology class $[F']$ was nonzero and the first Betti number of Y'_Γ was nonzero. Somewhat later the first author refined this result to

1.2. **Proposition.** *Assume Y_Γ is arithmetic and contains an (immersed) totally geodesic codimension one submanifold*

$$F \to Y_\Gamma. \tag{1.2.1}$$

Then, there exists a finite index subgroup $\Gamma' \subset \Gamma$ such that the map (1.2.1) *lifts to an embedding $F \hookrightarrow Y_{\Gamma'}$ and the dual class $[F] \in H^1(Y_{\Gamma'})$ is non zero.*

This result was the first of a series of results on non-vanishing of cohomology classes in hyperbolic manifolds, see [11] for the best known results in that direction. In this paper we investigate to which extent classes dual to totally geodesic submanifolds generate the whole cohomology. We work with *congruence* hyperbolic manifolds.

1.3. First we recall the general definition of congruence hyperbolic manifolds of simple type. Let F be a totally real field and $\mathbb{A}$ the ring of adeles of F. Let V be a nondegenerate quadratic space over F with $\dim_F V = m$. We assume that $G = \mathrm{SO}(V)$ is compact at all but one infinite place. We denote by v_0 the infinite place where $\mathrm{SO}(V)$ is non compact and assume that $G(F_{v_0}) = \mathrm{SO}(p,1)$.

Consider the image $\Gamma = \Gamma_K$ in $\mathrm{SO}(p,1)_0$ of the intersection $G(F) \cap K$, where K is a compact-open subgroup of $G(\mathbb{A}_f)$ the group of finite adèlic points of G. According to a classical theorem of Borel and Harish-Chandra, it is a lattice in $\mathrm{SO}(p,1)_0$. It is a cocompact lattice if and only if G is anisotropic over F. If Γ is sufficiently deep, i.e. K is a sufficiently small compact-open subgroup of $G(\mathbb{A}_f)$, then Γ is moreover torsion-free.

The special orthogonal group $\mathrm{SO}(p)$ is a maximal compact subgroup of $\mathrm{SO}(p,1)_0$, and the quotient $\mathrm{SO}(p,1)_0/\mathrm{SO}(p)$ — the associated symmetric space — is isometric to the p-dimensional hyperbolic space D.

A *compact congruence hyperbolic manifold of simple type* is a quotient $Y_K = \Gamma\backslash D$ with $\Gamma = \Gamma_K$ a torsion-free congruence subgroup obtained as above. $\Gamma\backslash D$ is a p-dimensional *congruence* hyperbolic manifold. In general, a hyperbolic manifold is *arithmetic* if it shares a common finite cover with a congruence hyperbolic manifold.

1.4. Compact congruence hyperbolic manifolds of simple type contain many (immersed) totally geodesic codimension one submanifolds to which Proposition 1.2 applies. In fact: to any totally positive definite sub-quadratic space $U \subset V$ of dimension $n \leq p$ we associate a totally geodesic (immersed) submanifold $c(U, K)$ of codimension n in Y_K. Set $H = \mathrm{SO}(U^{\perp})$ so that $H(F_{v_0}) = \mathrm{SO}(p-n, 1)$. There is a natural morphism $H \to G$. Recall that we can realize D as the set of negative lines in V_{v_0}. We then let D_H be the subset of D consisting of those lines which lie in $U_{v_0}^{\perp}$. Let Γ_U be the image of $H(F) \cap K$ in $\mathrm{SO}(p-n, 1)_0$. The cycle $c(U, K)$ is the image of the natural map

$$\Gamma_U \backslash D_H \to \Gamma \backslash D.$$

It defines a cohomology class $[c(U, K)] \in H^n(Y_K, \mathbb{Q})$.

The following theorem can thus be thought as a converse to Proposition 1.2.

1.5. **Theorem.** *Suppose $n < \frac{p}{3}$. Let Y_K be a p-dimensional compact congruence hyperbolic manifold of simple type. Then $H^n(Y_K, \mathbb{Q})$ is spanned by the Poincaré duals of classes of totally geodesic (immersed) submanifolds of codimension n.*

Remark. Note that this result is an analogue in constant negative curvature of the results that totally geodesic flat subtori span the homology groups of flat tori (zero curvature) and totally-geodesic subprojective spaces span the homology with $\mathbb{Z}/2$-coefficients for the real projective spaces (constant positive curvature).

1.6. There are results for local coefficients analogous to Theorem 1.5 that are important for the deformation theory of locally homogeneous structures on hyperbolic manifolds. Let $\rho : \Gamma \to \mathrm{SO}(p, 1)$ be the inclusion. Suppose G is $\mathrm{SO}(p+1, 1)$ resp. $\mathrm{GL}(p+1, \mathbb{R})$. In each case we have a natural inclusion $\iota : \mathrm{SO}(p, 1) \to G$. For the case $G = \mathrm{SO}(p+1, 1)$ the image of ι is the subgroup leaving the first basis vector of $\mathbb{R}^{p+2}$ fixed, in the second the inclusion is the "identity". The representation $\widetilde{\rho} = \iota \circ \rho : \Gamma \to G$ is no longer rigid (note that $\widetilde{\rho}(\Gamma)$ has infinite covolume in G). Though there was some earlier work this was firmly established in the early 1980's by Thurston, who discovered the "Thurston bending deformations", which are non-trivial deformations $\widetilde{\rho}_t, t \in \mathbb{R}$ associated to *embedded* totally geodesic hypersurfaces $C_U = c(U, K)$ where $\dim(U) = 1$, see [39] §5, for an algebraic description of these deformations. It is known that the Zariski tangent space to the real algebraic variety $\mathrm{Hom}(\Gamma, G)$ of representations at the point $\widetilde{\rho}$ is the space of one cocycles $Z^1(\Gamma, \mathbb{R}^{p+1})$ in the first case and $Z^1(\Gamma, \mathcal{H}^2(\mathbb{R}^{p+1}))$ in the second case. Here $\mathcal{H}^2(\mathbb{R}^{p+1}))$ denotes the space of harmonic (for the Minkowski metric) degree two polynomials on $\mathbb{R}^{p+1}$. Also trivial deformations correspond to 1-coboundaries. Then Theorem 5.1 of [39] proves that the tangent vector to the curve $\widetilde{\rho}_t$ at $t = 0$ is cohomologous to the Poincaré dual of the embedded hypersurface C_U equipped with the coefficient u in the first case and the harmonic projection of $u \otimes u$ in the second where u is a suitable vector in U determining the parametrization of the curve $\widetilde{\rho}_t$.

We then have

1.7. **Theorem.** *Suppose $p \geq 4$. Let $\Gamma = \Gamma_K$ be a cocompact congruence lattice of simple type in $\mathrm{SO}(p, 1)_0$. Then $H^1(\Gamma, \mathbb{R}^{p+1})$ resp. $H^1(\Gamma, \mathcal{H}^2(\mathbb{R}^{p+1}))$ are spanned by the Poincaré duals of (possibly non-embedded) totally-geodesic hypersurfaces with coefficients in $\mathbb{R}^{p+1}$ resp. $\mathcal{H}^2(\mathbb{R}^{p+1})$.*

Remark. First, we remind the reader that the above deformation spaces of representations are locally homeomorphic to deformation spaces of locally homogeneous structures. In the first case, a hyperbolic structure on a compact manifold Y is a fortiori a flat conformal structure and a neighborhood of $\widetilde{\rho}$ in the first space of representations (into $\mathrm{SO}(p+1,1)$) is homeomorphic to a neighborhood of Y in the space of (marked) flat conformal structures. In the second case (representations into $\mathrm{PGL}(p+1,\mathbb{R})$) a neighborhood of $\widetilde{\rho}$ is homeomorphic to a neighborhood of the hyperbolic manifold in the space of (marked) flat real projective structures. Thus it is of interest to describe a neighborhood of $\widetilde{\rho}$ in these two cases. By the above theorem we know the infinitesimal deformations of $\widetilde{\rho}$ are spanned modulo coboundaries by the Poincaré duals of totally-geodesic hypersurfaces with coefficients. The first obstruction (to obtaining a curve of structures or equivalently a curve of representations) can be nonzero, see [39] who showed that the first obstruction is obtained by intersecting the representing totally geodesic hypersurfaces with coefficients. By Theorem 1.22 we can compute the first obstruction as the restriction of the wedge of *holomorphic* vector-valued one-forms on $Y^{\mathbb{C}}$ (see the second paragraph of §1.21 below for the definition of the latter). This suggests the higher obstructions will be zero and in fact the deformation spaces will be cut out from the above first cohomology groups by the vector-valued quadratic equations given by the first obstruction (see [30]).

1.8. Theorems 1.5 and 1.7 bear a strong ressemblance to the famous *Hodge conjecture* for complex projective manifolds: Let Y be a projective complex manifold. Then every rational cohomology class of type (n,n) on Y is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of Y.

Hyperbolic manifolds are not complex (projective) manifolds, so that Theorem 1.5 is not obviously related to the Hodge conjecture. We may nevertheless consider the congruence locally symmetric varieties associated to orthogonal groups $\mathrm{O}(p,2)$. These are connected Shimura varieties and as such are projective complex manifolds. As in the case of real hyperbolic manifolds, one may associate special algebraic cycles to orthogonal subgroups $\mathrm{O}(p-n,2)$ of $\mathrm{O}(p,2)$.

The proof of the following theorem now follows the same lines as the proof of Theorem 1.5.

1.9. **Theorem.** *Let Y be a connected compact Shimura variety associated to the orthogonal group $\mathrm{O}(p,2)$. Let n be an integer $< \frac{p+1}{3}$. Then every rational cohomology class of type (n,n) on Y is a linear combination with rational coefficients of the cohomology classes Poincaré dual to complex subvarieties of Y.*

Note that the complex dimension of Y is p. Hodge theory provides $H^{2n}(Y)$ with a pure Hodge structure of weight $2n$ and we more precisely prove that $H^{n,n}(Y)$ is defined over $\mathbb{Q}$ and that every rational cohomology class of type (n,n) on Y is a linear combination with rational coefficients of the cup product with some power of the Lefschetz class of cohomology classes associated to the special algebraic cycles corresponding to orthogonal subgroups.[1]

It is very important to extend Theorem 1.9 to the noncompact case since, for the case $\mathrm{O}(2,19) \cong \mathrm{O}(19,2)$ (and the correct isotropic rational quadratic forms of signature $(2,19)$ depending on a parameter g, the genus), the resulting noncompact

[1]We should note that $H^{2n}(Y)$ is of pure (n,n)-type when $n < p/4$ but this is no longer the case in general when $n \geq p/4$.

locally symmetric spaces are now the moduli spaces of quasi-polarized $K3$ surfaces $\mathcal{K}_g$ of genus g. In Theorem 1.15 of this paper we state a theorem for general orthogonal groups, $\mathrm{O}(p,q)$ that as a special case extends Theorem 1.9 to the *noncompact* case by proving that the cuspidal projections, see subsection 1.11, onto the cuspidal Hodge summand of type (n,n) of the Poincaré-Lefschetz duals of special cycles span the *cuspidal* cohomology of Hodge type (n,n) for $n < \frac{p+1}{3}$.

In [7] (with our new collaborator Zhiyuan Li), we extend this last result from the cuspidal cohomology to the reduced L^2-cohomology, see [7], Theorem 0.3.1 and hence, by a result of Zucker, to the entire cohomology groups $H^{n,n}(Y)$. In [7], we apply this extended result to the special case $\mathrm{O}(19,2)$ and prove that the Noether-Lefschetz divisors (the special cycles $c(U,K)$ with $\dim(U)=1$) generate the Picard variety of the moduli spaces $\mathcal{K}_g, g \geq 2$ thereby giving an affirmative solution of the Noether-Lefschetz conjecture formulated by Maulik and Pandharipande in [57]. We note that using work of Weissauer [85] the result that special cycles span the second homology of the noncompact Shimura varieties associated to $\mathrm{O}(n,2)$ for the special case $n=3$, was proved earlier by Hoffman and He [33]. In this case the Shimura varieties are Siegel modular threefolds and the special algebraic cycles are Humbert surfaces and the main theorem of [33] states that the Humbert surfaces rationally generate the Picard groups of Siegel modular threefolds.

Finally, we also point out that in [8] we prove the Hodge conjecture — as well as its generalization in the version first formulated (incorrectly) by Hodge — away from the middle dimensions for Shimura varieties uniformized by complex balls. The main ideas of the proof are the same, although the extension to unitary groups is quite substantial; moreover, the extension is a more subtle. In the complex case sub-Shimura varieties do not provide enough cycles, see [8] for more details.

1.10. **A general theorem.** As we explain in Sections 12 and 13, Theorems 1.5, 1.7 and 1.9 follow from Theorem 11.10 (see also Theorem 1.15 of this Introduction) which is the main result of our paper. It is concerned with general (i.e. not necessarily compact) arithmetic congruence manifolds $Y = Y_K$ associated to $G = \mathrm{SO}(V)$ as above but such that $G(F_{v_0}) = \mathrm{SO}(p,q)$ with $p+q=m$. Roughly speaking, in low degree it characterizes the subspace of the cuspidal cohomology spanned by cup-products of cuspidal projections of classes of special cycles and invariant forms in terms of a refined Hodge decomposition of the cuspidal cohomology. In the noncompact quotient case we first have to explain how to project the Poincaré-Lefschetz dual form of the special cycle onto the space of cusp forms. This is done in the next paragraph — for another construction not using Franke's Theorem see Section 10, §10.4. We define the refined Hodge decomposition of the cuspidal cohomology we just alluded to in §1.12 below.

1.11. **The cuspidal projection of the class of a special cycle with coefficients.** In case Y_K is not compact the special cycles (with coefficients) are also not necessarily compact. However, they are properly embedded and hence we may consider them as Borel-Moore cycles or as cycles relative to the Borel-Serre boundary of Y_K. The Borel-Serre boundaries of the special cycles with coefficients have been computed in [26]. The smooth differential forms on Y_K that are Poincaré-Lefschetz dual to the special cycles are not necessarily L^2 but by [22] they are cohomologous to forms with automorphic form coefficients (uniquely up to coboundaries of such forms) which can then be projected onto cusp forms since any automorphic form may be decomposed into an Eisenstein component and a cuspidal component. We will abuse notation and call the class of the resulting cusp form the *cuspidal projection of the class of the special cycle*. Since this cuspidal projection is L^2 it has a harmonic projection which can then be used to defined the refined Hodge type(s) of the cuspidal projection and hence of the original class according to the next subsection.

1.12. **The refined Hodge decomposition.** As first suggested by Chern [15] the decomposition of exterior powers of the cotangent bundle of D under the action of the holonomy group, i.e. the maximal compact subgroup of G, yields a natural notion of *refined Hodge decomposition* of the cohomology groups of the associated locally symmetric spaces. Recall that

$$D = \mathrm{SO}_0(p,q)/(\mathrm{SO}(p) \times \mathrm{SO}(q))$$

and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding (complexified) Cartan decomposition. As a representation of $\mathrm{SO}(p,\mathbb{C}) \times \mathrm{SO}(q,\mathbb{C})$ the space $\mathfrak{p}$ is isomorphic to $V_+ \otimes V_-^*$ where $V_+ = \mathbb{C}^p$ (resp. $V_- = \mathbb{C}^q$) is the standard representation of $\mathrm{SO}(p,\mathbb{C})$ (resp. $\mathrm{SO}(q,\mathbb{C})$). The refined Hodge types therefore correspond to irreducible summands in the decomposition of $\wedge^\bullet \mathfrak{p}^*$ as a $(\mathrm{SO}(p,\mathbb{C}) \times \mathrm{SO}(q,\mathbb{C}))$-module. In the case of the group $\mathrm{SU}(n,1)$ (then D is the complex hyperbolic space) it is an exercise to check that one recovers the usual Hodge-Lefschetz decomposition. But in general the decomposition is much finer and in our orthogonal case it is hard to write down the full decomposition of $\wedge^\bullet \mathfrak{p}$ into irreducible modules. Note that, as a $\mathrm{GL}(V_+) \times \mathrm{GL}(V_-)$-module, the decomposition is already quite complicated. We have (see [23, Equation (19), p. 121]):

$$\wedge^R (V_+ \otimes V_-^*) \cong \bigoplus_{\mu \vdash R} S_\mu(V_+) \otimes S_{\mu^*}(V_-)^*. \tag{1.12.1}$$

Here we sum over all partitions of R (equivalently Young diagram of size $|\mu| = R$) and μ^* is the conjugate partition (or transposed Young diagram).

It nevertheless follows from work of Vogan-Zuckerman [80] that very few of the irreducible submodules of $\wedge^\bullet \mathfrak{p}^*$ can occur as refined Hodge types of non-trivial coholomogy classes. The ones which can occur (and do occur non-trivially for some Γ) are understood in terms of cohomological representations of G. We review cohomological representations of G in Section 5. We also explain in particular how to associate to each cohomological representation π of G a *strongly primitive* refined Hodge type, see Definition 5.7. These refined Hodge type correspond to irreducible sub-$\mathrm{SO}(p) \times \mathrm{SO}(q)$-modules

$$S_{[\mu]}(V_+) \otimes S_{[\mu^*]}(V_-)^* \subset S_\mu(V_+) \otimes S_{\mu^*}(V_-)^*$$

in (1.12.1) for some special kind of partitions μ, see [9] where these special partitions are called *orthogonal*. The first degree where these refined Hodge types can occur is $R = |\mu|$. We will use the notation H^μ for the space of the cohomology in degree $R = |\mu|$ corresponding to this special Hodge type.

Note that since $\wedge^\bullet \mathfrak{p} = \wedge^\bullet (V_+ \otimes V_-^*)$ the group $\mathrm{SL}(q) = \mathrm{SL}(V_-)$ acts on $\wedge^\bullet \mathfrak{p}^*$. In this paper we will be mainly concerned with elements of $(\wedge^\bullet \mathfrak{p}^*)^{\mathrm{SL}(q)}$ — that is elements that are trivial on the V_--side, because the (cuspidal projections of the) cohomology classes dual to the special cycles all lie in the corresponding subspace of $H^\bullet_{\rm cusp}$. In the decomposition (1.12.1) the module $S_{\mu^*}(V_-)$ is the *trivial* representation of $\mathrm{SL}(V_-)$ precisely when μ is the partition $q+\ldots+q$ (n times); in that case we use the notation $\mu = n\times q$. It follows that we have:

$$(\wedge^\bullet \mathfrak{p}^*)^{\mathrm{SL}(q)} \cong \oplus_{n=0}^p S_{n\times q}(V_+)\otimes S_{q\times n}(V_-)^* = \oplus_{n=0}^p S_{n\times q}(V_+)\otimes (\wedge^q V_-)^n.$$

Note that in general it is *strictly* contained in $(\wedge^\bullet \mathfrak{p}^*)^{\mathrm{SO}(q)}$. If q is even there exists an invariant element

$$e_q \in (\wedge^q \mathfrak{p}^*)^{\mathrm{SO}(p)\times \mathrm{SL}(q)},$$

the *Euler class/form* (see subsection 5.13.1 for the definition). We define $e_q=0$ if q is odd.

1.13. **The refined Hodge decomposition of special cycles with coefficients.** We also consider general local systems of coefficients. Let λ be a dominant weight for $\mathrm{SO}(p,q)$ with at most $n\leq p/2$ nonzero entries and let $E(\lambda)$ be the associated finite dimensional representation of $\mathrm{SO}(p,q)$.

The reader will verify that the subalgebra $\wedge^\bullet(\mathfrak{p}^*)^{\mathrm{SL}(q)}$ of $\wedge^\bullet(\mathfrak{p}^*)$ is invariant under $K_\infty = \mathrm{SO}(p)\times\mathrm{SO}(q)$. Hence, we may form the associated subbundle

$$F = \mathrm{SO}_0(p,q)\times_{K_\infty} (\wedge^\bullet(\mathfrak{p}^*)^{\mathrm{SL}(q)}\otimes E(\lambda))$$

of the bundle

$$\mathrm{SO}_0(p,q)\times_{K_\infty}(\wedge^\bullet(\mathfrak{p}^*)\otimes E(\lambda))$$

of exterior powers of the cotangent bundle of D twisted by $E(\lambda)$. The space of sections of F is invariant under the Laplacian and hence under harmonic projection, compare [15, bottom of p. 105]. In case $E(\lambda)$ is trivial the space of sections of F is a subalgebra of the algebra of differential forms. In general there is still a notion of refined Hodge types (see Definition 5.7) that now correspond to irreducible sub-K_∞-modules of $\wedge^\bullet(\mathfrak{p}^*)\otimes E(\lambda)$ obtained as Cartan products of the modules $S_{[\mu]}(V_+)\otimes S_{[\mu^*]}(V_-)^*$ considered above with $E(\lambda)$. We shall therefore denote by $H^\mu(\cdot,E(\lambda))$ the corresponding cohomology groups.

We denote by $H^\bullet_{\rm cusp}(Y,E(\lambda))^{\rm SC}$ the subspace (subalgebra if $E(\lambda)$ is trivial) of $H^\bullet_{\rm cusp}(Y,E(\lambda))$ corresponding to F. Note that when $q=1$ we have $H^\bullet_{\rm cusp}(Y,E(\lambda))^{\rm SC} = H^\bullet_{\rm cusp}(Y,E(\lambda))$ and when $q=2$ we have

$$H^\bullet_{\rm cusp}(Y,E(\lambda))^{\rm SC} = \oplus_{n=0}^p H^{n,n}_{\rm cusp}(Y,E(\lambda)).$$

As above we may associate to n-dimensional totally positive sub-quadratic spaces of V special cycles of codimension nq in Y with coefficients in the finite dimensional representation $E(\lambda)$.

1.14. **Proposition.** *The projection in $H^\bullet_{\rm cusp}(Y,E(\lambda))$ of a special cycle with coefficients in $E(\lambda)$ belongs to the subspace $H^\bullet_{\rm cusp}(Y,E(\lambda))^{\rm SC}$.*

See Lemma 8.6.

It follows from Proposition 5.16 that

$$H^\bullet_{\rm cusp}(Y,E(\lambda))^{\rm SC} = \oplus_{r=0}^{[p/2]}\oplus_{k=0}^{p-2r} e_q^k H^{r\times q}_{\rm cusp}(Y,E(\lambda)). \tag{1.14.1}$$

(Compare with the usual Hodge-Lefschetz decomposition.) We call $H^{n\times q}_{\rm cusp}(Y,E(\lambda))$ the *primitive part* of $H^{nq}_{\rm cusp}(Y,E(\lambda))^{\rm SC}$. We see then that if q is odd the above special classes have *pure refined Hodge type* and if q is even each such class is the sum of at most $n+1$ refined Hodge types. In what follows we will consider the primitive part of the special cycles i.e. their projections into the subspace associated to the refined Hodge type $n \times q$.

The notion of refined Hodge type is a local Riemannian geometric one. However since we are dealing with locally symmetric spaces there is an equivalent global definition in terms of automorphic representations. Let $A_{\mathfrak{q}}(\lambda)$ be the cohomological Vogan-Zuckerman $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ where $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$ whose associated Levi subgroup L is isomorphic to $\mathrm{U}(1)^n \times \mathrm{SO}(p-2n, q)$. Let $H^{nq}_{\mathrm{cusp}}(Y, E(\lambda))_{A_{\mathfrak{q}}(\lambda)}$ denote the space of cuspidal harmonic nq-forms such that the corresponding automorphic representations of the adelic orthogonal group have distinguished (corresponding to the noncompact factor) infinite component equal to the unitary representation corresponding to $A_{\mathfrak{q}}(\lambda)$. Then we have

$$H^{nq}_{\mathrm{cusp}}(Y, E(\lambda))_{A_{\mathfrak{q}}(\lambda)} = H^{n\times q}_{\mathrm{cusp}}(Y, E(\lambda)).$$

Now Theorem 11.10 reads as:

1.15. **Theorem.** *Suppose $p > 2n$ and $m - 1 > 3n$. Then the space $H^{n\times q}_{\mathrm{cusp}}(Y, E(\lambda))$ is spanned by the cuspidal projections of classes of special cycles.*

Remark. See Subsection 1.11 for the definition of the cuspidal projection of the class of a special cycle. In case we make the slightly stronger assumption $p > 2n + 1$ it is proved in [26], see Remark 1.2, that the form of Funke-Millson is *square integrable.* We can then immediately project it into the space of cusp forms and arrive at the cuspidal projection without first passing to an automorphic representative.

In degree $R < \min(m - 3, pq/4)$ one may deduce from the Vogan-Zuckerman classification of cohomological representations that $H^{R}_{\mathrm{cusp}}(Y, E(\lambda))$ is generated by cup-products of invariant forms with primitive subspaces $H^{n\times q}_{\mathrm{cusp}}(Y, E(\lambda))$ or $H^{p\times n}_{\mathrm{cusp}}(Y, E(\lambda))$. Exchanging the role of p and q we may therefore apply Theorem 1.15 to prove the following:

1.16. **Corollary.** *Let R be an integer $< \min(m - 3, pq/4)$. Then the full cohomology group $H^{R}_{\mathrm{cusp}}(Y, E)$ is generated by cup-products of classes of totally geodesic cycles and invariant forms.*

Beside proving Theorem 1.15 we also provide strong evidence for the following:

1.17. **Conjecture.** *If $p = 2n$ or $m-1 \leq 3n$ the space $H^{n\times q}_{\mathrm{cusp}}(Y, E(\lambda))$ is* not *spanned by projections of classes of special cycles.*

In the special case $p = 3$, $q = n = 1$ we give an example of a cuspidal class of degree one in the cohomology of Bianchi hyperbolic manifolds which does not belong to the subspace spanned by classes of special cycles, see Proposition 16.15.

1.18. **Organisation of the paper.** The proof of Theorem 11.10 is the combination of three main steps.

The first step is the work of Kudla-Millson [48] — as extended by Funke and Millson [25]. It relates the subspace of the cohomology of locally symmetric spaces associated to orthogonal groups generated by special cycles to certain cohomology classes associated to the "special theta lift" using vector-valued adelic Schwartz functions with a fixed vector-valued component at infinity. More precisely, the special theta lift restricts the general theta lift to Schwartz functions that have at the distinguished infinite place where the orthogonal group is noncompact the fixed Schwartz function $\varphi_{nq,[\lambda]}$ taking values in the vector space $S_{\lambda}(\mathbb{C}^n)^* \otimes \wedge^{nq}(\mathfrak{p})^* \otimes$

$S_{[\lambda]}(V)$, see §8, at infinity. The Schwartz functions at the other infinite places are Gaussians (scalar-valued) and at the finite places are scalar-valued and otherwise arbitrary. The main point is that $\varphi_{nq,[\lambda]}$ is a relative Lie algebra cocycle for the orthogonal group allowing one to interpret the special theta lift cohomologically.

The second step, accomplished in Theorem 11.5 and depending essentially on Theorem 8.23, is to show that the intersections of the images of the general theta lift and the special theta lift just described with the subspace of the cuspidal automorphic forms that have infinite component the Vogan-Zuckerman representation $A_{\mathfrak{q}}(\lambda)$ coincide (of course the first intersection is potentially larger). In other words the special theta lift accounts for all the cohomology of type $A_{\mathfrak{q}}(\lambda)$ that may be obtained from theta lifting. This is the analogue of the main result of the paper of Hoffman and He [33] for the special case of $\mathrm{SO}(3,2)$ and our arguments are very similar to theirs. Combining the first two steps, we show that, in low degree (small n), *all* cuspidal cohomology classes of degree nq and type $A_{\mathfrak{q}}(\lambda)$ that can by obtained from the general theta lift coincide with the span of the special chomology classes dual to the special cycles of Kudla-Millson and Funke-Millson.

The third step (and it is here that we use Arthur's classification [6]) is to show that in low degree (small n) any cohomology class in $H^{nq}_{\mathrm{cusp}}(Y, E(\lambda))_{A_{\mathfrak{q}}(\lambda)}$ can be obtained as a projection of the class of a theta series. In other words, we prove the low-degree cohomological surjectivity of the general theta lift (for cuspidal classes whose refined Hodge type is that associated to $A_{\mathfrak{q}}(\lambda)$). In particular in the course of the proof we obtain the following (see Theorem 9.10):

1.19. **Theorem.** *Assume that V is anisotropic and let n be an integer such that $p > 2n$ and $m - 1 > 3n$. Then the global theta correspondence induces an isomorphism between the space of cuspidal holomorphic Siegel modular forms, of weight $S_\lambda(\mathbb{C}^n)^* \otimes \mathbb{C}_{-\frac{m}{2}}$ at v_0 and weight $\mathbb{C}_{-\frac{m}{2}}$ at all the others infinite places, on the* connected *Shimura variety associated to the symplectic group Sp_{2n}/F and the space*

$$H^{nq}(\mathrm{Sh}^0(G), E(\lambda))_{A_{\mathfrak{q}}(\lambda)} = \varinjlim_K H^{nq}(Y_K, E(\lambda))_{A_{\mathfrak{q}}(\lambda)}.$$

Combining the two steps we find that in low degree the space

$$\varinjlim_K H^{nq}(Y_K, E(\lambda))_{A_{\mathfrak{q}}(\lambda)}$$

is spanned by images of duals of special cycles. From this we deduce (again for small n) that $H^{nq}_{\mathrm{cusp}}(Y, E(\lambda))_{A_{\mathfrak{q}}(\lambda)}$ is spanned by totally geodesic cycles.

The injectivity part of the previous theorem is not new. It follows from Rallis inner product formula [71]. In our case it is due to Li, see [54, Theorem 1.1]. The surjectivity is the subject of [65, 29] that we summarize in section 2. In brief a cohomology class (or more generally any automorphic form) is in the image of the theta lift if its partial L-function has a pole far on the right. This condition may be thought of as asking that the automorphic form — or rather its lift to $\mathrm{GL}(N)$ — is very non-tempered in all but a finite number of places. To apply this result we have to relate this global condition to the local condition that our automorphic form is of a certain cohomological type at infinity.

1.20. This is where the deep theory of Arthur comes into play. We summarize Arthur's theory in Section 3. Very briefly: Arthur classifies automorphic representations of classical groups into global packets. Two automorphic representations belong to the same packet if their partial L-functions are the same i.e. if the local components of the two automorphic representations are isomorphic almost everywhere. Moreover in loose terms: Arthur shows that if an automorphic form is very non tempered at one place then it is very non tempered almost everywhere. To conclude we therefore have to study the cohomological representations at infinity and show that those we are interested in are very non-tempered, this is the main issue of section 6. Arthur's work on the endoscopic classification of representations of classical groups relates the automorphic representations of the orthogonal groups to the automorphic representations of $\mathrm{GL}(N)$ twisted by some outer automorphism θ. Note however that the relation is made through the stable trace formula for the orthogonal groups (twisted by an outer automorphism in the even case) and the stable trace formula for the twisted (non connected) group $\mathrm{GL}(N) \rtimes \langle \theta \rangle$.

Thus, as pointed above, our work uses the hypothesis made in Arthur's book. The twisted trace formula has now been stabilized (see [81] and [63]). As opposed to the case of unitary groups considered in [8], there is one more hypothesis to check. Indeed: in Arthur's book there is also an hypothesis about the twisted transfer at the Archimedean places which, in the case of orthogonal groups, is only partially proved by Mezo. This is used by Arthur to find his precise multiplicity formula. We do not use this precise multiplicity formula but we use the fact that a discrete twisted automorphic representation of a twisted $\mathrm{GL}(N)$ is the transfer from a stable discrete representation of a unique endoscopic group. So we still have to know that: *at a real place, the transfer of the stable distribution which is the sum of the discrete series in one Langlands packet is the twisted trace of an elliptic representation of* $\mathrm{GL}(N)$ *normalized using a Whittaker functional as in Arthur's book.*

Mezo [58] has proved this result up to a constant which could depend on the Langlands' packet. Arthur's [6, §6.2.2] suggests a local-global method to show that this constant is equal to 1. This is worked out in [3] appendix.

1.21. Part 4 is devoted to applications. Apart from those already mentioned, we deduce from our results and recent results of Cossutta [17] and Cossutta-Marshall [18] an estimate on the growth of the small degree Betti numbers in congruence covers of hyperbolic manifolds of simple type. We also deduce from our results an application to the non-vanishing of certain periods of automorphic forms.

We finally note that the symmetric space D embeds as a totally geodesic and totally real submanifold in the Hermitian symmetric space $D^{\mathbb{C}}$ associated to the unitary group $\mathrm{U}(p,q)$. Also there exists a representation $E(\lambda)^{\mathbb{C}}$ of $\mathrm{U}(p,q)$ whose restriction of $\mathrm{O}(p,q)$ contains the irreducible representation $E(\lambda)$. Hence there is a $\mathrm{O}(p,q)$ homomorphism from $E(\lambda)^{\mathbb{C}}|\mathrm{O}(p,q)$ to $E(\lambda)$. As explained in §8.5 the form $\varphi_{nq,\lambda}$ is best understood as the restriction of a holomorphic form $\psi_{nq,\lambda}$ on $D^{\mathbb{C}}$. Now any $Y = Y_K$ as in §1.10 embeds as a totally geodesic and totally real submanifold in a connected Shimura variety $Y^{\mathbb{C}}$ modelled on $D^{\mathbb{C}}$. And the proof of Theorem 1.15 implies:

1.22. **Theorem.** *Suppose $p > 2n$ and $m - 1 > 3n$. Then the space $H^{n\times q}_{\mathrm{cusp}}(Y, E(\lambda))$ is spanned by the restriction of* holomorphic *forms in $H^{nq,0}_{\mathrm{cusp}}(Y^{\mathbb{C}}, E(\lambda)^{\mathbb{C}})$.*

As holomorphic forms are easier to deal with, we hope that this theorem may help to shed light on the cohomology of the non-Hermitian manifolds Y.

1.23. **More comments.** General arithmetic manifolds associated to $\mathrm{SO}(p,q)$ are of two types: The simple type and the non-simple type. In this paper we only deal with the former, i.e. arithmetic manifolds associated to a quadratic space V of signature (p,q) at one infinite place and definite at all other infinite places. Indeed: the manifolds constructed that way contain totally geodesic submanifolds associated to subquadratic spaces. But when $m = p+q$ is even there are other constructions of arithmetic lattices in $\mathrm{SO}(p,q)$ commensurable with the group of units of an appropriate skew-hermitian forms over a quaternion field (see e.g. [55, Section 2]. Note that when $m = 4, 8$ there are further constructions that we won't discuss here.

Arithmetic manifolds of non-simple type do not contain as many totally geodesic cycles as those of simple type and Theorem 1.15 cannot hold. For example the real hyperbolic manifolds constructed in this way in [55] do not contain codimension 1 totally geodesic submanifolds. We should nevertheless point out that there is a general method to produce nonzero cohomology classes for these manifolds: As first noticed by Raghunathan and Venkataramana, these manifolds can be embedded as totally geodesic and totally real submanifolds in unitary arithmetic manifolds of *simple type*, see [70]. On the latter a general construction due to Kazhdan and extended by Borel-Wallach [13, Chapter VIII] produces nonzero holomorphic cohomology classes as theta series. Their restrictions to the totally real submanifolds we started with can produce nonzero cohomology classes and Theorem 1.22 should still hold in that case. This would indeed follow from our proof modulo the natural extension of [29, Theorem 1.1 (1)] for unitary groups of skew-hermitian forms over a quaternion field.

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Part 1. **Automorphic forms**

2. Theta liftings for orthogonal groups: some background

2.1. **Notations.** Let F be a totally real number field and $\mathbb{A}$ the ring of adeles of F. Let V be a nondegenerate quadratic space over F with $\dim_F V = m$.

2.2. **The theta correspondence.** Let X be a symplectic F-space with $\dim_F X = 2p$. We consider the tensor product $X \otimes V$. It is naturally a symplectic F-space and we let $\mathrm{Sp}(X \otimes V)$ be the corresponding symplectic F-group. Then $(\mathrm{O}(V), \mathrm{Sp}(X))$ forms a reductive dual pair in $\mathrm{Sp}(X \otimes V)$, in the sense of Howe [34]. We denote by $\mathrm{Mp}(X)$ the metaplectic double cover of $\mathrm{Sp}(X)$ if m is odd or simply $\mathrm{Sp}(X)$ is m is even but as $\mathrm{Mp}(X)$ is not an algebraic group we have to be more precise: $\mathrm{Mp}(X, \mathbb{A})$ is a non split two fold cover of $\mathrm{Sp}(X, \mathbb{A})$ which contains canonicaly $\mathrm{Sp}(X, F)$. At each local place, v of F we denote by $\mathrm{Mp}(X, F_v)$ the reciprocal image of $\mathrm{Sp}(X, F_v)$. The group $\mathrm{Mp}(X, F_v)$ is a non split two fold cover of $\mathrm{Sp}(X, F_v)$ and if v is non archimedean and of residual characteristic different from 2, there is a canonical splitting of $\mathrm{Sp}(X, \mathcal{O}_v)$ in $\mathrm{Mp}(X, F_v)$. In this way, it is easy to define automorphic forms on $\mathrm{Mp}(X, \mathbb{A})$. An automorphic form, ϕ, on $\mathrm{Mp}(X, \mathbb{A})$ is called genuine if it satisfies $\phi(zg) = z\phi(g)$ for any $z \in \{\pm 1\}$ in the kernel of the map from $\mathrm{Mp}(X, \mathbb{A})$ onto $\mathrm{Sp}(X, \mathbb{A})$. We will only consider genuine automorphic forms on $\mathrm{Mp}(X, \mathbb{A})$. Any unipotent radical N of a parabolic subgroup of $\mathrm{Sp}(X)$ is such that $N(\mathbb{A})$

has a canonical lift in $\mathrm{Mp}(X, \mathbb{A})$. This allows us to define the notion of cuspidal automorphic forms on $\mathrm{Mp}(X, \mathbb{A})$.

For a non-trivial additive character ψ of $\mathbb{A}/F$, we may define the oscillator representation ω_ψ. We recall that our field is totally real. It is an automorphic representation of the metaplectic double cover $\widetilde{\mathrm{Sp}}(X \otimes V)$ of $\mathrm{Sp} = \mathrm{Sp}(X \otimes V)$, which is realized in the Schrödinger model. We have $\mathrm{Sp}(X \otimes V)(F \otimes \mathbb{R}) \cong \mathrm{Sp}_{2pm}(\mathbb{R})^{[F:\mathbb{Q}]}$. The maximal compact subgroup of $\mathrm{Sp}_{2pm}(\mathbb{R})$ is $\mathrm{U} = \mathrm{U}_{pm}$, the unitary group in pm variables. We denote by $\widetilde{\mathrm{U}}$ its preimage in $\widetilde{\mathrm{Sp}}_{2pm}(\mathbb{R})$. The associated space of smooth vectors of ω is the Bruhat-Schwartz space $\mathcal{S}(V(\mathbb{A})^p)$. The $(\mathfrak{sp}, \widetilde{\mathrm{U}})$-module associated to ω is made explicit by the realization of ω known as the Fock model that we will brefly review in §7. Using it, one sees that the $\widetilde{\mathrm{U}}^{[F:\mathbb{Q}]}$-finite vectors in ω is the subspace $\mathbf{S}(V(\mathbb{A})^p) \subset \mathcal{S}(V(\mathbb{A})^p)$ obtained by replacing, at each infinite place, the Schwartz space by the *polynomial Fock space* $\mathbf{S}(V^p) \subset \mathcal{S}(V^p)$, i.e. the image of holomorphic polynomials on $\mathbb{C}^{pm}$ under the intertwining map from the Fock model of the oscillator representation to the Schrödinger model.

2.3. We denote by $\mathrm{O}_m(\mathbb{A})$, $\mathrm{Mp}_{2p}(\mathbb{A})$ and $\widetilde{\mathrm{Sp}}_{2pm}(\mathbb{A})$ the adelic points of respectively $\mathrm{O}(V)$, $\mathrm{Mp}(X)$ and $\widetilde{\mathrm{Sp}}(X \otimes V)$. The global metaplectic group $\widetilde{\mathrm{Sp}}_{2pm}(\mathbb{A})$ acts in $\mathcal{S}(V(\mathbb{A})^p)$ via ω. For each $\phi \in \mathcal{S}(V(\mathbb{A})^p)$ we form the theta function

$$\theta_{\psi,\phi}(x) = \sum_{\xi \in V(F)^p} \omega_\psi(x)(\phi)(\xi) \tag{2.3.1}$$

on $\widetilde{\mathrm{Sp}}_{2pm}(\mathbb{A})$. There is a natural homomorphism

$$\mathrm{O}_m(\mathbb{A}) \times \mathrm{Mp}_{2p}(\mathbb{A}) \to \widetilde{\mathrm{Sp}}_{2pm}(\mathbb{A})$$

which is described with great details in [38]. We pull the oscillator representation ω_ψ back to $\mathrm{O}_m(\mathbb{A}) \times \mathrm{Mp}_{2p}(\mathbb{A})$. Then $(g, g') \mapsto \theta_{\psi,\phi}(g', g)$ is a smooth, slowly increasing function on $\mathrm{O}(V)\backslash\mathrm{O}_m(\mathbb{A}) \times \mathrm{Mp}(X)\backslash\mathrm{Mp}_{2p}(\mathbb{A})$; see [84, 34].

2.4. **The global theta lifting.** We denote by $\mathcal{A}^c(\mathrm{Mp}(X))$ the set of irreducible cuspidal automorphic representations of $\mathrm{Mp}_{2p}(\mathbb{A})$, which occur as irreducible subspaces in the space of cuspidal automorphic functions in $L^2(\mathrm{Mp}(X)\backslash\mathrm{Mp}_{2p}(\mathbb{A}))$. For a $\pi' \in \mathcal{A}^c(\mathrm{Mp}(X))$, the integral

$$\theta^f_{\psi,\phi}(g) = \int_{\mathrm{Mp}(X)\backslash\mathrm{Mp}_{2p}(\mathbb{A})} \theta_{\psi,\phi}(g, g') f(g') dg', \tag{2.4.1}$$

with $f \in H_{\pi'}$ (the space of π'), defines an automorphic function on $\mathrm{O}_m(\mathbb{A})$: the integral (2.4.1) is well defined, and determines a slowly increasing function on $\mathrm{O}(V)\backslash\mathrm{O}_m(\mathbb{A})$. We denote by $\Theta^V_{\psi,X}(\pi')$ the space of the automorphic representation generated by all $\theta^f_{\psi,\phi}(g)$ as ϕ and f vary, and call $\Theta^V_{\psi,X}(\pi')$ the ψ-theta lifting of π' to $\mathrm{O}_m(\mathbb{A})$. Note that, since $\mathbf{S}(V(\mathbb{A})^p)$ is dense in $\mathcal{S}(V(\mathbb{A})^p)$ we may as well let ϕ vary in the subspace $\mathbf{S}(V(\mathbb{A})^p)$.

We can similarly define $\mathcal{A}^c(\mathrm{O}(V))$ and $\Theta^X_{\psi,V}$ the ψ-theta correspondence from $\mathrm{O}(V)$ to $\mathrm{Mp}(X)$.

2.5. It follows from [65] and from [38, Theorem 1.3] that if $\Theta^V_{\psi,X}(\pi')$ contains nonzero cuspidal automorphic functions on $\mathrm{O}_m(\mathbb{A})$ then the representation of $\mathrm{O}_m(\mathbb{A})$ in $\Theta^V_{\psi,X}(\pi')$ is irreducible (and cuspidal). We also denote by $\Theta^V_{\psi,X}(\pi')$ the corresponding element of $\mathcal{A}^c(\mathrm{O}(V))$. In that case it moreover follows from [65] and [38, Theorem 1.1] that

$$\Theta^X_{\psi^{-1},V}(\Theta^V_{\psi,X}(\pi')) = \pi'.$$

We say that a representation $\pi \in \mathcal{A}^c(\mathrm{O}(V))$ *is in the image of the cuspidal ψ-theta correspondence from a smaller group* if there exists a symplectic space X with $\dim X \leq m$ and a representation $\pi' \in \mathcal{A}^c(\mathrm{Mp}(X))$ such that

$$\pi = \Theta^V_{\psi,X}(\pi').$$

2.6. The main technical point of this paper is to prove that if $\pi \in \mathcal{A}^c(\mathrm{O}(V))$ is such that its *local* component at infinity is "sufficiently non-tempered" (this has to be made precise) then the *global* representation π is in the image of the cuspidal ψ-theta correspondence from a smaller group.

As usual we encode local components of π into an L-function. In fact we only consider its *partial L-function* $L^S(s,\pi) = \prod_{v\notin S} L(s,\pi_v)$ where S is a sufficiently big finite set of places such that π_v is unramified for each $v \notin S$. For such a v we define the local factor $L(s,\pi_v)$ by considering the Langlands parameter of π_v.

Remark. We will loosely identify the partial L-function of π and that of its restriction to $\mathrm{SO}(V)$. However we should note that the restriction of π_v to the special orthogonal group may be reducible: If $v \notin S$ we may associate to the Langlands parameter of π_v representations from the principal series of the special orthogonal group. Each of these has a unique unramified subquotient and the restriction of π_v to the special orthogonal group is then the sum of the non-isomorphic subquotients.[2] Anyway: the local L-factor is the same for each summand of the restriction as it only depends on the Langlands parameter.

We may generalize these definitions to form the partial L-functions $L^S(s,\pi\times\eta)$ for any automorphic character η.

Now the following proposition is a first important step toward the proof that a "sufficiently non-tempered" automorphic representation is in the image of the cuspidal ψ-theta correspondence from a smaller group. It is symmetric to [49, Theorem 7.2.5] and is the subject of [65] and [29, Theorem 1.1 (1)]; it is revisited and generalized in [27].

2.7. **Proposition.** *Let $\pi \in \mathcal{A}^c(\mathrm{O}(V))$ and let η be a quadratic character of $F^*\backslash\mathbb{A}^*$. Let a be a nonnegative integer with $a+1 \equiv m \bmod 2$. We assume that the partial L-function $L^S(s,\pi\times\eta)$ is holomorphic in the half-plane $\mathrm{Re}(s) > \frac{1}{2}(a+1)$ and has a pole in $s = \frac{1}{2}(a+1)$. Let $p = \frac{1}{2}(m-a-1)$ and X be a symplectic F-space with $\dim X = 2p$.*

Then there exists an automorphic sign character ϵ of $\mathrm{O}_m(\mathbb{A})$ (a character of $\mathrm{O}_m(\mathbb{A})$ trivial on $\mathrm{O}_m(F)\mathrm{SO}_m(\mathbb{A})$) such that the ψ^{-1}-theta lifting of $(\pi\otimes\eta)\otimes\epsilon$ to $\mathrm{Mp}_{2p}(\mathbb{A})$ does not vanish.

The big second step to achieve our first goal will rely on Arthur's theory.

[2]There are at most two such non-isomorphic subquotients.

3. Arthur's theory

3.1. **Notations.** Let F be a number field, $\mathbb{A}$ its ring of adeles and $\Gamma_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Let V be a nondegenerate quadratic space over F with $\dim_F V = m$. We let G be the special orthogonal group $\mathrm{SO}(V)$ over F. We set $\ell = [m/2]$ and $N = 2\ell$.

The group G is an inner form of a quasi-split form G^*. As for now Arthur's work only deals with quasi-split groups. We first describe the group G^* according to the parity of m and briefly recall the results of Arthur we shall need. We recall from the introduction that Arthur's work relies on extensions to the twisted case of two results which have only been proved so far in the case of connected groups: the first is the stabilization of the twisted trace formula for the two groups $\mathrm{GL}(N)$ and $\mathrm{SO}(2n)$, see [6, Hypothesis 3.2.1]. The second is Shelstad's strong spectral transfer of tempered archimedean characters. Taking these for granted we will explain how to deal with non-quasi-split groups in the next section.

3.2. We first assume that $m = N+1$ is odd. Then the special orthogonal group G is an inner form of the *split* form $G^* = \mathrm{SO}(m)$ over F associated to the symmetric bilinear form whose matrix is

$$J = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}.$$

The (complex) dual group of G^* is $G^\vee = \mathrm{Sp}(N, \mathbb{C})$ and ${}^L G = G^\vee \times \Gamma_F$.

3.3. We now assume that $m = N$ is even. We let $\mathrm{SO}(N)$ be the *split* orthogonal group over F associated to the symmetric bilinear form whose matrix is J. The *quasi-split* forms of $\mathrm{SO}(N)$ are parametrized by morphisms $\Gamma_F \to \mathbb{Z}/2\mathbb{Z}$, which by class field theory correspond to characters η on $F^* \backslash \mathbb{A}^*$ such that $\eta^2 = 1$ — *quadratic Artin characters*. We denote by $\mathrm{SO}(N, \eta)$ the outer twist of the split group $\mathrm{SO}(N)$ determined by η: the twisting is induced by the action of Γ_F on the Dynkin diagram via the character η.

When $m = N$ is even, there exists a quadratic Artin character η such that G is an inner form of the *quasi-split* group $G^* = \mathrm{SO}(N, \eta)$. The (complex) dual group of G^* is then $G^\vee = \mathrm{SO}(N, \mathbb{C})$ and ${}^L G = G^\vee \rtimes \Gamma_F$, where Γ_F acts on $G^\vee$ by an order 2 automorphism — trivial on the kernel of η — and fixes a splitting, see [10, p. 79] for an explicit description.

Remark. Let v be an infinite real place of F such that $G(F_v) \cong \mathrm{SO}(p,q)$ with $m = p+q$ even so that $m = 2\ell$. Then η_v is trivial if and only if $(p-q)/2$ is even. We are lead to the following dichotomy for real orthogonal groups: if $(p-q)/2$ is odd, $\mathrm{SO}(p,q)$ is an inner form of $\mathrm{SO}(\ell-1, \ell+1)$ and if $(p-q)/2$ is even, $\mathrm{SO}(p,q)$ is an inner form of $\mathrm{SO}(\ell,\ell)$ (split over $\mathbb{R}$).

3.4. **Global Arthur parameters.** In order to extend the classification [62] of the discrete automorphic spectrum of $\mathrm{GL}(N)$ to the classical groups, Arthur represents the discrete automorphic spectrum of $\mathrm{GL}(N)$ by a set of formal tensor products

$$\Psi = \mu \boxtimes R,$$

where μ is an irreducible, unitary, cuspidal automorphic representation of $\mathrm{GL}(d)$ and R is an irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension n, for positive integers

d and n such that $N = dn$. For any such Ψ, we form the induced representation

$$\mathrm{ind}(\mu|\cdot|^{\frac{1}{2}(n-1)}, \mu|\cdot|^{\frac{1}{2}(n-3)}, \ldots, \mu|\cdot|^{\frac{1}{2}(1-n)})$$

(normalized induction from the standard parabolic subgroup of type $(d, \ldots, d)$). We then write Π_Ψ for the unique irreducible quotient of this representation.

We may more generally associate an automorphic representation Π_Ψ of $\mathrm{GL}(N)$ to a formal sum of formal tensor products:

$$\Psi = \mu_1 \boxtimes R_1 \boxplus \ldots \boxplus \mu_r \boxtimes R_r \quad (3.4.1)$$

where each μ_j is an irreducible, unitary, cuspidal automorphic representation of $\mathrm{GL}(d_i)/F$, R_j is an irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension n_j and $N = n_1 d_1 + \ldots + n_r d_r$.

Now consider the outer automorphism:

$$\theta : x \mapsto J^t x^{-1} J = J^t x^{-1} J^{-1} \quad (x \in \mathrm{GL}(N)).$$

This induces an action $\Pi_\Psi \mapsto \Pi_\Psi^\theta$ on the set of representations Π_Ψ. If Ψ is as in (3.4.1), set

$$\Psi^\theta = \mu_1^\theta \boxtimes R_1 \boxplus \ldots \boxplus \mu_r^\theta \boxtimes R_r.$$

Then $\Pi_\Psi^\theta = \Pi_{\Psi^\theta}$.

Arthur's main result [6, Theorem 1.5.2] (see also [5, Theorem 30.2]) then parametrizes the discrete automorphic spectrum of G^* by formal sum of formal tensor products Ψ as in (3.4.1) such that:

1. the $\mu_j \boxtimes R_j$ in (3.4.1) are all distinct,
2. for each j, $\mu_j^\theta \boxtimes R_j = \mu_j \boxtimes R_j$,
3. for each j, the parity of the dimension of R_j is determined by μ_j and G^*.

3.5. **Local Arthur parameters.** Assume that $k = F_v$ is local and let W_k' be its Weil-Deligne group. We can similarly define parameters over k. We define a local parameter over k as a formal sum of formal tensor product (3.4.1) where each μ_j is now a tempered irreducible representation of $\mathrm{GL}(d_j, k)$ that is square integrable modulo the center.[3] The other components R_j remain irreducible representations of $\mathrm{SL}_2(\mathbb{C})$. To each $\mu_j \boxtimes R_j$ we associate the unique irreducible quotient Π_j of

$$\mathrm{ind}(\mu_j|\cdot|^{\frac{1}{2}(n_j-1)}, \mu_j|\cdot|^{\frac{1}{2}(n_j-3)}, \ldots, \mu_j|\cdot|^{\frac{1}{2}(1-n_j)})$$

(normalized induction from the standard parabolic subgroup of type $(d_j, \ldots, d_j)$). We then define Π_Ψ as the induced representation

$$\mathrm{ind}(\Pi_1 \otimes \ldots \otimes \Pi_r)$$

(normalized induction from the standard parabolic subgroup of type $(n_1 d_1, \ldots, n_r d_r)$). It is irreducible and unitary. Finally, the local parameters for the classical group G are those parameters which (up to conjugation) factorize through the dual group of $G(k)$. Then, in particular, the associated representation, Π_Ψ, of $\mathrm{GL}(N, k)$ ($N = \sum_{i \in [1,r]} d_i n_i$) is theta-stable: $\Pi_\Psi \circ \theta \cong \Pi_\Psi$. It is proved by Arthur, that

[3]Because we do not know that the extension to $\mathrm{GL}(N)$ of Ramanujan's conjecture is valid, we do not know that the local components of automorphic representations of $\mathrm{GL}(N)$ are indeed tempered. So that in principle the μ_j are not necessarily tempered: their central characters need not be unitary. This requires a minor generalization that Arthur addresses in [5, Remark 3 p. 247]. Anyway the approximation to Ramanujan's conjecture proved by Luo, Rudnick and Sarnak [56] is enough for our purposes and it makes notations easier to assume that each μ_j is indeed tempered.

a global parameter for G^* localizes at each place v of F in a local parameter for G^*.

3.6. **Local Arthur packets.** Assume that $k = F_v$ is local and that $G(k) = G^*(k)$ is quasi-split. We now recall how Arthur associates a finite packet of representations of $G(k)$ to a local parameter Ψ for the group G.

We say that two functions in $C_c^\infty(G(k))$ are stably equivalent if they have the same stable orbital integrals, see e.g. [52]. Thanks to the recent proofs by Ngo [69] of the fundamental lemma and Waldspurger's work [82], we have a natural notion of transfer $f \rightsquigarrow f^G$ from a test function $f \in C_c^\infty(\mathrm{GL}(N,k))$ to a representative f^G of a stable equivalence class of functions in $C_c^\infty(G(k))$ such that f and f^G are *associated* i.e. they have matching stable orbital integrals [4], see [43] for more details about twisted transfer. Over Archimedean places existence of transfer is due to Shelstad, see [76]; we note that in that case being K-finite is preserved by transfer. When $G = \mathrm{SO}(N,\eta)$ one must moreover ask that f^G is invariant under an outer automorphism α of G; we may assume that $\alpha^2 = 1$.

Let Ψ be a local parameter as above and $\mathcal{H}_{\Pi_\Psi}$ be the space of Π_Ψ. We fix an intertwining operator $A_\theta : \mathcal{H}_{\Pi_\Psi} \to \mathcal{H}_{\Pi_\Psi}$ ($A_\theta^2 = 1$) intertwining Π_Ψ and $\Pi_\Psi \circ \theta$.

When $G = \mathrm{SO}(N,\eta)$ we identify the irreducible representations π of $G(k)$ that are conjugated by α. Then trace$\pi(f^G)$ is well defined when f^G is as explained above.

The following proposition follows from [6, Theorem 2.2.1] (see also [5, Theorem 30.1]).

3.7. **Proposition.** *There exists a finite family $\prod(\Psi)$ of representations of $G(k)$, and some multiplicities $m(\pi) > 0$ ($\pi \in \prod(\Psi)$) such that, for associated f and f^G :*

$$\mathrm{trace}(\Pi_\Psi(f)A_\theta) = \sum_{\pi \in \prod(\Psi)} \varepsilon(\pi) m(\pi) \mathrm{trace}\ \pi(f^G), \tag{3.7.1}$$

where each $\varepsilon(\pi)$ is a sign $\in \{\pm 1\}$.

We remark that (3.7.1) uniquely determines $\prod(\Psi)$ as a set of representations-with-multiplicities; it also uniquely determines the signs $\varepsilon(\pi)$. In fact Arthur explicitly computes these signs for some particular choice of an intertwiner A_θ.

By the local Langlands correspondence, a local parameter Ψ for $G(k)$ can be represented as a homomorphism

$$\Psi : W_k' \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G. \tag{3.7.2}$$

Arthur associates to such a parameter the L-parameter $\varphi_\Psi : W_k' \to {}^L G$ given by

$$\varphi_\Psi(w) = \Psi\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right).$$

One key property of the local Arthur packet $\prod(\Psi)$ is that it contains all representations of Langlands' L-packet associated to φ_Ψ. This is proved by Arthur, see also [67, Section 6].

Ignoring the minor generalization needed to cover the lack of Ramanujan's conjecture, the global part of Arthur's theory (see [6, Corollary 3.4.3] when G is quasi-split and [6, Proposition 9.5.2] in general) now implies:

[4] Here the orbital integrals on the $\mathrm{GL}(N)$-side are *twisted* orbital integrals.

3.8. **Proposition.** *Let π be an irreducible automorphic representation of $G(\mathbb{A})$ which occurs (discretely) as an irreducible subspace of $L^2(G(F)\backslash G(\mathbb{A}))$. Then there exists a global Arthur parameter Ψ and a finite set S of places of F containing all Archimedean ones such that for all $v \notin S$, the group $G(F_v) = G^*(F_v)$ is quasi-split, the representation π_v is unramified and the L-parameter of π_v is φ_{Ψ_v}.*

Remark. Proposition 3.8 in particular implies that the $\mathrm{SL}_2(\mathbb{C})$ part of a local Arthur parameter has a *global* meaning. This puts serious limitations on the kind of non-tempered representations which can occur discretely: e.g. an automorphic representation π of $G^*(\mathbb{A})$ which occurs discretely in $L^2(G^*(F)\backslash G^*(\mathbb{A}))$ and which is non-tempered at one place v is non-tempered almost everywhere in particular each place $v \notin S$ (where S is as above).

The above remark explains how Arthur's theory will be used in our proof. This will be made effective through the use of L-functions.

3.9. **Application to L-functions.** Let π be an irreducible automorphic representation of $G(\mathbb{A})$ which occurs (discretely) as an irreducible subspace of $L^2(G(F)\backslash G(\mathbb{A}))$ and let

$$\Psi = \mu_1 \boxtimes R_1 \boxplus \ldots \boxplus \mu_r \boxtimes R_r$$

be its global Arthur parameter (Proposition 3.8). We factor each $\mu_j = \otimes_v \mu_{j,v}$ where v runs over all places of F. Let S be a finite set of places of F containing the set S of Proposition 3.8, and all v for which either one of $\mu_{j,v}$ or π_v is ramified. We can then define the formal Euler product

$$L^S(s,\Pi_\Psi) = \prod_{j=1}^{r} \prod_{v \notin S} L_v(s - \frac{n_j - 1}{2}, \mu_{j,v}) L_v(s - \frac{n_j - 3}{2}, \mu_{j,v}) \ldots L_v(s - \frac{1 - n_j}{2}, \mu_{j,v}).$$

Note that $L^S(s,\Pi_\Psi)$ is the partial L-function of a very special automorphic representation of $\mathrm{GL}(N)$ with $N = \sum_{j \in [1,r]} d_j n_j$ (here as above n_j is the dimension of the representation R_j and d_j is such that μ_j is an automorphic cuspidal représentation of $GL(d_j, \mathbb{A})$); it is the product of partial L-functions of the square integrable automorphic representations associated to the parameters $\mu_j \boxtimes R_j$. According to Jacquet and Shalika [37] $L^S(s,\Pi_\Psi)$, which is a product absolutely convergent for $\mathrm{Re}(s) \gg 0$, extends to a meromorphic function of s. It moreover follows from Proposition 3.8 and the definition of $L^S(s,\pi)$ that:

$$L^S(s,\pi) = L^S(s,\Pi_\Psi).$$

Remark. Given an automorphic character η we can similarly write $L^S(s, \eta \times \pi)$ as a product of L-functions associated to linear groups: replace μ_j by $\eta \otimes \mu_j$ in the above discussion (note that each μ_j is self-dual).

We can now relate Arthur's theory to Proposition 2.7:

3.10. **Lemma.** *Let $\pi \in \mathcal{A}^c(G)$ whose global Arthur parameter Ψ is a sum*

$$\Psi = \left(\boxplus_{(\rho,b)} \rho \boxtimes R_b \right) \boxplus \eta \boxtimes R_a$$

where η is a selfdual (quadratic) automorphic character and for each pair (ρ, b), either $b < a$ or $b = a$ and $\rho \not\cong \eta$. Then the partial L-function $L^S(s, \eta \times \pi)$ — here S is a finite set of places containing the set S of Proposition 3.8 and all the places where η ramifies — is holomorphic in the half-plane $\mathrm{Re}(s) > (a+1)/2$ and it has a simple pole in $s = (a+1)/2$.

Proof. Writing $L^S(s,\eta\times\pi)$ explicitly on a right half-plane of absolute convergence; we get a product of $L^S(s-(a-1)/2,\eta\times\eta)$ by factors $L^S(s-(b-1)/2,\eta\times\rho)$. Our hypothesis on a forces $b\le a$ and if $b=a$, $\rho\not\simeq\eta$. The conclusion of the lemma follows. □

3.11. **Infinitesimal character.** Let π_{v_0} be the local Archimedean factor of a representation $\pi\in\mathcal{A}^c(G)$ with global Arthur parameter Ψ. We may associate to Ψ the parameter $\varphi_{\Psi_{v_0}}:\mathbb{C}^*\to G^\vee\subset \mathrm{GL}(N,\mathbb{C})$ given by

$$\varphi_{\Psi_{v_0}}(z)=\Psi_{v_0}\left(z,\begin{pmatrix}(z\overline{z})^{1/2} & \\ & (z\overline{z})^{-1/2}\end{pmatrix}\right).$$

Being semisimple, it is conjugate into the maximal torus

$$T^\vee=\{\mathrm{diag}(x_1,\dots,x_\ell,x_\ell^{-1},\dots,x_1^{-1})\}$$

of $G^\vee$. We may therefore write $\varphi_{\Psi_{v_0}}=(\eta_1,\dots,\eta_\ell,\eta_\ell^{-1},\dots,\eta_1^{-1})$ where each η_j is a character $z\mapsto z^{P_i}\overline{z}^{Q_i}$. One easily checks that the vector

$$\nu_\Psi=(P_1,\dots,P_\ell)\in\mathbb{C}^\ell\cong\mathrm{Lie}(T)\otimes\mathbb{C}$$

is uniquely defined modulo the action of the Weyl group W of $G(F_{v_0})$. The following proposition is detailed in [11].

3.12. **Proposition.** *The infinitesimal character of π_{v_0} is the image of ν_Ψ in $\mathbb{C}^\ell/W$.*

Recall that the infinitesimal character ν_Ψ is said to be *regular* if it is not fixed by an element in the Weyl group. In particular we have $P_j\neq\pm P_k$ for all $j\neq k$ and $P_j\neq 0$ except eventually for one P_i in case m is even.

4. A surjectivity theorem for theta liftings

4.1. **Notations.** Let F be a number field and $\mathbb{A}$ be its ring of adeles. Fix ψ a non-trivial additive character of $\mathbb{A}/F$. Let V be a nondegenerate quadratic space over F with $\dim_F V=m$. We set $\ell=[m/2]$ and $N=2\ell$.

We say that a representation $\pi\in\mathcal{A}^c(\mathrm{SO}(V))$ — i.e. an irreducible cuspidal automorphic representation of $\mathrm{SO}(V)$ — is in the image of the cuspidal ψ-theta correspondence from a smaller group if there exists a symplectic space X with $\dim X\le N$ and an extension $\tilde\pi$ of π to $\mathrm{O}(V)$ such that $\tilde\pi$ is the image of a cuspidal automorphic form of $\mathrm{Mp}(X)$ by the ψ-theta correspondence. Here we assume that the extension exists but this is necessary in order that a cuspidal representation of $\mathrm{O}(V,\mathbb{A})$ containing π in its restriction to $\mathrm{SO}(V,\mathbb{A})$ is the image in the theta correspondence with a smaller group: let v be a place, the extension $\tilde\pi_v$ only exists if $\mathrm{O}(V,F_v)$ acts trivialy on the isomorphism class of π_v or equivalently if for an irreducible representation $\tilde\pi_v$ of $\mathrm{O}(V,F_v)$ containing π_v in its restriction, $\tilde\pi_v$ is not isomorphic to $\tilde\pi_v\otimes\epsilon_v$ where ϵ_v is the non trivial character of $\mathrm{O}(V,F_v)$ trivial on $\mathrm{SO}(V,F_v)$. Kudla and Rallis in the non Archimedean case in [50], and Sun and Zhu [77] in the Archimedean case (generalizing [50]), have proved that only one of the two representations $\tilde\pi_v$ or $\tilde\pi_v\otimes\epsilon_v$ can be a theta lift from a representation of the group $\mathrm{Mp}(X,F_v)$ if dim $X\le N$. Remark that if $\tilde\pi$ is a cuspidal representation of $\mathrm{O}(V,\mathbb{A})$ containing π and coming from a small theta lift, the same is true for any of its local component. In particular in that case, the restriction to $\mathrm{SO}(V,\mathbb{A})$ of $\tilde\pi$ is equal to π; moreover there is at most one such $\tilde\pi$.

Here we combine Propositions 2.7 and 3.8 to prove the following result which is the main automorphic ingredient in our work.

We say that a global representation $\pi \in \mathcal{A}^c(\mathrm{SO}(V))$ is *highly non-tempered* if its global Arthur parameter Ψ contains a factor $\eta \boxtimes R_a$ where η is a quadratic character and $3a > m-1$ and, if v be a place of F, we say that π is *highly non-tempered at the place* v if its local Arthur parameter at the place v contains a factor $\eta_v \boxtimes R_a$ with η_v a local quadratic character and $3a > m-1$.

4.2. **Theorem.** *Let $\pi \in \mathcal{A}^c(\mathrm{SO}(V))$. Assume that π is highly non-tempered at the place v_0 and that π_{v_0} has a regular infinitesimal character. Then there exists an automorphic quadratic character χ such that $\pi \otimes \chi$ is in the image of the cuspidal ψ-theta correspondence from a smaller group associated to a symplectic space of dimension $m-a-1$.*

Proof. Let $G = \mathrm{SO}(V)$ and let $\tilde{\pi} \in \mathcal{A}^c(\mathrm{O}(V))$ be an irreducible representation containing π in its restriction to $\mathrm{SO}(V, \mathbb{A})$. Recall from Remark 2.6 that the partial L-function $L^S(s, \tilde{\pi}) = L^S(s, \pi)$ and see the remark below the next lemma.

4.3. **Lemma.** *The global Arthur parameter of π is a sum*

$$\Psi = \left(\boxplus_{(\rho,b)} \rho \boxtimes R_b\right) \boxplus \eta \boxtimes R_a$$

where η is a selfdual (quadratic) automorphic character and each pair (ρ, b) consists of a (selfdual) cuspidal automorphic representation ρ of some $\mathrm{GL}(d_\rho)$ and a positive integer $b < a$ such that $\sum_{(\rho,b)} bd_\rho + a = N$.

Proof. The global Arthur parameter of π has to localize at the place v_0 in a parameter containing a factor $\eta_0 \boxtimes R_a$ where η_0 is a quadratic character. So the global parameter is necessarily a sum $\Psi = \delta \boxtimes R_a \boxplus \boxplus_{(\rho,b)} \rho \boxtimes R_b$ (without multiplicity) where δ_{v_0} contains η_0. Denote by d_0 the dimension of the representation δ. Then

$$ad_0 + \sum_{(\rho,b)} d_\rho b = N.$$

We have $N = m-1$ if m is odd and certainly by the hypothesis $3a > m-1$, $d_0 \leq 2$ in that case; if $d_0 = 2$, the localization δ_{v_0} of δ is the sum of two quadratic characters of $\mathbb{R}^*$ and they both contribute to the infinitesimal character, according to §3.11 by $\left(\frac{a-1}{2}, \frac{a-3}{2}, \dots, \frac{1}{2}\right)$. This is in contradiction with the hypothesis that the infinitesimal character is regular. We conclude that $d_0 = 1$.

If m is even, then $N = m$, we only have $d_0 \leq 3$. Assume first that $d_0 = 3$ which is only possible if $3a = m$: in that case the localization of δ is either the sum of three characters, η_0, μ, μ^{-1} with μ a unitary character or the sum of η_0 with the parameter of a discrete series of $\mathrm{GL}(2, \mathbb{R})$. In these two cases, δ_{v_0} is orthogonal and a has to be odd. This is a contradiction with the equality $3a = m$ because, here, m is even. We rule out $d_0 = 2$ as above: $\delta_{v_0} \boxtimes R_a$ will contribute to the infinitesimal character by two copies of $\left(\frac{a-1}{2}, \frac{a-3}{2}, \dots, 1, 0\right)$ in contradiction with the regularity of the infinitesimal character.

We have proved, so far, that $d_0 = 1$, which mean that δ is a quadractic character, denoted form now on as η. This also implies that $a \equiv m+1$ modulo 2 because $\eta \otimes R_a$ has to be orthogonal (resp. symplectic) if the dual group of $\mathrm{SO}(V)$ is orthogonal (resp. symplectic). In particular a is odd if m is even. The hypothesis $3a > m-1$ therefore implies that $3a > N$ in any cases.

We now want to prove that there is no $b \geq a$ occuring in the second sum. Indeed: suppose by contradiction that such a (ρ, b) occurs. The inequality $3a > N$ implies that $d_\rho = 1$. Now since $d_\rho = 1$ the automorphic representation ρ is also a quadratic character and $b \equiv a$ modulo 2. The two factors (η, a), (ρ, b) contribute to the infinitesimal character of π_{v_0} by respectively $\left(\frac{a-1}{2}, \frac{a-3}{2}, \dots\right)$ and $\left(\frac{b-1}{2}, \frac{b-3}{2}, \dots\right)$ and the infinitesimal character of π_{v_0} cannot be regular in contradiction with our hypothesis. □

Remarks. 1. The stronger hypothesis $a > \ell = N/2$ (without any hypothesis on the infinitesimal character of π_{v_0}) directly implies that $b < a$ if (ρ, b) appears in the parameter of π.

2. Assume that m is even. We have just seen that the global parameter contains $\eta \boxtimes R_a$ with η a quadratic character and a an odd integer. The same is true at each local place v of F, the localization of that parameter contains $\eta_v \boxtimes R_a$. This implies that the conjugacy class of that parameter under $\mathrm{SO}(V, \mathbb{C})$ coincide with the conjugacy class under $\mathrm{O}(V, \mathbb{C})$ (see [6] the discussion following (1.5.5)). In particular if π_v is unramified it is isomorphic to its image under $\mathrm{O}(V, F_v)$ and the restriction of $\tilde{\pi}_v$ to $\mathrm{SO}(V, F_v)$ coincides with π_v.

4.4. Let η be the automorphic character given by Lemma 4.3. Then Lemma 3.10 implies that for some finite set of places S, the partial L-function $L^S(s, \pi \times \eta)$ is holomorphic in the half-plane $\mathrm{Re}(s) > (a+1)/2$ and has a simple pole in $s = (a+1)/2$. Adding a finite set of places to S we may assume that this also holds for the partial L-function $L^S(s, \tilde{\pi} \times \eta)$.

Now let $p = \frac{1}{2}(m - a - 1)$ and X be a symplectic F-space with $\dim X = 2p$. Proposition 2.7 implies that there exists an automorphic sign character ϵ of $\mathrm{O}_m(\mathbb{A})$ such that the ψ^{-1}-theta lifting of $(\tilde{\pi} \otimes \eta) \otimes \epsilon$ to $\mathrm{Mp}_{2p}(\mathbb{A})$ does not vanish.

4.5. Here we prove that $\pi' := \Theta^X_{\psi^{-1}, V}((\tilde{\pi} \otimes \eta) \otimes \epsilon)$ is cuspidal. Let π'_0 be the first (non-zero) occurrence of the ψ^{-1}-theta lifting of $(\tilde{\pi} \otimes \eta) \otimes \epsilon$ in the Witt tower of the symplectic spaces. By the Rallis theta tower property [71], then π'_0 is cuspidal. Let $2p_0$ be the dimension of the symplectic space corresponding to π'_0. We want to prove that $p_0 = p$. By the unramified correspondence we know $\tilde{\pi}_v$ in all but finitely many places v. There corresponds to $\tilde{\pi}_v$ a *unique* (see e.g. [64]) Arthur packet Ψ_v which contains π_v — the restriction of $\tilde{\pi}_v$ to $\mathrm{SO}(V)(F_v)$ (see the remark above). And Ψ_v contains a factor $\eta_v \otimes R_{a'}$ with $a' = m - 2p_0 - 1$. In particular $a' \geq a$. As in the proof of Lemma 3.10 writing explicitly the partial L-function $L^S(s, \pi \times \eta)$ on a right half-plane of absolute convergence, we get a product of $L^S(s - (a'-1)/2, \eta \times \eta)$ by factors which are holomorphic in $(a'+1)/2$. This forces the partial L-function $L^S(s, \pi \times \eta)$ to have a pole in $s = (a'+1)/2$ and it follows from Lemma 3.10 that $a' \leq a$. Finally $a = a'$ and $p_0 = m - a - 1$.

4.6. The main theorem of [66] and [38, Theorem 1.2] now apply to the representation $(\tilde{\pi} \otimes \eta) \otimes \epsilon$ to show that

$$\Theta^V_{\psi, X}(\Theta^X_{\psi^{-1}, V}((\tilde{\pi} \otimes \eta) \otimes \epsilon)) = (\tilde{\pi} \otimes \eta) \otimes \epsilon.$$

In otherwords: $\Theta^V_{\psi, X}(\pi') = (\tilde{\pi} \otimes \eta) \otimes \epsilon$. This concludes the proof of Theorem 4.2 with $\chi = \eta \otimes \epsilon$. □

Part 2. Local computations

5. Cohomological unitary representations

5.1. **Notations.** Let p and q be two non-negative integers with $p+q=m$. In this section $G = \mathrm{SO}_0(p,q)$ and $K = \mathrm{SO}(p)\times\mathrm{SO}(q)$ is a maximal compact subgroup of G. We let $\mathfrak{g}_0$ the real Lie algebra of G and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition associated to the choice of the maximal compact subgroup K. We denote by θ the corresponding Cartan involution. If $\mathfrak{l}_0$ is a real Lie algebra we denote by $\mathfrak{l}$ its complexification $\mathfrak{l} = \mathfrak{l}_0 \otimes \mathbb{C}$.

5.2. **Cohomological $(\mathfrak{g}, K)$-modules.** Let (π, V_π) be an irreducible unitary $(\mathfrak{g}, K)$-module and E be a finite dimensional irreducible representation of G. We say that (π, V_π) is *cohomological (w.r.t. the local system associated to E)* if it has nonzero $(\mathfrak{g}, K)$-cohomology $H^\bullet(\mathfrak{g}, K; V_\pi \otimes E)$.

Cohomological $(\mathfrak{g}, K)$-modules are classified by Vogan and Zuckerman in [80]: Let $\mathfrak{t}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$. A *θ-stable parabolic subalgebra* $\mathfrak{q} = \mathfrak{q}(X) \subset \mathfrak{g}$ is associated to an element $X \in i\mathfrak{t}_0$. It is defined as the direct sum

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u},$$

of the centralizer $\mathfrak{l}$ of X and the sum $\mathfrak{u}$ of the positive eigenspaces of $\mathrm{ad}(X)$. Since $\theta X = X$, the subspaces $\mathfrak{q}$, $\mathfrak{l}$ and $\mathfrak{u}$ are all invariant under θ, so

$$\mathfrak{q} = \mathfrak{q} \cap \mathfrak{k} \oplus \mathfrak{q} \cap \mathfrak{p},$$

and so on.

The Lie algebra $\mathfrak{l}$ is the complexification of $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$. Let L be the connected subgroup of G with Lie algebra $\mathfrak{l}_0$. Fix a positive system $\Delta^+(\mathfrak{l})$ of the roots of $\mathfrak{t}$ in $\mathfrak{l}$. Then $\Delta^+(\mathfrak{g}) = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{u})$ is a positive system of the roots of $\mathfrak{t}$ in $\mathfrak{g}$. Now extend $\mathfrak{t}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and choose $\Delta^+(\mathfrak{g}, \mathfrak{h})$ a positive system of roots of $\mathfrak{h}$ in $\mathfrak{g}$ such that its restriction to $\mathfrak{t}$ gives $\Delta^+(\mathfrak{g})$. Let ρ be half the sum of the roots in $\Delta^+(\mathfrak{h}, \mathfrak{g})$ and $\rho(\mathfrak{u} \cap \mathfrak{p})$ half the sum of the roots in $\mathfrak{u} \cap \mathfrak{p}$. A one-dimensional representation $\lambda : \mathfrak{l} \to \mathbb{C}$ is admissible if it satisfies the following two conditions:

1. λ is the differential of a unitary character of L,
2. if $\alpha \in \Delta(\mathfrak{u})$, then $\langle \alpha, \lambda_{|\mathfrak{t}} \rangle \geq 0$.

Given $\mathfrak{q}$ and an admissible λ, let $\mu(\mathfrak{q}, \lambda)$ be the representation of K of highest weight $\lambda_{|\mathfrak{t}} + 2\rho(\mathfrak{u} \cap \mathfrak{p})$. We will abbreviate by $\mu(\mathfrak{q})$ the K-module $\mu(\mathfrak{q}, 0)$.

The following proposition is due to Vogan and Zuckerman [80, Theorem 5.3 and Proposition 6.1].

5.3. **Proposition.** *Assume that λ is zero on the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{h}$. There exists a unique irreducible unitary $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ such that:*

1. *$A_{\mathfrak{q}}(\lambda)$ contains the K-type $\mu(\mathfrak{q}, \lambda)$.*
2. *$A_{\mathfrak{q}}(\lambda)$ has infinitesimal character $\lambda + \rho$.*

Vogan and Zuckerman (see [80, Theorem 5.5 and 5.6]) moreover prove:

5.4. **Proposition.** *Let (π, V_π) be an irreducible unitary $(\mathfrak{g}, K)$-module and E be a finite dimensional irreducible representation of G. Suppose $H^\bullet(\mathfrak{g}, K; V_\pi \otimes E) \neq 0$. Then there is a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$, such that:*

1. *$E/\mathfrak{u}E$ is a one-dimensional unitary representation of L; write $-\lambda : \mathfrak{l} \to \mathbb{C}$ for its differential.*

(2) $\pi \cong A_{\mathfrak{q}}(\lambda)$. *Moreover, letting* $R = \dim(\mathfrak{u} \cap \mathfrak{p})$, *we have:*

$$\begin{aligned} H^{\bullet}(\mathfrak{g}, K; V_{\pi} \otimes E) &\cong H^{\bullet - R}(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k}, \mathbb{C}) \\ &\cong \mathrm{Hom}_{\mathfrak{l} \cap \mathfrak{k}}(\wedge^{\bullet - R}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}). \end{aligned}$$

(3) *As a consequence of (2) we have*

$$H^R(\mathfrak{g}, K; V_{\pi} \otimes E) \cong \mathrm{Hom}_{\mathfrak{l} \cap \mathfrak{k}}(\wedge^0(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}) \cong \mathbb{C} \tag{5.4.1}$$

We can now prove the following proposition.

5.5. **Proposition.** *The representation* $\mu(\mathfrak{q}, \lambda)$ *of* K *has the following properties*

1. $\mu(\mathfrak{q}, \lambda)$ *is the only representation of* K *common to both* $\wedge^R(\mathfrak{p}) \otimes E^*$ *and* $A_{\mathfrak{q}}(\lambda)$.
2. $\mu(\mathfrak{q}, \lambda)$ *occurs with multiplicity one in* $\wedge^R(\mathfrak{p}) \otimes E^*$.
3. $\mu(\mathfrak{q}, \lambda)$ *occurs with multiplicity one in* $A_{\mathfrak{q}}(\lambda)$.

Proof. From [80], Proposition 5.4(c) we have

$$H^R(\mathfrak{g}, K; V_{\pi} \otimes E) \cong \mathrm{Hom}_K(\wedge^R(\mathfrak{p}), A_{\mathfrak{q}}(\lambda) \otimes E).$$

But combining this equation with (5.4.1) we conclude

$$\dim \mathrm{Hom}_K(\wedge^R(\mathfrak{p}), A_{\mathfrak{q}}(\lambda) \otimes E) = 1.$$

This last equation implies all three statements of the proposition. □

Let $e(\mathfrak{q})$ be a generator of the line $\wedge^R(\mathfrak{u} \cap \mathfrak{p})$. Then $e(\mathfrak{q})$ is the highest weight vector of an irreducible representation $V(\mathfrak{q})$ of K contained in $\wedge^R\mathfrak{p}$ (and whose highest weight is thus necessarily $2\rho(\mathfrak{u} \cap \mathfrak{p})$). It follows that $V(\mathfrak{q})$ is the unique occurrence of $\mu(\mathfrak{q})$ in $\wedge^R(\mathfrak{p})$. We will refer to the special K-types $\mu(\mathfrak{q})$ as *Vogan-Zuckerman K-types*. Let $V(\mathfrak{q}, \lambda)$ denote the Cartan product of $V(\mathfrak{q})$ and E^* (this means the irreducible submodule of the tensor product $V(\mathfrak{q}) \otimes E^*$ with highest weight the sum $2\rho(\mathfrak{u} \cap \mathfrak{p}) + \lambda$). By definition $V(\mathfrak{q}, \lambda)$ occurs in $\wedge^R(\mathfrak{u} \cap \mathfrak{p}) \otimes E^*$ and hence, by (2) of Proposition 5.5 above it is the unique copy of $\mu(\mathfrak{q}, \lambda)$ in $\wedge^R(\mathfrak{u} \cap \mathfrak{p}) \otimes E^*$. From the discussion immediately above and Proposition 5.5 we obtain

5.6. **Corollary.** *Any nonzero element* $\omega \in \mathrm{Hom}_K(\wedge^R\mathfrak{p} \otimes E^*, A_{\mathfrak{q}}(\lambda))$ *factors through the isotypic component* $V(\mathfrak{q}, \lambda)$.

5.7. **Definition.** We will say that the subspace $V(\mathfrak{q}, \lambda) \subset \wedge^R(\mathfrak{u} \cap \mathfrak{p}) \otimes E^*$ is the *strongly primitive refined Hodge type* associated to $A_{\mathfrak{q}}(\lambda)$.

5.8. We will make geometric use of the isomorphism of Proposition 5.4(2). In doing so we will need the following lemmas which are essentially due to Venkataramana [79].

We let T be the torus of K whose Lie algebra is $\mathfrak{t}_0$. The action of T on the space $\mathfrak{p}$ is completely reducible and we have a decomposition

$$\mathfrak{p} = (\mathfrak{u} \cap \mathfrak{p}) \oplus (\mathfrak{l} \cap \mathfrak{p}) \oplus (\mathfrak{u}^- \cap \mathfrak{p}),$$

where the element $X \in \mathfrak{t}$ acts by strictly positive (resp. negative) eigenvalues on $\mathfrak{u} \cap \mathfrak{p}$ (resp. $\mathfrak{u}^- \cap \mathfrak{p}$) and by zero eigenvalue on $\mathfrak{l} \cap \mathfrak{p}$. Now using the Killing form, the inclusion map $\mathfrak{l} \cap \mathfrak{p} \to \mathfrak{p}$ induces a *restriction* map $\mathfrak{p} \to \mathfrak{l} \cap \mathfrak{p}$ and we have the following:

5.9. **Lemma.** *Consider the restriction map* $B : [\wedge^\bullet \mathfrak{p}]^T \to [\wedge^\bullet(\mathfrak{l} \cap \mathfrak{p})]^T$ *and the cup-product map* $A : [\wedge^\bullet \mathfrak{p}]^T \to \wedge^\bullet \mathfrak{p}$ *given by* $y \mapsto y \wedge e(\mathfrak{q})$. *Then the kernels of* A *and* B *are the same.*

Proof. This is [79, Lemma 1.3]. Note that although it is only stated there for Hermitian symmetric spaces, the proof goes through without any modification. □

Now consider the restriction map

$$[\wedge^\bullet \mathfrak{p}]^K \to [\wedge^\bullet(\mathfrak{l} \cap \mathfrak{p})]^{K \cap L}. \tag{5.9.1}$$

An element $c \in [\wedge^\bullet \mathfrak{p}]^K$ defines — by cup-product — a linear map in

$$\mathrm{Hom}_K(\wedge^\bullet \mathfrak{p}, \wedge^\bullet \mathfrak{p})$$

that we still denote by c. The following lemma is essentially the same as [79, Lemma 1.4].

5.10. **Lemma.** *Let* $c \in [\wedge^\bullet \mathfrak{p}]^K$. *Then we have:*

$$c(V(\mathfrak{q})) = 0 \Leftrightarrow c \in \mathrm{Ker}\left([\wedge^\bullet \mathfrak{p}]^K \to [\wedge^\bullet(\mathfrak{l} \cap \mathfrak{p})]^{K \cap L}\right).$$

Proof. As a K-module $V(\mathfrak{q})$ is generated by $e(\mathfrak{q})$. We therefore deduce from the K-invariance of c that:

$$c(V(\mathfrak{q})) = 0 \Leftrightarrow c \wedge e(\mathfrak{q}) = 0.$$

The second equation is equivalent to the fact that c belongs to the kernel of the map B of Lemma 5.9. But $B(c) = 0$ if and only if c belongs to the kernel of the restriction map

$$[\wedge^\bullet \mathfrak{p}]^K \to [\wedge^\bullet(\mathfrak{l} \cap \mathfrak{p})]^{K \cap L}.$$

This concludes the proof. □

5.11. In the notation of [14], we may choose a Killing-orthogonal basis ε_i of $\mathfrak{h}^*$ such that the positive roots are those roots $\varepsilon_i \pm \varepsilon_j$ with $1 \leq i < j \leq \ell$ as well as the roots ε_i ($1 \leq i \leq \ell$) if m is odd. The finite dimensional irreducible representations of G are parametrized by a highest weight $\lambda = (\lambda_1, \dots, \lambda_\ell) = \lambda_1 \varepsilon_1 + \dots + \lambda_\ell \varepsilon_\ell$ such that λ is dominant (i.e. $\lambda_1 \geq \dots \geq \lambda_{\ell-1} \geq |\lambda_\ell|$ and $\lambda_\ell \geq 0$ if m is odd) and integral (i.e. every $\lambda_i \in \mathbb{Z}$).

In the applications we will mainly be interested in the following examples.

Examples. 1. The group $G = \mathrm{SO}_0(n,1)$ and $\lambda = 0$. Then for each integer $q = 0, \dots, \ell - 1$ the K-representation $\wedge^q \mathfrak{p}$ is just $\wedge^q \mathbb{C}^n$ with $K = \mathrm{SO}(n)$; it is irreducible and we denote it τ_q. In addition, if $n = 2\ell$, $\wedge^\ell \mathfrak{p}$ decomposes as a sum of two irreducible representations τ_ℓ^+ and τ_ℓ^-. From this we get that for each integer $q = 0, \dots, \ell - 1$ there exists exactly one irreducible $(\mathfrak{g}, K)$-module (π_q, V_q) such that $H^q(\mathfrak{g}, K; V_q) \neq 0$. In addition, if $n = 2\ell$, there exists two irreducible $(\mathfrak{g}, K)$-module $(\pi_q^\pm, V_q^\pm)$ such that $H^\ell(\mathfrak{g}, K; V_\ell^\pm) \neq 0$. Moreover:

$$H^k(\mathfrak{g}, K; V_q) = \begin{cases} 0 \text{ if } k \neq q, n-q, \\ \mathbb{C} \text{ if } k = q \text{ or } n-q \end{cases}$$

and, if $n = 2\ell$,

$$H^k(\mathfrak{g}, K; V_\ell^\pm) = \begin{cases} 0 \text{ if } k \neq \ell, \\ \mathbb{C} \text{ if } k = \ell. \end{cases}$$

The Levi subgroup $L \subset G$ associated to (π_q, V_q) ($q = 0, \dots, \ell - 1$) is $L = C \times \mathrm{SO}_0(n - 2q, 1)$ where $C \subset K$.

2. The group $G = \mathrm{SO}_0(n,1)$ and $\lambda = (1,0,\dots,0)$ is the highest weight of its standard representation in $\mathbb{C}^m$. Then there exists a unique $(\mathfrak{g},K)$-module (π, V_π) such that $H^1(\mathfrak{g},K; V_\pi \otimes \mathbb{C}^m) \neq 0$. The Levi subgroup $L \subset G$ associated to (π, V_π) is $L = C \times \mathrm{SO}_0(n-2,1)$ where $C \subset K$

3. The group $G = \mathrm{SO}_0(n,2)$ and $\lambda = 0$. Then $K = \mathrm{SO}(n) \times \mathrm{SO}(2)$ and $\mathfrak{p} = \mathbb{C}^n \otimes (\mathbb{C}^2)^*$ where $\mathbb{C}^n$ (resp. $\mathbb{C}^2$) is the standard representation of $\mathrm{SO}(n)$ (resp. $\mathrm{SO}(2)$). Given an integer $r \leq n/2$ we let $A_{r,r}$ be the cohomological representation whose associated Levi subgroup $L \subset G$ is $L = C \times \mathrm{SO}_0(n-2r,2)$ where $C \subset K$.

We denote by $\mathbb{C}^+$ and $\mathbb{C}^-$ the $\mathbb{C}$-span of the vectors $e_1 + ie_2$ and $e_1 - ie_2$ in $\mathbb{C}^2$. The two lines $\mathbb{C}^+$ and $\mathbb{C}^-$ are left stable by $\mathrm{SO}(2)$. This yields a decomposition $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ which corresponds to the decomposition given by the natural complex structure on $\mathfrak{p}_0$. For each non-negative integer q the K-representation $\wedge^q \mathfrak{p} = \wedge^q(\mathfrak{p}^+ \oplus \mathfrak{p}^-)$ decomposes as the sum:

$$\wedge^q \mathfrak{p} = \bigoplus_{a+b=q} \wedge^a \mathfrak{p}^+ \otimes \wedge^b \mathfrak{p}^-.$$

The K-representations $\wedge^a \mathfrak{p}^+ \otimes \wedge^b \mathfrak{p}^-$ are not irreducible in general: there is at least a further splitting given by the Lefschetz decomposition:

$$\wedge^a \mathfrak{p}^+ \otimes \wedge^b \mathfrak{p}^- = \bigoplus_{k=0}^{\min(a,b)} \tau_{a-k,b-k}.$$

One can check that for $2(a+b) < n$ each K-representation $\tau_{a,b}$ is irreducible. Moreover in the range $2(a+b) < n$ only those with $a = b$ can occur as a K-type of a cohomological module. For each non-negative integer r such that $4r < n$, $A_{r,r}$ is the unique cohomological module that satisfies:

$$H^q(\mathfrak{g},K; A_{r,r}) = \begin{cases} \mathbb{C} \text{ if } q = 2r + 2k \ (0 \leq k \leq n-2r), \\ 0 \text{ otherwise.} \end{cases}$$

Moreover: $H^{2r}(\mathfrak{g},K; A_{r,r}) = H^{r,r}(\mathfrak{g},K; A_{r,r})$. See e.g. [32, §1.5] for more details.

5.12. We now consider the general case $G = \mathrm{SO}_0(p,q) = \mathrm{SO}_0(V)$, where V is a real quadratic space of dimension m and signature (p,q) two non-negative integers with $p + q = m$. We denote by $(,)$ the non-degenerate quadratic form on V and let v_α, $\alpha = 1,\dots,p$, v_μ, $\mu = p+1,\dots,m$, be an orthogonal basis of V such that $(v_\alpha, v_\alpha) = 1$ and $(v_\mu, v_\mu) = -1$. We denote by V_+ (resp. V_-) the span of $\{v_\alpha \ : \ 1 \leq \alpha \leq p\}$ (resp. $\{v_\mu \ : \ p+1 \leq \mu \leq m\}$). As a representation of $\mathrm{SO}(p) \times \mathrm{SO}(q) = \mathrm{SO}(V_+) \times \mathrm{SO}(V_-)$, the space $\mathfrak{p}$ is isomorphic to $V_+ \otimes (V_-)^*$.

First recall that, as a $\mathrm{GL}(V_+) \times \mathrm{GL}(V_-)$-module, we have (see [23, Equation (19), p. 121]):

$$\wedge^R (V_+ \otimes V_-^*) \cong \bigoplus_{\mu \vdash R} S_\mu(V_+) \otimes S_{\mu^*}(V_-)^*. \tag{5.12.1}$$

Here $S_\mu(\cdot)$ denotes the Schur functor (see [24]), we sum over all partition of R (equivalently Young diagram of size $|\mu| = R$) and μ^* is the conjugate partition (or transposed Young diagram).

We will see that as far as we are concerned with special cycles, we only have to consider the decomposition of the submodule $\wedge^R(V_+ \otimes V_-^*)^{\mathrm{SL}(V_-)}$. Then each Young diagram μ which occurs in (5.12.1) is of type $\mu = (q,\dots,q)$.

Following [24, p. 296], we may define the harmonic Schur functor $S_{[\mu]}(V_+)$ as the image of $S_\mu(V_+)$ by the $\mathrm{SO}(V_+)$-equivariant projection of $V_+^{\otimes R}$ onto the harmonic tensors. From now on we suppose that μ has at most $\frac{p}{2}$ (positive) parts. The representation $S_{[\mu]}(V_+)$ is irreducible with highest weight μ. And Littlewood gives a formula for the decomposition of $S_\mu(V_+)$ as a representation of $\mathrm{SO}(V_+)$ by restriction (see [24, Eq. (25.37), p. 427]):

5.13. **Proposition.** *The multiplicity of the finite dimensional* $\mathrm{SO}(V_+)$*-representation* $S_{[\nu]}(V_+)$ *in* $S_\mu(V_+)$ *equals*

$$\sum_\xi \dim \mathrm{Hom}_{\mathrm{GL}(V_+)}(S_\mu(V_+), S_\nu(V_+) \otimes S_\xi(V_+)),$$

where the sum is over all nonnegative integer partitions ξ *with rows of* even *length.*

5.13.1. *The Euler form.* In what follows $\{v_\alpha : 1 \leq \alpha \leq p\}$ is an orthonormal basis of V_+. It is well known (see e.g. [31, Theorem 5.3.3]) that $[\mathrm{Sym}^q(V_+)]^{\mathrm{SO}(V_+)}$ is trivial if q is odd and 1-dimensional generated by

$$\sum_{\sigma \in \mathfrak{S}_q} \sigma \cdot \theta_q \tag{5.13.1}$$

where $\theta = \sum_{\alpha=1}^p v_\alpha \otimes v_\alpha$ and

$$\theta_q = \underbrace{\theta \otimes \cdots \otimes \theta}_{\ell} = \sum_{\alpha_1, \ldots, \alpha_\ell} v_{\alpha_1} \otimes v_{\alpha_1} \otimes \ldots \otimes v_{\alpha_\ell} \otimes v_{\alpha_\ell},$$

if $q = 2\ell$ is even. Note that

$$\wedge^q (V_+ \otimes V_-^*)^{\mathrm{SO}(V_+) \times \mathrm{SL}(V_-)} \cong [\mathrm{Sym}^q(V_+)]^{\mathrm{SO}(V_+)} \otimes \wedge^q (V_-)^*.$$

It is therefore trivial if q is odd and 1-dimensional if q is even. Using the isomorphism of (5.13.1) we obtain a generator of $[\wedge^q \mathfrak{p}]^{\mathrm{SO}(V_+) \times \mathrm{SL}(V_-)}$ as the image of $\sum_{\sigma \in \mathfrak{S}_q} \sigma \cdot \theta_q$ under the above isomorphism. The associated invariant q-form on D is called the *Euler form* e_q. The Euler form e_q is zero if q is odd and for $q = 2\ell$ is expressible in terms of the curvature two-forms $\Omega_{\mu,\nu} = \sum_{\alpha=1}^p (v_\alpha \otimes v_\mu^*) \wedge (v_\alpha \otimes v_\nu^*)$ by the formula

$$e_q = \sum_{\sigma \in \mathfrak{S}_q} \mathrm{sgn}(\sigma) \Omega_{p+\sigma(1), p+\sigma(2)} \wedge \ldots \wedge \Omega_{p+\sigma(2\ell-1), p+\sigma(2\ell)} \in \wedge^q \mathfrak{p}. \tag{5.13.2}$$

As in the above example it more generally follows from [31, Theorem 5.3.3] that we have:

5.14. **Proposition.** *The subspace* $[\wedge^\bullet \mathfrak{p}]^{\mathrm{SO}(V_+) \times \mathrm{SL}(V_-)}$ *is the subring of* $\wedge^\bullet \mathfrak{p}$ *generated by the Euler class* e_q.

Remark. Proposition 5.14 implies that

$$[\wedge^{nq} \mathfrak{p}]^{\mathrm{SO}(V_+) \times \mathrm{SL}(V_-)} = \mathbb{C} \cdot e_q^n.$$

It is consistent with Proposition 5.13 since one obviously has:

$$\sum_\xi \dim \mathrm{Hom}_{\mathrm{GL}(V_+)}(S_{n \times q}(V_+), S_\xi(V_+)) = \begin{cases} 0 & \text{if } q \text{ is odd} \\ 1 & \text{if } q \text{ is even} \end{cases}.$$

5.15. Let $V_r \subset \wedge^{rq}\mathfrak{p}$ $(0 \leq r \leq p/2)$ denote the realization of the irreducible K-type μ_r isomorphic to

$$S_{[r\times q]}(V_+) \otimes (\wedge^q V_-)^r \subset \wedge^{rq}(V_+ \otimes V_-^*).$$

Remark. The K-type μ_r is the Vogan-Zuckerman K-type $\mu(\mathfrak{q}_r)$ where $\mathfrak{q}_r$ is any θ-stable parabolic subalgebra with corresponding Levi subgroup $L = C \times \mathrm{SO}_0(p-2r, q)$ with $C \subset K$. In particular we may as well take the maximal one where $L = \mathrm{U}(r) \times \mathrm{SO}(p-2r, q)$, see §8.10 where μ_r is analysed in detail.

Wedging with the Euler class defines a linear map in

$$\mathrm{Hom}_{\mathrm{SO}(V_+)\times \mathrm{SL}(V_-)}(\wedge^\bullet \mathfrak{p}, \wedge^\bullet \mathfrak{p}).$$

We still denote by e_q the linear map. Note that under the restriction map (5.9.1) to $\mathfrak{l} \cap \mathfrak{p}$ the Euler class e_q restricts to the Euler class in $\wedge^q(\mathfrak{l}\cap\mathfrak{p})$. It then follows from Lemma 5.10 that if q is even $e_q^k(V_r)$ is a *non-trivial* K-type in $\wedge^{(r+k)q}\mathfrak{p}$ if and only if $k \leq p-2r$. This leads to the following:

5.16. **Proposition.** *The irreducible K-types μ_r are the only Vogan-Zuckerman K-types that occur in the subring*

$$[\wedge^\bullet \mathfrak{p}]^{\mathrm{SL}(V_-)} = \oplus_{n=0}^p S_{n\times q}(V_+) \otimes (\wedge^q V_-)^n.$$

Moreover:

$$\mathrm{Hom}_{\mathrm{SO}(V_+)\times \mathrm{SO}(V_-)}\left(V_r, [\wedge^{nq}\mathfrak{p}]^{\mathrm{SL}(V_-)}\right) = \begin{cases} \mathbb{C}\cdot e_q^{n-r} & \text{if } n = r, \dots, p-r \\ 0 & \text{otherwise.}\end{cases}$$

Proof. Cohomological representations of $G = \mathrm{SO}_0(p,q)$, or equivalently their lowest K-types $\mu(\mathfrak{q})$, are parametrized in [9, §2.2] by certain types of Young diagram ν so that

$$\mu(\mathfrak{q}) \cong S_{[\nu]}(V_+) \otimes S_{[\nu^*]}(V_-)^*.$$

Here we are only concerned with K-types which have trivial $\mathrm{SO}(V_-)$-representation. Therefore $\nu = r \times q$ for some integer r, so that $S_{[\nu^*]}(V_-) = (\wedge^q V_-)^{\otimes r}$. This proves the first assertion of the proposition.

We now consider the decomposition of

$$[\wedge^\bullet \mathfrak{p}]^{\mathrm{SL}(V_-)} = \oplus_{n=0}^p S_{n\times q}(V_+) \otimes (\wedge^q V_-)^n$$

into irreducibles. By (Poincaré) duality it is enough to consider $S_{n\times q}(V_+)\otimes(\wedge^q V_-)^n$ with $n \leq p/2$. It then follows from Proposition 5.13 that the multiplicity of $S_{[r\times q]}(V_+)$ in $S_{n\times q}(V_+)$ equals

$$\sum_\xi \dim \mathrm{Hom}_{\mathrm{GL}(V_+)}(S_{n\times q}(V_+), S_{r\times q}(V_+) \otimes S_\xi(V_+)),$$

where the sum is over all nonnegative integer partitions ξ with rows of even length. Each term of the sum above is a Littlewood-Richardson coefficient. The Littlewood-Richardson rule states that $\dim \mathrm{Hom}_{\mathrm{GL}(V_+)}(S_{n\times q}(V_+), S_{r\times q}(V_+) \otimes S_\xi(V_+))$ equals the number of Littlewood-Richardson tableaux of shape $(n\times q)/(r\times q) = (n-r)\times q$ and of weight ξ, see e.g. [23]. The point is that the shape $(n\times q)/(r\times q)$ is an $n-r$ by q rectangle and it is immediate that there is only one semistandard filling of an n by q rectangle that satifies the reverse lattice word condition, see [23], §5.2, page

63, (the first row must be filled with ones, the second with twos etc.). We conclude that

$$\begin{aligned}\dim \mathrm{Hom}_{\mathrm{GL}(V_+)}(S_{n\times q}(V_+), S_{r\times q}(V_+) \otimes S_\xi(V_+)) \\ = \dim \mathrm{Hom}_{\mathrm{GL}(V_+)}(S_{(n-r)\times q}(V_+), S_\xi(V_+)).\end{aligned}$$

The multiplicity of $S_{[r\times q]}(V_+)$ in $S_{n\times q}(V_+)$ therefore equals 0 is q is odd and 1 if q is even. Since in the last case

$$e_q^{n-r}(V_r) \cong S_{[r\times q]}(V_+) \otimes (\wedge^q V_-)^n \subset [\wedge^{nq} V_+ \otimes V_-^*]^{\mathrm{SL}(V_-)} = S_{n\times q}(V^+) \otimes (\wedge^q V_-)^n,$$

this concludes the proof. □

Remark. Proposition 5.16 implies the decomposition (1.14.1) of the Introduction.

We conclude the section by showing that the Vogan-Zuckerman types μ_r are the only K-types to give small degree cohomology.

5.17. **Proposition.** *Consider a cohomological module $A_{\mathfrak{q}}(\lambda)$. Suppose that $R = \dim(\mathfrak{u} \cap \mathfrak{p})$ is strictly less than both $p+q-3$ and $pq/4$. Then: either $L = C \times \mathrm{SO}_0(p-2n, q)$ with $C \subset K$ and $R = nq$ or $L = C \times \mathrm{SO}_0(p, q-2n)$ with $C \subset K$ and $R = np$.*

Proof. Suppose by contradiction that L contains as a direct factor the group $\mathrm{SO}_0(p-2a, q-2b)$ for some positive a and b. Then $R \geq ab + b(p-2a) + a(q-2b)$. And writing:

$$\begin{aligned}ab + b(p-2a) + a(q-2b) - (p+q-3) \\ = (a-1)(b-1) + (b-1)(p-2a-1) + (a-1)(q-2b-1)\end{aligned}$$

we conclude that $R \geq p+q-3$ except perhaps if $p = 2a$ or $q = 2b$. But in that last case $R \geq pq/4$. □

6. Cohomological Arthur packets

For archimedean v, local Arthur packets $\prod(\Psi)$ should coincide with the packets constructed by Adams, Barbasch and Vogan [1]. Unfortunately this is still unproved. We may nevertheless build upon their results to obtain a conjectural description of all the real Arthur packets which contain a cohomological representation.

6.1. **Notations.** Let p and q be two non-negative integers with $p+q=m$. We set $\ell = [m/2]$ and $N = 2\ell$. In this section $G = \mathrm{SO}(p,q)$, K is a maximal compact subgroup of G and $K_0 = \mathrm{SO}(p) \times \mathrm{SO}(q)$. We let $\mathfrak{g}_0$ the real Lie algebra of G and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition associated to the choice of the maximal compact subgroup K. We denote by θ the corresponding Cartan involution. If $\mathfrak{l}_0$ is a real Lie algebra we denote by $\mathfrak{l}$ its complexification $\mathfrak{l} = \mathfrak{l}_0 \otimes \mathbb{C}$.

We finally let $W_{\mathbb{R}}$ be the Weil group of $\mathbb{R}$

6.2. **The Adams-Johnson packets.** Let $\mathfrak{l} \subset \mathfrak{g}$ be the Levi component of a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$. Then $\mathfrak{l}$ is defined over $\mathbb{R}$ and we let L be the corresponding connected subgroup of G. Let $T \subset L$ be a θ-stable Cartan subgroup of G (so that T is also a Cartan subgroup of L). We fix $\lambda : \mathfrak{l} \to \mathbb{C}$ an admissible one-dimensional representation. We will identify λ with its highest weight in $\mathfrak{t}^*$ that is its restriction to $\mathfrak{t}$.

Adams and Johnson have studied the particular family — to be called *Adams-Johnson parameters* — of local Arthur parameters

$$\Psi : W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$

such that

(1) Ψ factors through ${}^L L$, that is $\Psi : W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \stackrel{\Psi_L}{\to} {}^L L \to {}^L G$ where the last map is the canonical extension [75, Proposition 1.3.5] of the injection $L^\vee \subset G^\vee$, and
(2) φ_{Ψ_L} is the L-parameter of a unitary character of L whose differential is λ.

The restriction of the parameter Ψ to $\mathrm{SL}_2(\mathbb{C})$ therefore maps $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ to a principal unipotent element in $L^\vee \subset G^\vee$.

6.3. The packet $\prod_{\mathrm{AJ}}(\Psi)$ constructed by Adams and Johnson takes the following form. Let $W(\mathfrak{g}, \mathfrak{t})^\theta$ (resp. $W(\mathfrak{l}, \mathfrak{t})^\theta$) be those elements of the Weyl group of $\mathfrak{g}$ which commute with θ. And let $W(G,T)$ be the real Weyl group of G. The representations in $\prod_{\mathrm{AJ}}(\Psi)$ are parametrized by the double cosets

$$S = W(\mathfrak{l}, \mathfrak{t})^\theta \backslash W(\mathfrak{g}, \mathfrak{t})^\theta / W(G,T).$$

For any $w \in S$, the Lie subalgebra $\mathfrak{l}_w = w \mathfrak{l} w^{-1}$ is still defined over $\mathbb{R}$, and is the Levi subalgebra of the θ-stable parabolic subalgebra $\mathfrak{q}_w = w \mathfrak{q} w^{-1}$. Let $L_w \subset G$ be the corresponding connected group. The representations in $\prod_{\mathrm{AJ}}(\Psi)$ are the irreducible unitary representations of G whose underlying $(\mathfrak{g}, K_0)$-modules are the Vogan-Zuckerman modules $A_{\mathfrak{q}_w}(w\lambda)$.

We call *Adams-Johnson packets* the packets associated to Adams-Johnson parameters as above.

6.4. **Conjecture.** (1) *For any Adams-Johnson parameter Ψ one has $\prod(\Psi) = \prod_{\mathrm{AJ}}(\Psi)$. In otherwords, the Arthur packet associated to Ψ coincides with the Adams-Johnson packet associated to Ψ.*
(2) *The only Arthur packets that contain a cohomological representation of G are Adams-Johnson packets.*

Remark. Adams-Johnson packets are a special case of the packets defined in [1]. In otherwords: calling *ABV packets* the packets defined in [1], any Adams-Johnson packet is an ABV packet defined by a parameter Ψ of a particular form. According to J. Adams (private communication) part 2 of Conjecture 6.4 with "ABV packets" in place of "Arthur packets" is certainly true. It would therefore be enough to prove the natural extension of part 1 of Conjecture 6.4 to any ABV packet. In chapter 26 of [1], Adams, Barbasch and Vogan prove that ABV packets are compatible with endoscopic lifting.[5] Part 1 of Conjecture 6.4 amounts to a similar statement for twisted endoscopy. This seems still open but is perhaps not out of reach.

[5] In particular, it would be enough to check Conjecture 6.4 only for quasi-split groups.

6.5. We now give a more precise description of the possible Adams-Johnson parameters: The Levi subgroups L corresponding to a θ-stable parabolic subalgebra of $\mathfrak{g}$ are of the forms

$$\mathrm{U}(p_1,q_1)\times\ldots\times\mathrm{U}(p_r,q_r)\times\mathrm{SO}(p_0,q_0) \tag{6.5.1}$$

with $p_0+2\sum_j p_j=p$ and $q_0+2\sum_j q_j=q$. We let $m_j=p_j+q_j$ $(j=0,\ldots,r)$. Then $m_0=p_0+q_0$ has the same parity as m; we set $\ell_0=[m_0/2]$ and $N_0=2\ell_0$. Here and after we assume that $p_0q_0\neq 0$. The parameter Ψ_L corresponding to such an L maps $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ to a principal unipotent element in each factor of $L^\vee\subset G^\vee$. The parameter Ψ_L therefore contains a $\mathrm{SL}_2(\mathbb{C})$ factor of the maximal dimension in each factor of $L^\vee$. These factors consist of $\mathrm{GL}(m_j)$, $j=1,\ldots,r$, and the group $G^\vee_{N_0}=\mathrm{SO}(N_0,\mathbb{C})$ or $\mathrm{Sp}(N_0,\mathbb{C})$ (according to the parity of m). In any case the biggest possible $\mathrm{SL}_2(\mathbb{C})$ representation in $G^\vee_{N_0}$ is R_{m_0-1}.[6] We conclude that the restriction of Ψ to $\mathbb{C}^*\times\mathrm{SL}_2(\mathbb{C})$ decomposes as:

$$(\mu_1\otimes R_{m_1}\oplus\mu_1^{-1}\otimes R_{m_1})\oplus\ldots\oplus(\mu_r\otimes R_{m_r}\oplus\mu_r^{-1}\otimes R_{m_r})\oplus\Psi_0 \tag{6.5.2}$$

where the μ_j are unitary characters of $W_\mathbb{R}$ and

$$\Psi_0=\begin{cases}\chi\otimes R_{m_0-1} & \text{if } m_0 \text{ is odd,}\\ \chi\otimes R_{m_0-1}\oplus\chi'\otimes R_1 & \text{if } m_0 \text{ is even.}\end{cases} \tag{6.5.3}$$

Here χ and χ' are quadratic characters of $W_\mathbb{R}$.

6.6. Now consider a cohomological representation π of G. Its underlying $(\mathfrak{g},K_0)$-module is a Vogan-Zuckerman module $A_\mathfrak{q}(\lambda)$. It follows from their construction that we may choose $\mathfrak{q}$ so that the group L has no compact (non-abelian) simple factors; in that case $\mathfrak{q}$ and the unitary character of L whose differential is λ are uniquely determined up to conjugation by K_0. The group L is therefore as in (6.5.1) with either $p_jq_j\neq 0$ or $(p_j,q_j)=(1,0)$ or $(0,1)$. So that:

$$L^\vee\cong\mathrm{GL}(1)^s\times\mathrm{GL}(m_1)\times\ldots\times\mathrm{GL}(m_t)\times G^\vee_{N_0}$$

where each m_j, $j=1,\ldots,t$, is >1, $r=s+t$ and $m_{t+1}=\ldots=m_r=1$. We let Ψ be the Adams-Johnson parameter associated to L and λ as in the preceeding paragraph.

6.7. **Lemma.** (1) *The cohomological representation π belongs to $\prod_{\mathrm{AJ}}(\Psi)$.*

(2) *If $\pi\in\prod_{\mathrm{AJ}}(\Psi')$ for some Adams-Johnson parameter Ψ', then Ψ' contains*

$$(\mu_1\otimes R_{m_1}\oplus\mu_1^{-1}\otimes R_{m_1})\oplus\ldots\oplus(\mu_t\otimes R_{m_t}\oplus\mu_t^{-1}\otimes R_{m_t})\oplus\Psi_0$$

as a direct factor.

Proof. 1. The representation π belongs to $\prod_{\mathrm{AJ}}(\Psi)$ by definition of the Adams-Johnson packets. Note that the Langlands parametrization of Vogan-Zuckerman modules is given by [80, Theorem 6.16]. In the case of unitary groups this is detailed in [10] where the parametrization is moreover related to Arthur parameters. Everything works similarly for orthogonal groups.

2. If $\pi\in\prod_{\mathrm{AJ}}(\Psi')$ for some Adams-Johnson parameter Ψ' we let L' be the Levi subgroup attached to the parameter Ψ'. The underlying $(\mathfrak{g},K_0)$-module of π is isomorphic to some Vogan-Zuckerman module associated to a θ-stable parabolic algebra whose associated Levi subgroup is L'_w for some $w\in S$. It first follows from the Vogan-Zuckerman construction that L and L'_w can only differ by compact

[6] Note that R_a is symplectic if a is even and orthogonal if a is odd.

factors. But it follows from [2, Lemma 2.5] that all the L'_w ($w \in S$) are inner forms of each other so that the dual group of L'_w may be identified to $L'^\vee$ which therefore contains $\mathrm{GL}(m_1) \times \ldots \times \mathrm{GL}(m_t) \times G^\vee_{N_0}$ as direct factor. And Lemma 6.7 follows from §6.5. □

In particular Lemma 6.7 implies that if π is a cohomological representation of $\mathrm{SO}(p,q)$ associated to a Levi subgroup $L = \mathrm{SO}(p-2r,q) \times \mathrm{U}(1)^r$ with $p > 2r$ and $m-1 > 3r$, then if π is contained in a (local) Adams-Johnson packet $\prod_{\mathrm{AJ}}(\Psi)$, the (local) parameter Ψ contains a factor $\eta \boxtimes R_{m-2r-1}$ where η is a quadratic character and $3(m-2r-1) > m-1$. Theorem 4.2 therefore implies the following:

6.8. **Proposition.** *Assume Conjecture 6.4. Let $\pi \in \mathcal{A}^c(\mathrm{SO}(V))$ and let v_0 be an infinite place of F such that $\mathrm{SO}(V)(F_{v_0}) \cong \mathrm{SO}(p,q)$. Assume that π_{v_0} is a cohomological representation of $\mathrm{SO}(p,q)$ associated to a Levi subgroup $L = \mathrm{SO}(p-2r,q) \times \mathrm{U}(1)^r$ with $p > 2r$ and $m-1 > 3r$. Then: there exists an automorphic character χ such that $\pi \otimes \chi$ is in the image of the cuspidal ψ-theta correspondence from a smaller group associated to a symplectic space of dimension $2r$.*

While this paper was under refereeing process a proof of Conjecture 6.4 has appeared, see [3]. In this paper we prove the a priori weaker result.

6.9. **Proposition.** *Let $\pi \in \mathcal{A}^c(\mathrm{SO}(V))$ and let v_0 be an infinite place of F such that $\mathrm{SO}(V)(F_{v_0}) \cong \mathrm{SO}(p,q)$. Assume that π_{v_0} is a cohomological representation of $\mathrm{SO}(p,q)$ associated to a Levi subgroup $L = \mathrm{SO}(p-2r,q) \times \mathrm{U}(1)^r$ with $p > 2r$ and $m-1 > 3r$. Then: the (global) Arthur parameter Ψ of π is highly non-temptered, i.e. contains a factor $\eta \boxtimes R_a$ where η is a quadratic character and $a \geq m-2r-1$ and in particular $3a > m-1$.*

Since the reader may want to directly use Conjecture 6.4 — now a theorem — we delay the proof of Proposition 6.9 until the Appendix (see Proposition 17.2 and the remark following it). It relies on the theory of exponents.

As we will explain later in Remark 8.31, a cohomological representation as in Proposition 6.9 does not occur in Howe's theta correspondence from a symplectic group smaller than Sp_{2r}. Applying Theorem 4.2 we therefore obtain:

6.10. **Corollary.** *Let $\pi \in \mathcal{A}^c(\mathrm{SO}(V))$ and let v_0 be an infinite place of F such that $\mathrm{SO}(V)(F_{v_0}) \cong \mathrm{SO}(p,q)$. Assume that π_{v_0} is a cohomological representation of $\mathrm{SO}(p,q)$ associated to a Levi subgroup $L = \mathrm{SO}(p-2r,q) \times \mathrm{U}(1)^r$ with $p > 2r$ and $m-1 > 3r$. Then: there exists an automorphic character χ such that $\pi \otimes \chi$ is in the image of the cuspidal ψ-theta correspondence from a smaller group associated to a symplectic space of dimension $2r$.*

As we have explained above, the Arthur's parameter of π contains the factor $\eta \boxtimes R_a$ with $a \geq m-2r-1$. Theorem 4.2 therefore implies that π comes from a square integrable representation of a smaller group in a dual pair and gives the Arthur parameter of that representation; we keep the parameter of π except for the factor $\eta \bigotimes R_{m-2r-1}$ which disappears. And this fixes the size of the group as in the corollary.

6.11. *Remark.* We believe that the hypothesis $m-1 > 3r$ is optimal. This is indeed the case if $p=3$, $q=1$ and $r=1$, see Proposition 16.15.

We now provide support for the above remark.

6.12. In the remaining part of this section we assume that Conjecture 6.4 — which relates the Adams-Johnson packets and the Arthur packets containing at least one cohomological representations — holds. In case m is odd we moreover assume that the representations of the metaplectic groups are classified with Arthur-like parameters (see below). Assuming that we will prove:

If $3r \geq m-1$ there exist cohomological cuspidal representations which are not in the image of the θ correspondence from $\mathrm{Mp}(2r)$ or $\mathrm{Sp}(2r)$ even up to a twist by an automorphic character.

6.13. We assume that $m = N+1$ is odd and the field is totally real.

A particular case of Arthur's work is the classification of square integrable representation of $\mathrm{SL}(2,F)$ using $\mathrm{GL}(3,F)$; this can be also covered by the known Gelbart-Jacquet correspondance between GL(2) and GL(3). We therefore take it for granted.

We define $F_2 = F \otimes \mathbb{Q}_2$ to be the completion of F at the places of residual characteristic 2.

Let τ_2 be a cuspidal irreducible self-dual representation of $\mathrm{GL}(3,F_2)$ which comes from a representation of $\mathrm{SL}(2,F_2)$ or, in other terms, whose L-parameter factorizes through $\mathrm{SO}(3,\mathbb{C})$. We denote by $\tilde{\tau}_2$ the corresponding representation of $\mathrm{SL}(2,F_2)$. We fix $\tilde{\tau}$ a cuspidal irreducible representation of $\mathrm{SL}(2,F)$ whose F_2 component is $\tilde{\tau}_2$ and which is a discrete series at the archimedean places. We go back to $\mathrm{GL}(3,F)$ denoting by τ the automorphic representation corresponding to $\tilde{\tau}$; because of the condition on the F_2-component τ is necessarily cuspidal.

For each $i \in \{1,\dots,3r-m+1\}$, we also fix a cuspidal irreducible representation ρ_i of $\mathrm{GL}(2,F)$. We assume that these representations are distinct and that each ρ_i is of symplectic type, i.e. its local parameter is symplectic. In other words each ρ_i is coming — by the Langlands-Arthur functoriality — from $\mathrm{SO}(3,F)$, equivalently $L(s,\rho_i,\mathrm{Sym}^2)$ has a pole at $s=1$. We moreover assume that at each archimedean place $v|\infty$, each representation $\rho_{i,v}$ belongs to the discrete series. We consider the Arthur parameter

$$\Psi = \left(\boxplus_{i=1}^{3r-m+1} \rho_i \boxtimes R_1\right) \boxplus \tau \boxtimes R_{m-1-2r}. \tag{6.13.1}$$

This is the Arthur parameter of a packet of representations of $G = \mathrm{SO}(V)$.

6.14. We now look more precisely at Ψ at a real place. As a morphism of $W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$ in $\mathrm{Sp}(N,\mathbb{C})$ it is the sum:

$$\Psi_v = \left(\boxplus_{i=1}^{3r-m+1} \phi_{\rho_{i,v}} \boxtimes R_1\right) \boxplus \phi_{\tilde{\tau}_v} \boxtimes R_{m-2r-1} \boxplus 1 \boxtimes R_{m-2r-1},$$

where for $i = 1,\dots,3r-m+1$, $\phi_{\rho_{i,v}}$ is the Langlands parameter of the discrete series $\rho_{i,v}$ and $\phi_{\tilde{\tau}_v}$ is the Langlands parameter of $\tilde{\tau}_v$; it takes value in $\mathrm{GL}(2,\mathbb{C})$. All such local parameters may be obtained by the above construction. By a suitable choice of data we can therefore assume that Ψ_v coincides with an Adams-Johnson parameter. In particular the representations attached to it are all cohomological for some fixed system of coefficients. Let λ_v be the corresponding highest weight.

We may moreover assume the Adams-Johnson parameter Ψ_v is associated to the Levi subgroup $L = \mathrm{SO}(m)$ if $v \neq v_0$ and $L = \mathrm{U}(1)^{3r-m+1} \times \mathrm{U}(m-2r-1) \times \mathrm{SO}(p-2r,q)$ if $v = v_0$.[7] We also ask that $\lambda_v = 0$ if $v \neq v_0$, and fix $\lambda_v = \lambda$ if $v = v_0$. It follows from (the proof of) Lemma 6.7(2) that the trivial representation π_v of $G(F_v)$

[7]Recall that $G(F_{v_0}) = \mathrm{SO}(p,q)$ and that $G(F_v) = \mathrm{SO}(m)$ if $v \neq v_0$.

is contained in $\prod_{\rm AJ}(\Psi_v)$ if $v \neq v_0$ and that the cohomological representation π_{v_0} of $G(F_{v_0}) = \mathrm{SO}(p,q)$ whose underlying $(\mathfrak{g}, K_0)$-module is the Vogan-Zuckerman module $A_{\mathfrak{q}}(\lambda)$, with $\mathfrak{q}$ a θ-stable parabolic subalgebra with real Levi component isomorphic to $\mathrm{U}(1)^r \times \mathrm{SO}(p-2r,q)$, is contained in $\prod_{\rm AJ}(\Psi_{v_0})$. We fix these local components π_v.

6.15. The multiplicity formula to construct a global square integrable representation of $G = \mathrm{SO}(V)$ from the local components is still the subject of work in progress of Arthur; but we can anticipate that we have enough freedom at the finite places to construct a square integrable representation π in the global Arthur's packet and with local component at the archimedean places, the component we have fixed.

We want to show that the representation π is certainly not obtained via theta correspondence from a cuspidal representation of a metaplectic group $\mathrm{Mp}(2n)$ with $2n \leq 2r$. To do that we continue to anticipate some results: Here we anticipate that *the square integrable representations of the metaplectic group can also be classified as those of the symplectic group but using* $\mathrm{Sp}(2n,\mathbb{C})$ *as dual group*; after work by Adams, Adams-Barbash, Renard, Howard this is work in progress by Wen Wei Li.

To prove that π is not a theta lift we can now argue by contradiction: Let σ be a cuspidal irreducible representation of $\mathrm{Mp}(2n)$ (with $2n \leq 2r < p$) such that π is a theta lift of σ. Write $\boxplus_{i=1}^{v} \sigma_i \boxtimes R_{n_i}$ for the Arthur-like parameter attached to σ. To simplify matters assume that V is an orthogonal form of discriminant 1 at each place (otherwise we would have to twist by the quadratic character which corresponds by class field theory to this discriminant). Consider the parameter:

$$\boxplus_{i \in [1,v]} \sigma_i \boxtimes R_{n_i} \boxplus 1 \boxtimes R_{m-1-2n}. \tag{6.15.1}$$

Here we use the fact that $m-1-2n \geq m-1-2r \geq p-2r \geq 1$.

The local theta correspondence is known at each place where σ and π are both unramified, see [68]. This implies that at every unramified place π_v is necessarily the unramified representation in the local Arthur packet associated to the parameter (6.15.1). But by definition, π_v is also in the local packet associated to (6.13.1); this implies that (6.13.1) and (6.15.1) define automorphic (isobaric) representations of $\mathrm{GL}(N)$ which are isomorphic almost everywhere. These automorphic representation are therefore isomorphic and (6.13.1) must coincide with (6.15.1). This is in contradiction with the fact that there is no factor $\eta \boxtimes R_{m-1-2n}$ – for some automorphic quadratic character η of $\mathrm{GL}(1)$ — in (6.13.1).

6.16. We now assume that $m = N$ is even. We moreover assume that $p - 2r > 1$.

We then do the same: First construct τ as above. For each $i \in \{1, \dots, 3r-m+1\}$ we fix a cuspidal irreducible representation of $\mathrm{GL}(2,F)$ of orthogonal type, this means that $L(s, \rho_i, \wedge^2)$ has a pole at $s = 1$; we can impose any discrete series at the archimedean places we want. The Arthur parameter we look at is now:

$$\boxplus_{i=1}^{3r-m+1} \rho_i \boxtimes R_1 \boxplus \tau \boxtimes R_{m-1-2r} \boxplus \eta \boxtimes R_1, \tag{6.16.1}$$

where η is a suitable automorphic quadratic character of $\mathrm{GL}(1)$ in such a way that this parameter is relevant for the quasisplit form of $\mathrm{SO}(V)$, see [6].

Now we argue as above to construct a global representation π of $\mathrm{SO}(V)$ in this packet which is as we want at the archimedean places. We construct (6.15.1) as above and here we have not to anticipate more results than those announced by Arthur. But to conclude we have to make sure that $m - 1 - 2n > 1$ because there is a factor $\eta \boxtimes R_1$ in (6.16.1). Nevertheless: as we have hypothesised that

$$m - 1 - 2n \geq m - 1 - 2r \geq p - 2r > 1,$$

we obtain a contradiction which proves that π is not a θ-lift of a cuspidal representation of $\mathrm{Sp}(2n)$ with $n \leq r$.

7. The polynomial Fock model for the dual pair $\mathrm{O}(p,q) \times \mathrm{Sp}_{2n}(\mathbb{R})$

In this section we will describe the polynomial Fock model for the action of $(\mathfrak{so}(p,q) \times \mathfrak{sp}(2n,\mathbb{R}), \big(\widetilde{\mathrm{O}(p) \times \mathrm{O}}(q) \times \mathrm{MU}_n)$ in the oscillator representation associated to the dual pair $\mathrm{O}(p,q) \times \mathrm{Sp}_{2n}(\mathbb{R})$. A secondary goal will be to understand *geometrically* the half-determinant twists i.e. by a power of $\det^{1/2}$, see Definition 7.1 immediately below, for the actions of the maximal compact subgroups $\widetilde{\mathrm{O}(p) \times \mathrm{O}(q)}$ and MU_n.

7.1. **Definition.** In what follows $\det^{1/2} = \det^{1/2}_{\mathrm{U}_n}$ will denote the unique character of MU_n with square equal to the pull-back of the character det from U_n. For a subgroup $H \subset \mathrm{U}_n$ we will let $\det^{1/2}_H$ denote the restriction of $\det^{1/2}_{\mathrm{U}_n}$ to the inverse image of H under the covering $\mathrm{MU}_n \to \mathrm{U}_n$.

The geometric interpretation is useful to understand the dependence of the sign of the exponent of the half-determinant twist on the positive definite complex structure $J_{V\otimes W}$ — see below. The sign of the half-determinant twist also depends on the choice of embedding of $\mathrm{U}_n \to \mathrm{Sp}_{2n}(\mathbb{R})$. Two different choices were made in [8] and [25]. We will use the choice of [25]. We now recall the formula of Funke-Millson in [25].

7.2. **Definition.** We embed U_n into $\mathrm{Sp}_{2n}(\mathbb{R})$ by the map f given by

$$f(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad a+ib \in \mathrm{U}(n). \tag{7.2.1}$$

We then embed MU_n into $\mathrm{Mp}_{2n}(\mathbb{R})$ by the map $\widetilde{f}$ which is the double cover of f.

We now derive the formulas for the action of the above maximal compact subgroups in the polynomial Fock model for the dual pair

$$(\mathfrak{so}(p,q) \times \mathfrak{sp}(2n,\mathbb{R}), \big(\widetilde{\mathrm{O}(p) \times \mathrm{O}(q)}\big) \times \mathrm{MU}_n).$$

7.2.1. *The polynomial Fock model for a tensor product of a quadratic space V and symplectic space W.* Let W be a real vector space of dimension $2n$ with a symplectic form $\langle\ ,\ \rangle_W$. Let V be a real vector space and $(,)$ be a non-degenerate quadratic form of signature (p,q) on V. Recall we have set $m = p+q$. We then let $\langle\ ,\ \rangle$ denote the non-degenerate skew-symmetric form $(,) \otimes \langle , \rangle_W$ on $V \otimes W$. We note that the space $V \otimes W$ has dimension $2n(p+q) = 2nm$. We will restrict the action of $\mathrm{Mp}_{2nm}(\mathbb{R})$ to the induced two-fold cover of $\mathrm{O}(p,q) \times \mathrm{Sp}_{2n}(\mathbb{R})$. We will use two different notations for the covering groups of subgroups of $\mathrm{Mp}_{2nm}(\mathbb{R})$. We will use a superscript tilde as in $\widetilde{\mathrm{O}(p,q)}$ for the coverings of subgroups of $\mathrm{O}(p,q)$, i.e. for groups attached to the first factor and continue using the letter M as in MU_n for subgroups of $\mathrm{Mp}_{2n}(\mathbb{R})$. We will also use subscripts for the dimension parameters, e.g. $2n$ as immediately above, for the second family but not for the first. We will use the symbol K to denote the maximal compact subgroup $\widetilde{\mathrm{O}(p) \times \mathrm{O}(q)}$ of $\widetilde{\mathrm{O}(p,q)}$.

We note that the multiplication in the group $\widetilde{\mathrm{O}(p,q)}$ induces a two-fold covering map $\mu_{p,q} : \widetilde{\mathrm{O}(p)} \times \widetilde{\mathrm{O}(q)} \to \widetilde{\mathrm{O}(p) \times \mathrm{O}(q)}$ given by

$$\mu_{p,q}\big((((k_1,1),z_1),((1,k_2),z_2))\big) = ((k_1,k_2),z_1z_2) \tag{7.2.2}$$

See [8] §4.8.2 for more details. We will use this map to identify $\widetilde{\mathrm{O}(p) \times \mathrm{O}(q)}$ with the above quotient of $\widetilde{\mathrm{O}(p)} \times \widetilde{\mathrm{O}(q)}$.

In what follows we will assume the unitary group U_n is embedded in $\mathrm{Sp}_{2n}(\mathbb{R})$ by the embedding f of Equation (7.2.1). Choose a symplectic basis $\{x_1,\cdots,x_n,y_1,\cdots,y_n\}$ of W and let J_W be the positive complex structure given by

$$J_W x_i = y_i \text{ and } J_W y_i = -x_i.$$

Write $W\otimes\mathbb{C} = W' \oplus W''$ where W' (resp. W'') is the $+i$ (resp. $-i$)-eigenspace of J_W operating on $W\otimes\mathbb{C}$. We define complex bases $\{\tilde{w}'_1\ldots,\tilde{w}'_n\}$ and $\{\tilde{w}''_1,\ldots,\tilde{w}''_n\}$ for W' and W'' respectively by

$$\tilde{w}'_j = x_j - iy_j \quad \text{and} \quad \tilde{w}''_j = x_j + iy_j \quad \text{for } 1 \le j \le n.$$

We let $z'_i, 1 \le j \le n$ resp. $z''_i, 1 \le j \le n$ be the associated coordinates for W' resp. W'', so $z'_j \in (W')^* \cong W''$. We make U_n act on W' and W'' using the above coordinates. The following lemma is immediate.

7.3. **Lemma.** *The induced action of* U_n *on* W' *is equivalent to the dual of the standard action and the action on* W'' *is equivalent to the standard action.*

Remark. It is critical in the construction of the unitary Fock model that the symmetric form b_J on W' given by $b_J(x,y) = \langle x, J_W y\rangle$ is positive definite. Hence if we change the sign of the symplectic structure then we have to change the sign of J_W and the actions on W' and W'' would become respectively the standard action and the dual of the standard action.

We then have the fundamental

7.4. **Proposition.** *The subgroup* MU_n *of* $\mathrm{Mp}_{2n}(\mathbb{R})$ *embedded according to Equation* (7.2.1) *acts on the space* $\mathrm{Pol}(W')$ *of polynomial functions on* W' *by*

$$\omega(k')f(w') = \det(k')^{\frac{1}{2}} f((k')^{-1}w') \tag{7.4.1}$$

and on $\mathrm{Pol}(W'')$ *by the dual action, hence by*

$$\omega(k')f(w'') = \det(k')^{-\frac{1}{2}} f((k')^{-1}w''). \tag{7.4.2}$$

We now give a geometric proof of Proposition 7.4 by explaining how the half-determinant twists in Equations (7.4.1) and (7.4.2) come from the "metaplectic correction" in the theory of Geometric Quantization — see Woodhouse [86], Chapter 10. In the geometric description of the Fock model the action is not on polynomial functions but rather on *polynomial half-forms*, i.e. polynomial sections of the unique square-root $\mathcal{L}_{W'}$ of the canonical bundle $\mathcal{K}_{W'}$ of W'. The following lemma follows from a covering space argument.

7.5. **Lemma.** *The action of* U_n *on the line bundle* $\mathcal{K}_{W'}$ *lifts to an action of* MU_n *on the line bundle* $\mathcal{L}_{W'}$.

Now note that $\tau = dz_1' \wedge \cdots \wedge dz_n'$ is a nowhere zero section of $\mathcal{K}_{W'}$. Since W' is contractible it follows from the long exact sequence of cohomology associated to the short exact sequence of sheaves $\mu_2 \to \mathcal{L}_{W'} \to \mathcal{K}_{W'}$ (the second map is the squaring map $v \to v \otimes v$) that there exists a nowhere zero section of η_0 of $\mathcal{L}_{W'}$ (unique up to sign) such that $\eta_0 \otimes \eta_0 = \tau$. We will choose one of the square roots and denote it by $\sqrt{dz_1' \wedge \cdots \wedge dz_n'}$. Hence any polynomial section η of $\mathcal{L}_{W'}$ may be written

$$\eta = f(z_1', \cdots, z_n')\sqrt{dz_1' \wedge \cdots \wedge dz_n'}.$$

Proposition 7.4 then follows from

7.6. **Lemma.** *The group* MU_n *acts on the half-form* $\sqrt{dz_1' \wedge \cdots \wedge dz_n'}$ *by the character* $\det^{\frac{1}{2}}$.

Proof. Note first that since any character of a connected group has at most one square-root for $k' \in \mathrm{U}_n$

$$\omega(k')\eta = \det(k')^{\frac{1}{2}}\eta \iff \omega(k')(\eta \otimes \eta) = \det(k')(\eta \otimes \eta). \tag{7.6.1}$$

Hence it suffices to prove that for the action of U_n on holomorphic n-forms ν on W' we have

$$\omega(k')\nu = \det(k')\nu.$$

But this is obvious, for example take $\nu = dz_1' \wedge \cdots \wedge dz_n'$. □

In the paragraph above, we have chosen a complex structure J_W compatible with $\langle\ ,\ \rangle_W$ and commuting with $\mathrm{U}_n = \mathrm{U}(W)$. We now choose a basis for V. Let v_α, $\alpha = 1, \ldots, p$, v_μ, $\mu = p+1, \ldots, q$, be an orthogonal basis of V such that $(v_\alpha, v_\alpha) = 1$ and $(v_\mu, v_\mu) = -1$. We denote by V_+ (resp. V_-) the real vector space spanned by $\{v_\alpha : 1 \le \alpha \le p\}$ (resp. $\{v_\mu : p+1 \le \mu \le m\}$). Let θ_V be the Cartan involution of $\mathrm{O}(p,q)$ associated to the splitting $V = V_+ + V_-$. Then $J = J_{V \otimes W} = \theta_V \otimes J_W$ is a positive complex structure on $V \otimes W$. For this complex structure we then have (see §4.15 of [8]).

7.7. **Lemma.** (1) $(V \otimes W)' \cong (V_+ \otimes W') + (V_- \otimes W'')$
(2) $(V \otimes W)'' \cong (V_+ \otimes W'') + (V_- \otimes W')$

The underlying vector space for the polynomial Fock model $\mathrm{Pol}((V \otimes W)')$ for the pair $(\mathfrak{sp}_{2mn}, \widetilde{\mathrm{U}}_{mn})$ is then

$$\mathrm{Pol}((V_+ \otimes W') + (V_- \otimes W'') \cong \mathrm{Sym}(V_+ \otimes W'' + V_- \otimes W').$$

7.7.1. *Coordinates on* $(V \otimes W)'$. We introduce linear functionals

$$\{z_{\alpha,j},\ z_{\mu,j} : 1 \le \alpha \le p, \quad p+1 \le \mu \le m, \quad 1 \le j \le n\}$$

on $V_+ \otimes W' + V_- \otimes W''$ by the formulas

$$z_{\alpha,j}(v \otimes w) = \frac{1}{2i}\langle v \otimes w, v_\alpha \otimes \tilde{w}_j'' \rangle, \quad 1 \le \alpha \le p, \quad 1 \le j \le n,$$

$$z_{\mu,j}(v \otimes w) = -\frac{1}{2i}\langle v \otimes w, v_\mu \otimes \tilde{w}_j' \rangle, \quad p+1 \le \mu \le m, \quad 1 \le j \le n.$$

Hence if we identify $V \otimes W'$ with V^n using the above basis for W' then for $\mathbf{x} = (x_1, \cdots, x_n)$ we have

$$x_j = \sum_{\alpha=1}^{p} z_{\alpha,j} v_\alpha + \sum_{\mu=p+1}^{p+q} z_{\mu,j} v_\mu.$$

We use these coordinates to identify the space $\mathrm{Sym}(V_+ \otimes W'' + V_- \otimes W')$ with the space $\mathcal{P} = \mathcal{P}(\mathbb{C}^{mn})$ of polynomials in mn variables $z_{1,j}, \dots, z_{m,j}$ $(j = 1, \dots, n)$. We will use $\mathcal{P}_+$ to denote the polynomials in the "positive" variables $z_{\alpha,j}, 1 \leq \alpha \leq p, 1 \leq j \leq n$. We will use the symbol $\mathcal{P}^{(\ell)}$ to denote the subspace of polynomials of degree ℓ and similarly for $\mathcal{P}_+^{(\ell)}$. It will be important to record the structure of $\mathcal{P}(\mathbb{C}^{mn})$ as a $K \times K'$-module. By Lemma 7.3 we know that the action of $\mathrm{GL}(n, \mathbb{C})$ on W'' is equivalent to the standard one. We will accordingly identify $\mathbb{C}^n$ with W'' (and $(\mathbb{C}^n)^*$ with W') as modules for $\mathrm{GL}(n, \mathbb{C})$. We can then replace W'' with $\mathbb{C}^n$ and W' by $(\mathbb{C}^n)^*$ (since we are only concerned here with the $\mathrm{GL}(n, \mathbb{C})$-module structure of W' and W''). We will use $e_1, \cdots, e_n$ to denote the standard basis of $\mathbb{C}^n$, accordingly under the above identification e_j corresponds to $\tilde{w}''_j$ for $1 \leq j \leq n$.

Remark. To be consistent with the previous section we should have written $z'_{\alpha,j}$ and $z'_{\mu,j}$. We have chosen to drop the primes here.

7.7.2. *The action of $K \times K'$ on* $\mathrm{Pol}((V \otimes W)')$. We now consider the dual pair

$$\mathrm{O}(p, q) \times \mathrm{Sp}_{2n}(\mathbb{R}) \subset \mathrm{Sp}_{2mn}(\mathbb{R}).$$

By restriction of the above representation of the pair $(\mathfrak{sp}_{2mn}, \widetilde{\mathrm{U}}_{mn})$ we get the polynomial Fock model for $\mathrm{O}(p, q) \times \mathrm{Sp}_{2n}(\mathbb{R})$ acting on $\mathrm{Pol}((V \otimes W)') \cong \mathrm{Sym}(V_+ \otimes W'' + V_- \otimes W') \cong \mathcal{P}(\mathbb{C}^{mn})$.

In the next Proposition we will use the isomorphism

$$(7.7.1)\quad \mathrm{Sym}(V_+ \otimes W'') \otimes \mathrm{Sym}(V_- \otimes W') \cong \mathcal{P}(\mathbb{C}^{pn} \otimes (\mathbb{C}^*)^{qn}) \cong \mathcal{P}(\mathbb{C}^{pn}) \otimes \mathcal{P}((\mathbb{C}^*)^{qn}).$$

Recall from the previous subsection that we consider $\mathrm{O}(\widetilde{p) \times \mathrm{O}}(q)$ to be a quotient by $\mathbb{Z}/2$ of the product $\widetilde{\mathrm{O}(p)} \times \widetilde{\mathrm{O}(q)}$. We then have

7.8. **Proposition.** (1) $\mathrm{MU}(n)$ *acts on* $\mathcal{P}(\mathbb{C}^{pn}) \otimes \mathcal{P}((\mathbb{C}^*)^{qn})$ *by the product of the standard action of* $\mathrm{U}(n)$ *on the first tensor factor with the dual of the standard action on the second tensor factor all twisted by* $\det^{\frac{p-q}{2}}$.

(2) *The group* $\mathrm{O}(\widetilde{p) \times \mathrm{O}}(q)$ *acts on* $\mathcal{P}(\mathbb{C}^{pn}) \otimes \mathcal{P}((\mathbb{C}^*)^{qn})$ *by the (outer) tensor product of representations* $\big(\det_{\mathrm{O}(p)}^{\frac{n}{2}} \otimes \rho_{\mathrm{O}(p)}\big) \boxtimes \big(\det_{\mathrm{O}(q)}^{-\frac{n}{2}} \otimes \rho_{\mathrm{O}(q)}\big)$ *where* $\rho_{\mathrm{O}(p)}$ *resp.* $\rho_{\mathrm{O}(q)}$ *denotes the standard action on the polynomials in the positive variables resp. the dual of the standard action on the polynomials in the negative variables.*

Proof. To prove Statement (1) of the Proposition, note after diagonalizing $(\ ,\ )$ using the above basis we have MU_n-equivariant isomorphisms of symplectic spaces

(1) $V_+ \otimes W'', \langle\ ,\ \rangle \cong (W'')^p, \langle\ ,\ \rangle_{W''}$;
(2) $V_- \otimes W'', \langle\ ,\ \rangle \cong (W'')^q, -\langle\ ,\ \rangle_{W''}$.

The part of Statement (1) concerning the action on the first tensor factor then follows because the determinant of the action on a direct sum is the product of the determinants of the actions on the factors.

To prove the part of Statement (1) concerning the action on the second tensor factor, note first that by the remark following Lemma 7.3 that since we have changed the sign of the symplectic form on W hence changed the sign of J and hence interchanged the $+i$ and $-i$ eigenspaces of J. Hence we have dualized the representation on W'' and in particular we have changed the sign of the exponent of the twist associated to each factor of the direct sum. Then we apply the argument of the preceding paragraph.

To prove Statement (2) we combine the results for the dual pair $\mathrm{U}(p,q)\times \mathrm{U}_{2n}(\mathbb{R})$ of [8], Part 1, Chapter 4 together with the seesaw pair

$$\begin{array}{ccc} \mathrm{U}(p,q) & & \mathrm{Sp}_{2n}(\mathbb{R}) \\ | & \times & | \\ \mathrm{O}(p,q) & & \mathrm{U}_n \end{array}$$

Lemma 4.21 of [8] proves the analogue of Statement (2) for the larger group $\mathrm{MU}(p)\times \mathrm{MU}(q)$. Then Statement (2) above follows by restriction using the seesaw pair and the fact that the half-determinant characters are natural for the inclusions of orthogonal groups into unitary groups.

The reason that the exponents of the half-determinant twists for the actions of MU_p and MU_q have different signs is because the map $\mu_{p,q} : \mathrm{MU}_p\times \mathrm{MU}_q \to \mathrm{Mp}_{2nm}$ involves a conjugation of the MU_q-factor, see §4.8.2 of [8]. □

Remark. The embedding of MU_{nm} into $\mathrm{Mp}_{2nm}(\mathbb{R})$ in Equation (7.2.1) is equal to the embedding of [8], §4.8.1, *preceded by complex conjugation*. This is why the signs of the exponents of the twists are the negatives of the corresponding signs in Lemma 4.21 of [8]. It does agree with the embedding of Funke-Millson, [25], §51, pg. 917.

Before stating the next corollary we recall $\widetilde{\mathrm{O}(p,q)}$ has a character $\det^{\frac{1}{2}}_{\mathrm{O}(p,q)}$ that restricts to $\det^{\frac{1}{2}}_{\mathrm{O}(p)} \boxtimes \det^{\frac{1}{2}}_{\mathrm{O}(q)}$ on $\widetilde{\mathrm{O}(p)\times \mathrm{O}}(q)$.

7.9. **Corollary.** *If we twist the restriction of the oscillator representation of* $\mathrm{Mp}_{2nm}(\mathbb{R})$ *to* $\widetilde{\mathrm{O}(p,q)}$ *by* $\det^{-\frac{n}{2}}_{\mathrm{O}(p,q)}$ *the resulting twisted representation descends to* $\mathrm{O}(p,q)$ *and the restriction of this representation to* $\mathrm{O}(p)\times \mathrm{O}(q)$ *is the standard action of* $\mathrm{O}(p)$ *on the positive variables and the dual of the standard action of* $\mathrm{O}(q)$ *twisted by* $\det^{-n}_{\mathrm{O}(q)}$ *on the negative variables.*

We will henceforth replace the oscillator representation restricted to $\widetilde{\mathrm{O}(p,q)}$ by its twist as above. From now on we will refer to the resulting representation of $\mathrm{O}(p,q)$ as the oscillator representation of $\mathrm{O}(p,q)$.

In what follows we will be mostly concerned with the subspace $\mathcal{P}_+$ of $\mathcal{P}$, that is, polynomials in the "positive" variables $z_{\alpha,j}$ alone. We then have the following Theorem

7.10. **Theorem.** (1) *The group* MU_n *acts on* $\mathcal{P}_+$ *by the standard action of* U_n *twisted by* $\det^{\frac{p-q}{2}}$.
(2) *The group* $\mathrm{O}(p)$ *acts on* $\mathcal{P}_+$ *by the standard action.*
(3) *The group* $\mathrm{O}(q)$ *acts on* $\mathcal{P}_+$ *by multiplication by the character* $\det^{-n}_{\mathrm{O}(q)}$.

We see from Proposition 7.8 (applied to the complexification of the action of U_n) that the variables $z_{\alpha,j}, 1\le \alpha\le p, 1\le j\le n$, transform (in j) according to the standard representation of $\mathrm{GL}(n,\mathbb{C})$ and the variables $z_{\mu,j}, p{+}1 \le \mu \le p{+}q, 1 \le j \le n$, transform in j according to the dual of the standard representation of $\mathrm{GL}(n,\mathbb{C})$.

It will be useful (both for our computations and to compare our formulas with those of [41]) to regard these variables as coordinate functions on the space $\mathrm{M}_{p,n}(\mathbb{C})$ of p by n matrices. The following lemma justifies this.

7.11. **Lemma.** *There is an isomorphism of $K \times K'$-modules*

$$\mathcal{P}_+ = \mathrm{Sym}(V_+ \otimes \mathbb{C}^n) \cong \mathrm{Pol}(\mathrm{M}_{p,n}(\mathbb{C})).$$

Proof. We have

$$\begin{aligned}\mathrm{Pol}(\mathrm{M}_{p,n}(\mathbb{C})) &\cong \mathrm{Pol}(\mathrm{Hom}(\mathbb{C}^n, V_+)) \cong \mathrm{Pol}(V_+ \otimes (\mathbb{C}^n)^*) \\ &\cong \mathrm{Sym}((V_+ \otimes (\mathbb{C}^n)^*)^*) \cong \mathrm{Sym}(V_+ \otimes \mathbb{C}^n).\end{aligned}$$

Here we use (as we will do repeatedly) that $V_+ \cong V_+^*$ as a K-module. □

To summarize, if we represent the action of the Weil representation ω in terms of the p by n matrix representation of the Fock model as we have just explained then we have

$$\mathcal{P}_+ = \mathrm{Pol}(\mathrm{M}_{p\times n}(\mathbb{C})))$$

and:

7.12. **Theorem.**

(1) *The action of the group* MU_n *induced by the oscillator representation on polynomials in the matrix variables is the tensor product of the character* $\det^{\frac{p-q}{2}}$ *with the action induced by the natural action on the rows (i.e. from the right) of the matrices. Note that each row has n entries.*
(2) *The action of the group* $\mathrm{O}(p)$ *is induced by the natural action on the columns (i.e. from the left) of the matrices.*
(3) *The group* $\mathrm{O}(q)$ *simply scales all polynomials by the central character* $\det_{\mathrm{O}(q)}^{-n}$.

7.13. **The operators in the infinitesimal polynomial Fock model.** The operators in the infinitesimal oscillator representation for the dual pair $\mathrm{O}(p,q)\times\mathrm{Sp}_{2n}(\mathbb{R})$ may be found in Kudla-Millson, [48]. We will very briefly review the formulas in Theorem 7.1 (b) of Kudla-Millson, loc. cit.

We will return to the general symplectic vector space W equipped with a positive definite complex structure J of the beginning of this section. We recall, see [48], pg. 150, that there is a canonical identification

$$S^2(W) \cong \mathfrak{sp}_{2n}(\mathbb{R}) \text{ and hence } S^2(W\otimes\mathbb{C}) \cong \mathfrak{sp}_{2n}(\mathbb{C}).$$

Since the vector space $W \otimes \mathbb{C}$ has a $\mathrm{GL}(n,\mathbb{C})$-invariant splitting

$$W \otimes \mathbb{C} = W' \oplus W''$$

with $\mathrm{GL}(n,\mathbb{C})$ acting by the standard representation on W'' and the dual of the standard representation on W' the symmetric product $S^2(W\otimes\mathbb{C})$ has a $\mathrm{GL}(n,\mathbb{C})$ -invariant invariant bigrading

$$S^2(W\otimes\mathbb{C}) = S^2(W'')\oplus(W''\otimes W')\oplus S^2(W') = S^{2,0}(W\otimes\mathbb{C})\oplus S^{1,1}(W\otimes\mathbb{C})\oplus S^{0,2}(W\otimes\mathbb{C}),$$

it is immediate that the induced bigrading of vector spaces

$$\mathfrak{sp}_{\mathbb{C}} = \mathfrak{sp}^{(1,1)} \oplus \mathfrak{sp}^{(2,0)} \oplus \mathfrak{sp}^{(0,2)}$$

is a bigrading of Lie algebras. In terms of the coordinates $z_1', \cdots, z_n'$ of the beginning of this section we have the corresponding bigrading of the operators in the infinitesimal polynomial Fock model. We will drop the superscript primes on the coordinates $z_1', \cdots, z_n'$ from now on (since we will need to redefine the meaning of these superscripts shortly).

$$\begin{aligned}\omega(\mathfrak{sp}^{(1,1)}) &= \operatorname{span}\left\{\frac{1}{2}\left(z_i\frac{\partial}{\partial z_j}+\frac{\partial}{\partial z_j}z_i\right)\right\},\\ \omega(\mathfrak{sp}^{(2,0)}) &= \operatorname{span}\{z_iz_j\},\\ \omega(\mathfrak{sp}^{(0,2)}) &= \operatorname{span}\left\{\frac{\partial^2}{\partial z_i\partial z_j}\right\}.\end{aligned}$$

Note that while $\mathfrak{sp}_{2mn}$ is a real Lie algebra, the $\mathfrak{sp}^{(a,b)}$ are complex subalgebras of $\omega(\mathfrak{sp}_{\mathbb{C}})$. In the Cartan decomposition

$$\mathfrak{sp}_{2mn} = \mathfrak{u}_{mn} \oplus \mathfrak{q} \tag{7.13.1}$$

we have

$$\omega(\mathfrak{u}_{\mathbb{C}}) = \mathfrak{sp}^{(1,1)} \quad \text{and} \quad \omega(\mathfrak{q}_{\mathbb{C}}) = \mathfrak{sp}^{(2,0)} \oplus \mathfrak{sp}^{(0,2)}.$$

7.13.1. *The action of $\mathfrak{sp}_{2n}(\mathbb{R})$ in the dual pair $\mathfrak{o}(p,q)\times\mathfrak{sp}_{2n}(\mathbb{R})$.* For i,j with $1\leq i,j\leq n$ we define operators on $\mathcal{P}_+$ by

(1) $\Delta_{ij}=\sum_{\alpha=1}^{p}\frac{\partial^2}{\partial z_{\alpha,i}\partial z_{\alpha,j}}$;
(2) $r_{ij}=\sum_{\alpha=1}^{p} z_{\alpha,i}z_{\alpha,j}$;
(3) $D_{ij}=\sum_{\alpha=1}^{p} z_{\alpha,i}\partial z_{\alpha,j}+z_{\alpha,j}\partial z_{\alpha,i}$.

From [48], Theorem 7.1 (b), we have (with $\lambda=\frac{1}{2i}$) the following formulas for the action on $\mathcal{P}_+$:

7.14. **Theorem.** *(1) $\omega(w_j'\circ w_k')=-2i\ \Delta_{jk}$;*
(2) $\omega(w_j''\circ w_k'')=\frac{i}{2}\ r_{jk}$;
(3) $\omega(w_j'\circ w_k'')=-i\ D_{jk}$.

In the above $w_1\circ w_2$, $(w_1,w_2\in W)$, denotes the symmetric product of w_1 and w_2. There are analogous formulas for the action of $\mathfrak{sp}_{2n}(\mathbb{R})$ on $\mathcal{P}_-$ which we leave to the reader to write down.

7.15. **Some conventions.** By abuse of notations we will use, in the following, the same symbols for corresponding objects and operators in both the Schwartz and Fock models.

We use the convention that indices $\alpha,\beta,\ldots$ run from 1 to p and indices $\mu,\nu,\ldots$ run from $p+1$ to m. In this numbering $K=\mathrm{SO}(p)\times\mathrm{SO}(q)$ acts so that for each j, the group $\mathrm{SO}(p)$ rotates the variables $z_{\alpha,j}$ and $\mathrm{SO}(q)$ rotates the variables $z_{\mu,j}$.

Note that $\mathfrak{p}\cong\mathbb{C}^p\otimes(\mathbb{C}^q)^*\cong \mathrm{M}_{p,q}(\mathbb{C})$, with our convention we let $\omega_{\alpha,\mu}$ be the linear form which maps an element of $\mathfrak{p}$ to its (α,μ)-coordinate.

Finally, for multi-indices $\underline{\alpha}=(\alpha_1,\ldots,\alpha_q)$ and $\underline{\beta}=(\beta_1,\ldots,\beta_\ell)$ we will write

$$\omega_{\underline{\alpha}}=\omega_{\alpha_1,p+1}\wedge\ldots\wedge\omega_{\alpha_q,p+q},$$

$$z_{\underline{\alpha},j}=z_{\alpha_1,j}\ldots z_{\alpha_q,j},$$

$$v_{\underline{\beta}}=v_{\beta_1}\otimes\ldots\otimes v_{\beta_\ell}.$$

7.16. We are interested in the reductive dual pair $\mathrm{O}(V)\times\mathrm{Mp}_{2n}(\mathbb{R})$ inside $\mathrm{Mp}_{2mn}(\mathbb{R})$. We suppose that $\mathrm{O}(V)$ and $\mathrm{Mp}_{2n}(\mathbb{R})$ are embedded in $\mathrm{Mp}_{2mn}(\mathbb{R})$ in such a way that the Cartan decomposition of $\mathfrak{sp}_{2mn}$ also induces Cartan decompositions of $\mathfrak{g}$ and $\mathfrak{g}'$. Then $\mathcal{P}$ is a $(\mathfrak{g}, K)$-module and a $(\mathfrak{g}', K')$-module. We will make use of the structure of $\mathcal{P}$ as a $(\mathfrak{g}\oplus\mathfrak{g}', K\cdot K')$-module. We first recall the definition of harmonics (see [36]).

The Lie algebra $\mathfrak{k} = \mathfrak{o}_p \times \mathfrak{o}_q$ of K is a member of a reductive dual pair $(\mathfrak{k}, \mathfrak{l}')$ where $\mathfrak{l}' = \mathfrak{sp}_{2n}\times\mathfrak{sp}_{2n}$. We can decompose

$$\mathfrak{l}' = \mathfrak{l}'^{(2,0)} \oplus \mathfrak{l}'^{(1,1)} \oplus \mathfrak{l}'^{(0,2)}, \quad \text{where } \mathfrak{l}'^{(a,b)} = \mathfrak{l}' \cap \mathfrak{sp}^{(a,b)}.$$

Then the harmonics are defined by

$$\mathcal{H} = \mathcal{H}(K) = \left\{ P \in \mathcal{P} \, : \, l(P) = 0 \text{ for all } l \in \mathfrak{l}'^{(0,2)} \right\}.$$

Remark. The space $\mathcal{H}$ is smaller that the usual space of harmonics $\mathcal{H}(G)$ in $\mathcal{P}$ associated to the "indefinite Laplacians", the latter being associated to the dual pair $(\mathfrak{g}, \mathfrak{g}')$ rather than $(\mathfrak{k}, \mathfrak{l}')$. The space $\mathcal{H}$ is easily described by separating variables: Let $\mathcal{P}_+$, resp. $\mathcal{P}_-$, be the space of all polynomial functions on $V_+\otimes W' \cong V_+^n \otimes \mathbb{C}$, resp. $V_-\otimes W'' \cong V_-^n\otimes\mathbb{C}$. It is naturally realized as a subspace of $\mathcal{P}(\mathbb{C}^{mn})$, namely the space of polynomials in the variables $z_{\alpha,j}$, resp. $z_{\mu,j}$. We obviously have $\mathcal{P} = \mathcal{P}_+\otimes\mathcal{P}_+$. Now denote by $\mathcal{H}_+$, resp. $\mathcal{H}_-$, the harmonic polynomials in V_+^n, resp. V_-^n, (the "pluriharmonics" in the terminology of Kashiwara-Vergne [41]). We then have:

$$\mathcal{H} = \mathcal{H}_+\otimes\mathcal{H}_-.$$

7.17. **Some special harmonic polynomials.** In this subsection we will introduce subspaces $\mathcal{H}'(V_+^n)$ and $\mathcal{H}''(V_+^n)$ of $\mathcal{H}_+$ which are closed under polynomial multiplication but not closed under the action of $K = \mathrm{O}(V_+)$.

We begin by introducing coordinates (that we will call "Witt coordinates") that will play a key role in what follows. The resulting coordinates $w'_{\alpha,j}$, $w''_{\alpha,j}$, t_j, $1\le\alpha\le[\frac{p}{2}]$, $1\le j\le n$, coincide (up to an exchange of order of the indices α, j) with the coordinates $x_{j\alpha}$, $y_{j\alpha}$, t_j, $1\le\alpha\le[\frac{p}{2}]$, $1\le j\le n$, of Kashiwara and Vergne [41] (Kashiwara and Vergne use i instead of j and ν instead of α). First we define an ordered *Witt basis* $\mathcal{B}$ for V_+. Let $p_0 = [p/2]$. We define an involution $\alpha\to\alpha'$ of the set $\{1, 2, \cdots, 2p_0\}$ by

$$\alpha' = 2p_0 - \alpha + 1.$$

If p is even define u'_α, u''_α $(1\le\alpha\le p_0)$ by

$$u'_\alpha = \frac{v_\alpha - i v_{\alpha'}}{\sqrt{2}}$$

and

$$u''_\alpha = \frac{v_\alpha + i v_{\alpha'}}{\sqrt{2}}.$$

Then $(u'_1, \cdots, u'_{p_0}, u''_1, \cdots, u''_{p_0})$ is the required ordered Witt basis. In case p is odd we define u'_α and u''_α $(1\le\alpha\le p_0)$ as above then add v_p as the last basis vector. In both cases we will use $\mathcal{B}$ to denote the above ordered basis.

We note that u'_α $(1\le\alpha\le p_0)$, and u''_α $(1\le\alpha\le p_0)$, are isotropic vectors which satisfy

$$(u'_\alpha, u'_\beta) = 0, \quad (u''_\alpha, u''_\beta) = 0 \text{ and } (u'_\alpha, u''_\beta) = \delta_{\alpha\beta} \quad \text{for all} \quad \alpha, \beta.$$

Of course in the odd case v_p is orthogonal to all the u'_α's and u''_β's.

Define coordinates

$$(w'_1, \cdots, w'_{p_0}, w''_{p_0}, \cdots, w''_1) \quad \text{for } \dim(V_+) \text{ even},$$

$$\text{resp. } (w'_1, \cdots, w'_{p_0}, w''_{p_0}, \cdots, w''_1, t) \quad \text{for } \dim(V_+) \text{ odd},$$

by

$$x = \sum_{\alpha=1}^{p_0} w'_\alpha(x) u'_\alpha + \sum_{\alpha=1}^{p_0} w''_\alpha(x) u''_\alpha \quad \text{in case } p \text{ is even}$$

and

$$x = \sum_{\alpha=1}^{p_0} w'_\alpha(x) u'_\alpha + \sum_{\alpha=1}^{p_0} w''_\alpha(x) u''_\alpha + t v_p \quad \text{in case } p \text{ is odd.}$$

The above coordinates on V_+ induce coordinates $w'_{\alpha,k}$, $w''_{\alpha,k}$, t_k on V_+^n for $1 \le \alpha \le p$, $1 \le k \le n$, and for $\mathbf{x} = (x_1, \cdots, x_n) \in V_+^n$ and $1 \le k \le n$ we have:

$$x_k = \sum_{\alpha=1}^{p_0} w'_{\alpha,k}(\mathbf{x}) u'_\alpha + \sum_{\alpha=1}^{p_0} w''_{\alpha,k}(\mathbf{x}) u''_\alpha \quad \text{in case } p \text{ is even}$$

and

$$x_k = \sum_{\alpha=1}^{p_0} w'_{\alpha,k}(\mathbf{x}) u'_\alpha + \sum_{\alpha=1}^{p_0} w''_{\alpha,k}(\mathbf{x}) u''_\alpha + t_k v_p \quad \text{in case } p \text{ is odd.}$$

We note the formulas

(1) $z_{\alpha,j}(\mathbf{x}) = (x_j, v_\alpha)$, $1 \le \alpha \le p, 1 \le j \le n$
(2) $w'_{\alpha,j}(\mathbf{x}) = (x_j, u''_\alpha)$, $1 \le \alpha \le p_0, 1 \le j \le n$
(3) $w''_{\alpha,j}(\mathbf{x}) = (x_j, u'_\alpha)$, $1 \le \alpha \le p_0, 1 \le j \le n$.

We find as a consequence that

(7.17.1) $\quad w'_{\alpha,j} = z_{\alpha,j} + i z_{\alpha',j}$ and $w''_{\alpha,j} = z_{\alpha,j} - i z_{\alpha',j}, \quad 1 \le \alpha \le p_0,\ 1 \le j \le n.$

It will be convenient to define $w'_{\alpha,j}$, resp. $w''_{\alpha,j}$, for α satisfying $p_0 + 1 \le \alpha \le 2p_0$ by

$$w'_{\alpha',j} = i w''_{\alpha,j}, \quad 1 \le \alpha \le 2p_0.$$

For both even and odd p we denote the algebra of polynomials in $w'_{\alpha,j}$ by $\mathcal{H}'(V_+^n)$ and the algebra of polynomials in $w''_{\alpha,j}$ by $\mathcal{H}''(V_+^n)$. The following lemma is critical in what follows.

7.18. **Lemma.** *The $\mathbb{C}$-algebras $\mathcal{H}'(V_+^n)$ and $\mathcal{H}''(V_+^n)$ of $\mathcal{P}_+$ lie in the vector space $\mathcal{H}_+$.*

Proof. The Laplacians $\Delta_{ij}, 1 \le i, j \le n$, on $\mathcal{P}_+$ of subsection 7.13.1, whose kernels define $\mathcal{H}_+$ are given in Witt coordinates by sums of *mixed* partials

$$\Delta_{ij} = \sum_{\alpha=1}^{p} \frac{\partial^2}{\partial w'_{\alpha,i} \partial w''_{\alpha,j}} + \frac{\partial^2}{\partial w'_{\alpha,j} \partial w''_{\alpha,i}} \text{ for } p \text{ even}$$

and

$$\Delta_{ij} = \sum_{\alpha=1}^{p} \frac{\partial^2}{\partial w'_{\alpha,i} \partial w''_{\alpha,j}} + \frac{\partial^2}{\partial w'_{\alpha,j} \partial w''_{\alpha,i}} + \frac{\partial^2}{\partial t_i \partial t_j} \text{ for } p \text{ odd.}$$

See [41], pages 22 and 26. $\square$

Remark. The only harmonic polynomials we will encounter in this Chapter will belong to the subring $\mathcal{H}''(V_+^n)$.

7.19. **The matrix $W''(\mathbf{x})$ and the harmonic polynomials $\Delta_k(\mathbf{x})$.** We will use $\mathbf{x}$ to denote an n-tuple of vectors, $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in V^n$. Let $W''(\mathbf{x})$ be the p_0 by n matrix with $(\alpha, j)^{\text{th}}$ entry $w''_{\alpha,j}(\mathbf{x})$, $1 \leq \alpha \leq p_0, 1 \leq j \leq n$, that is the coordinates of x_j relative to $u''_1, \cdots, u''_{p_0}$. Following the notation of [41] we let $\Delta_k(\mathbf{x})$ be the leading principal k by k minor of the matrix $W''(\mathbf{x})$ (by this we mean the determinant of the upper left k by k block). The polynomials $\Delta_k(\mathbf{x}), 1 \leq k \leq n$ belong to $\mathcal{H}''(V_+)$ and hence they belong to $\mathcal{H}$. For $1 \leq k \leq p_0$ we let $W''_k(\mathbf{x})$ be the submatrix obtained by taking the first k rows of $W''(\mathbf{x})$. We then have the following equation

$$W''_k(\mathbf{x}g) = W''_k(\mathbf{x})g, \quad g \in \mathrm{GL}(n, \mathbb{C}), 1 \leq k \leq p_0. \tag{7.19.1}$$

8. The classes of Kudla-Millson and Funke-Millson

In this section we will introduce the $(\mathfrak{so}(p,q), K)$-cohomology classes (with $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$) of Kudla-Millson and Funke-Millson, find explicit values for these classes on the highest weight vectors of the Vogan-Zuckerman special K-types $V(n, \lambda)$, see Proposition 8.15 and Proposition 8.17, and from those values and a result of Howe deduce the key result that the translates of the above classes under the universal enveloping algebra of the symplectic group span $\mathrm{Hom}_K(V(n,\lambda), \mathcal{P}(V^n))$ where $\mathcal{P}(V^n)$ is the polynomial Fock space, see Theorem 8.23. Our computations will give a new derivation of some of the formulas in [41].

Let V be a real quadratic space of dimension m and signature (p, q) two non-negative integers with $p + q = m$. Set $G = \mathrm{SO}_0(V)$ and $\mathfrak{g}$ be the complexified Lie algebra of G and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ its Cartan decomposition, where $\mathrm{Lie}(K) \otimes \mathbb{C} = \mathfrak{k}$. We let $\mathfrak{g}'$ be the complexified Lie algebra of $\mathrm{Sp}_{2n}(\mathbb{R})$.

In this section we make the assumption that

$$2n < p. \tag{8.0.1}$$

Equation (8.0.1) has two important consequences. Note that $\mathrm{rank}(G) = [\frac{m}{2}]$ and $\mathrm{rank}(\mathrm{SO}(V_+)) = [\frac{p}{2}]$. It then follows from (8.0.1) that we have:

1. $n \leq \mathrm{rank}(G)$.
2. $n \leq \mathrm{rank}(\mathrm{SO}(V_+))$.

As we will explain in Subsection 8.21, item (2) will imply we are in the first family for the two families of formulas in the paper of Kashiwara-Vergne, [41], concerning the action of $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{O}(V_+)$ on the (pluri)harmonic polynomials on V_+^n.

Let λ be a dominant weight for G expressed as in §5.11. Assume that λ has at most n nonzero entries. By supressing the last $m_0 - n$ zeroes the dominant weight λ for G gives rise to a dominant weight $\lambda_1 \geq \ldots \geq \lambda_n$ of $\mathrm{U}(n)$ (also to be denoted λ) and as such a finite dimensional irreducible representation $S_\lambda(\mathbb{C}^n)$ of $\mathrm{U}(n)$ and thus of K'. Here $S_\lambda(\mathbb{C}^n)$ denotes the Schur functor (see [24]); it occurs as an irreducible subrepresentation in $(\mathbb{C}^n)^{\otimes \ell}$ where $\ell = \lambda_1 + \ldots + \lambda_n$. We denote by ι_λ the inclusion $S_\lambda(\mathbb{C}^n) \to (\mathbb{C}^n)^{\otimes \ell}$.

Following [24, p. 296], we may define the harmonic Schur functor $S_{[\lambda]}(V)$ as the image of the classical Schur functor $S_\lambda(V)$ of V under the G-equivariant projection of $V^{\otimes \ell}$ onto the harmonic tensors. We denote by $\pi_{[\lambda]}$ the G-equivariant projection

$V^{\otimes \ell} \to S_{[\lambda]}(V)$. The representation $S_{[\lambda]}(V)$ is irreducible with highest weight λ. Note that all $S_{[\lambda]}(V)$ we will encounter are self-dual. This is obvious if m is odd as all representations of $\mathrm{SO}(m)$ are then self-dual, when m is even this follows from the fact that λ has $n < \mathrm{rank}(G)$ nonzero entries. In what follows, if V' is a representation of K' and k is an integer then $V'[k/2]$ will denote the representation of K' which is the tensor product of V' by the 1-dimensional representation $\det^{\frac{k}{2}}$.

8.1. **The relation between relative Lie algebra cochains and G-invariant vector-valued differential forms on $D = G/K$.** In §5 of [25] Funke and Millson construct a relative Lie algebra cocycle

$$\varphi_{nq,[\lambda]} \in \mathrm{Hom}_{K \times K'}(S_\lambda(\mathbb{C}^n)[m/2] \otimes \wedge^{nq}\mathfrak{p}, \mathcal{S}(V^n) \otimes S_{[\lambda]}(V)) \tag{8.1.1}$$

which takes values in the polynomial Fock space, more precisely in $\mathcal{P}_+^{(nq+\ell)} \otimes S_{[\lambda]}(V)$. However the same symbol $\varphi_{nq,[\lambda]}$ is used in Theorem 5.7 of Funke-Millson [25] where $\varphi_{nq,[\lambda]}$ is said to be a closed differential form on D. We will now restate Theorem 5.7 of Funke-Millson [25] more carefully. Then, in Lemma 8.3 we will explain the relation between cocycles and closed forms.

8.2. **Proposition.** *The cocycle $\varphi_{nq,[\lambda]}$ corresponds under the bijection of Lemma 8.3 to a $G \times G'$-invariant closed nq-form $\Phi_{nq,[\lambda]}$ on D with values in the tensor product of $\mathcal{S}(V^n) \otimes S_{[\lambda]}(V))$ (in the orthogonal variable) with the space of sections of the Mp_n-homogeneous vector bundle over the Siegel space $\mathbb{H}_n$ with fiber $(S_\lambda(\mathbb{C}^n)[m/2])^*$ (in the symplectic variable).*

Our goal in this subsection is to justify this abuse of terminology. In Lemma 8.3 below we will construct a natural isomorphism between the space of V-valued relative Lie algebra cochains φ and the space of invariant V-valued differential forms Φ on D for any representation $\rho : G \to \mathrm{Aut}(V)$ of a semisimple Lie group G. After this subsection we will no longer distinguish between φ and Φ.

Let $\Omega^\bullet(D, V)^G$ be the complex of invariant V-valued differential forms on D equipped with the usual exterior differential and $C^\bullet(\mathfrak{g}, K; V)$ be the complex of relative Lie algebra cochains. Let $\pi : G \to G/K$ be the quotient map. Recall that the tangent vectors to the right K-orbits in G are called vertical vectors — they are the kernel of the differential of π. Let $\Phi \in \Omega^N(D, V)^G$. Then $\pi^*\Phi$ is a left-G-invariant V-valued differential N- form on G which satisfies

(1) $\pi^*\Phi$ is basic — it annihilates vertical vectors
(2) $\pi^*\Phi$ is right K-invariant

$$R^*_{k^{-1}}\pi^*\Phi = \pi^*\Phi.$$

Since $\pi^*\Phi$ is left G-invariant it is determined by its value at the identity. We put φ equal to this value, $\pi^*\Phi|_e$. We then have:

8.3. **Lemma.** *The restriction of φ to $\wedge^N\mathfrak{p}$ is a relative Lie algebra N-cochain, that is $\varphi \in \mathrm{Hom}_K(\wedge^N\mathfrak{p}, V)$ where K acts on $\wedge^N\mathfrak{p}$ by the N-exterior power of the adjoint representation. Furthermore the map $\Phi \to (\pi^*\Phi|_e)|\wedge^N\mathfrak{p}$ is a bijection from G-invariant V-valued forms to V-valued relative Lie algebra cochains.*

Proof. Specializing the G to K in the left G-invariance property of $\pi^*\Phi$ we have $L^*_k(\pi^*\Phi) = \rho(k)\pi^*\Phi$ and hence evaluating $\pi^*\Phi$ at the identity we have

$$L^*_k(\varphi) = \rho(k)\varphi.$$

Combining this with the right K-invariance, Property (2) above we have

$$L_k^* R_{k^{-1}}^* \varphi = \rho(k)\varphi \text{ and hence } \mathrm{Ad}(k)^*\varphi = \rho(k)\varphi.$$

We have proved the required K-invariance property of φ.

But Property (1) above implies that φ descends to an element of $\mathrm{Hom}_K(\wedge^N(\mathfrak{g}/\mathfrak{k}), V)$ hence is determined by its restriction to to $\wedge^N \mathfrak{p}$.

The inverse map involves extending a relative Lie algebra cochain to a left G-invariant V-valued basic, right K-invariant form on G. Such a form automatically descends to D. With this the lemma is proved. □

We have now explained the isomorphism of graded vector spaces (but not the fact that the above map is a map of complexes which is harder, see [13], Chapter II, §1.1) in the following result which is fundamental for this paper and the previous work of Kudla-Millson and Funke-Millson.

8.4. **Theorem.** *The map $\Phi \to \pi^*\Phi|_e$ induces an isomorphism of complexes from $\Omega^\bullet(D, V)^G$ to $C^\bullet(\mathfrak{g}, K; V)$.*

Remark. Let $\mathbb{H}_n$ be the Siegel space of genus n so $\mathbb{H}_n = \mathrm{Sp}_n/\mathrm{U}_n$. In the applications that follow one refine the previous argument to go from cochains as above taking values in $\mathcal{S}(V^n) \otimes S_{[\lambda]}(V)$ now considered as a representation space for MU_n (with MU_n acting trivially on the second tensor factor) further tensored with the representation space $S_\lambda(V)[m/2]$ for MU_n to forms on $D \times \mathbb{H}_n$ which are sections of a homogeneous vector bundle on $\mathbb{H}_n$. Note that we now consider $\mathcal{P}_+$ as a model for $K \times K'$ and we have transferred the half-determinant twist to the second factor $S_\lambda(V)[m/2]$ — see below. We now give more details about the dependence on the symplectic group.

We now explain the transformation law for $\varphi_{nq,[\lambda]}$ under the action of MU_n, especially the half-determinant twist $\det(k')^{\frac{p+q}{2}}$. Equation (8.1.1) is equivalent (as far as the action of MU_n goes) to the equation

$$\omega(k')\varphi_{nq,[\lambda]} = \det(k')^{\frac{p+q}{2}} \varphi_{nq,[\lambda]} \rho_\lambda(k'), \quad k' \in \mathrm{MU}_n. \tag{8.4.1}$$

We now explain this transformation law.

First by Theorem 7.10 the action of U_n on $\mathcal{P}_+$ is the usual action of U_n twisted by $\det^{\frac{p-q}{2}}$. Hence it remains to prove that under the usual action of U_n we have

$$\omega(k')\varphi_{nq,[\lambda]} = \det(k')^q \varphi_{nq,[\lambda]} \rho_\lambda(k'), \quad k' \in \mathrm{U}_n. \tag{8.4.2}$$

Here ρ_λ is the representation of U_n with highest weight λ.

Now we will see in §8.7 that we have a product decomposition

$$\varphi_{nq,[\lambda]} = \varphi_{nq,0} \cdot \varphi_{0,[\lambda]}.$$

The first factor on the right in the previous equation is the cocycle of Kudla-Millson with trivial coefficients — it is easily checked that it transforms under U_n by $\det^q$ and it is clear that the second factor transforms by $\rho_\lambda(k')$.

The finite dimensional representation $S_\lambda(\mathbb{C}^n)[m/2]$ gives rise to a $\mathrm{Mp}_{2n}(\mathbb{R})$-equivariant hermitian vector bundle with fiber $S_\lambda(\mathbb{C}^n)$ on the Siegel (symmetric) space $\mathbb{H}_n = \mathrm{Mp}_{2n}(\mathbb{R})/K'$. We may therefore extend the relative Lie-algebra cocycle $\varphi_{nq,[\lambda]}$ to a closed G-invariant differential nq-form $\Phi_{nq,[\lambda]}$ on D with values in $\mathcal{S}(V^n) \otimes S_{[\lambda]}(V)$ (in the orthogonal variable) which is futher tensored with the space of sections of the homogeneous vector bundle over the Siegel upper-half space corresponding to the representation of K' given by $S_\lambda(\mathbb{C}^n)[m/2]$.

Remark. Intuitively, $\Phi_{nq,[\lambda]}$ is the tensor product of a section of the above homogeneous vector bundle on the Siegel space with the $\mathcal{S}(V^n)\otimes S_{[\lambda]}(V)$)-valued differential form on D which was previously denoted Φ. Unfortunately this is not quite correct, the tensor product must be replaced by the completed tensor product.

The above vector-valued differential forms are the generalization of the "scalar-valued" forms (actually oscillator representation valued differential forms) considered by Kudla and Millson [46, 47, 48] to the coefficient case. We now digress to explain how to construct the forms with trivial coefficients as restriction of forms associated to the unitary group $\mathrm{U}(p,q)$.

8.5. **Some special cocycles.** The natural embedding $\mathrm{O}(p,q)\subset \mathrm{U}(p,q)$ yields a totally real embedding of G/K into the Hermitian symmetric space U/L where $U=\mathrm{U}(p,q)$ and $L=\mathrm{U}(p)\times\mathrm{U}(q)$. The tangent space $\mathfrak{p}$ of G/K identifies with the holomorphic tangent space $\mathfrak{p}_U^{1,0}$ of U/L.

In [8, Section 5] we have introduced special $(\mathfrak{u}(p,q),L)$-cocycles $\psi_{aq,bq}$; here we will denote by ψ_{nq} the cocycle $\psi_{nq,0}$. In the case $n=1$, we have:

$$\psi_q=\sum_{\underline{\alpha}} z_{\underline{\alpha}}\otimes\omega_{\underline{\alpha}}\in \mathrm{Hom}_L(\wedge^{q,0}\mathfrak{p}_U,\mathcal{P}_+^{(q)}).$$

One important feature of the cocycle is that, interpreted as a differential form on U/L, the form ψ_q is closed, holomorphic and square integrable (hence harmonic) of degree q. For general n we have $\psi_{nq}=\psi_q\wedge\cdots\wedge\psi_q$.

In the Fock model the "scalar-valued" Schwartz form (Kudla-Millson form) $\varphi_{nq,0}$ is — up to a constant — the restriction of the holomorphic form ψ_{nq}. In the case $n=1$, it is given by the formula:

$$\varphi_{q,0}=\sum_{\underline{\alpha}} z_{\underline{\alpha}}\otimes\omega_{\underline{\alpha}}\in \mathrm{Hom}_K(\wedge^{q}\mathfrak{p},\mathcal{P}_+^{(q)}).$$

More generally $\varphi_{nq,0}$ is obtained as the (external) wedge-product:

$$\varphi_{nq,0}=\underbrace{\varphi_{q,0}\wedge\ldots\wedge\varphi_{q,0}}_{n\text{ times}}.$$

We will often write φ_{nq} instead of $\varphi_{nq,0}$.

Note that, interpreted as a differential form on G/K, the form $\varphi_{nq,0}$ is closed but not harmonic.

The reader will verify that there are analogous forms $\psi_{nq,\lambda}$ and restriction formulas for the general $\varphi_{nq,\lambda}$ with values in the *reducible* representation $S_\lambda(V)$ but there is no such form for the harmonic-valued projection $\varphi_{nq,[\lambda]}=\pi_{[\lambda]}\circ\varphi_{nq,\lambda}$ with values in the *irreducible* representation $S_{[\lambda]}(V)$.

8.6. **Lemma.** *The form $\varphi_{nq,[\lambda]}$ is a section of the bundle F of the Introduction.*

Proof. By construction the form ψ_{nq} factors through the subspace $[\wedge^{nq,0}\mathfrak{p}_U]^{\mathrm{SU}(q)}$ of $\mathrm{SU}(q)$-invariants in $\wedge^{nq,0}\mathfrak{p}_U$. In particular the form $\varphi_{nq,[\lambda]}$ defines a element in

$$\mathrm{Hom}_{K'}(S_\lambda(\mathbb{C}^n)[m/2],\mathrm{Hom}_K([\wedge^{nq}\mathfrak{p}]^{\mathrm{SL}(q)},\mathcal{P}\otimes S_{[\lambda]}(V))).$$

□

As announced in the Introduction it will therefore follow from Proposition 10.8 that the subspace of the cohomology $H^\bullet_{\text{cusp}}(X_K, S_{[\lambda]}(V))$ generated by special cycles is in fact contained in $H^\bullet_{\text{cusp}}(X_K, S_{[\lambda]}(V))^{\text{SC}}$. (This of course explains the notation.)

8.7. **Cocycles with coefficients.** In order to go from the forms with trivial coefficients to those with nontrivial coefficients one multiplies $\varphi_{nq,0}$ by a remarkable K'-invariant element $\varphi_{0,[\lambda]}$ of degree zero in the $(\mathfrak{g}, K)$-complex with values in $S_\lambda(\mathbb{C}^n)^* \otimes \mathcal{P}(V^n) \otimes S_{[\lambda]}(V)$. These elements (as λ varies) are projections of the basic element $\varphi_{0,\ell}$ whose properties will be critical to us. Thus (in the Fock model) we have

$$\varphi_{nq,[\lambda]} = \varphi_{nq,0} \cdot \varphi_{0,[\lambda]}.$$

Remark. We will think of $\varphi_{nq,0}$ as taking values in the polynomial half-forms on $V_+ \otimes W'$, hence in its transformation law under MU_n there is an additional half determinant twist by $\det^{\frac{p-q}{2}}$. On the other hand we will think of $\varphi_{0,[\lambda]}$ as taking values in the polynomial functions on $V_+ \otimes W'$ so there will be no twist by $\det^{\frac{p-q}{2}}$.

8.7.1. *The K'-equivariant family of zero $(\mathfrak{so}(p,q), K)$-cochains $\varphi_{0,\ell}$.* In [25] Funke and Millson define (here $\mathbb{C}^n$ is the standard representation of $\text{U}(n)$ and $T^\ell(\mathbb{C}^n)$ denotes the space of rank ℓ tensors)

$$\varphi_{0,\ell} \in \text{Hom}_{K' \times S_\ell}((T^\ell(\mathbb{C}^n)), \text{Hom}_K(\wedge^0 \mathfrak{p}, \mathcal{P}_+^{(\ell)} \otimes T^\ell(V_+)))$$

by

$$\varphi_{0,\ell}(e_I) = \sum_{\underline{\beta}} z_{\beta_1, i_1} \cdots z_{\beta_\ell, i_\ell} \otimes v_{\underline{\beta}} \tag{8.7.1}$$

(up to a constant factor) where $e_I = e_{i_1} \otimes \ldots \otimes e_{i_\ell}$ and $\underline{I} = (i_1, \cdots, i_\ell)$. Here K' acts on $T^\ell(\mathbb{C}^n)$ and $\mathcal{P}_+^{(\ell)}$ and the symmetric group S_ℓ acts on $T^\ell(\mathbb{C}^n)$ and $T^\ell(V_+)$. They also set:

$$\varphi_{nq,\ell} = \varphi_{nq,0} \cdot \varphi_{0,\ell} \in \text{Hom}_{K' \times S_\ell}(T^\ell(\mathbb{C}^n)[m/2], \text{Hom}_K(\wedge^{nq} \mathfrak{p}, \mathcal{P}_+^{(nq+\ell)} \otimes T^\ell(V)),$$

and

$$\varphi_{nq,[\lambda]} = (1 \otimes \pi_{[\lambda]}) \circ \varphi_{nq,\ell} \circ \iota_\lambda \in \text{Hom}_{K'}(S_\lambda(\mathbb{C}^n)[m/2], \text{Hom}_K(\wedge^{nq} \mathfrak{p}, \mathcal{P}_+^{(nq+\ell)} \otimes S_{[\lambda]}(V))).$$

In what follows it will be important to note that $\text{GL}(n, \mathbb{C})$ acts on $\mathcal{P}_+^{(nq+\ell)} = S^{nq+\ell}(\mathbb{C}^n \otimes V_+)$ by (the action induced by) the standard action of $\text{GL}(n, \mathbb{C})$. Also the map $\varphi_{0,\ell}$ has image contained in $\text{Hom}_K(\wedge^0 \mathfrak{p}, \mathcal{P}_+^\ell \otimes T^\ell(V_+))$. We will now rewrite $\varphi_{0,\ell}$ to deduce some remarkable properties that it posseses.

8.7.2. *Three properties of $\varphi_{0,\ell}$.* Note first that a map $\varphi : U \to W \otimes V$ corresponds to a map $\varphi^* : V^* \otimes U \to W$. Hence, using the isomorphism $V_+ \cong V_+^*$ we obtain

$$\varphi_{0,\ell}^* : T^\ell(V_+) \otimes T^\ell(\mathbb{C}^n) \to \mathcal{P}_+^{(\ell)}$$

by the formula

$$\varphi_{0,\ell}^*(v_{\underline{\beta}} \otimes e_I) = (\varphi_{0,\ell}(e_I), v_{\underline{\beta}}).$$

Equation (8.7.1) then becomes

$$\varphi_{0,\ell}^*(v_{\underline{\beta}} \otimes e_I) = z_{\beta_1, i_1} \cdots z_{\beta_\ell, i_\ell}. \tag{8.7.2}$$

Rearranging the tensor factors $\mathbb{C}^n$ and V_+ we may consider the map $\varphi_{0,\ell}^*$ as a map

$$\varphi_{0,\ell}^* : T^\ell(V_+ \otimes \mathbb{C}^n) \to \mathrm{Sym}^\ell(V_+ \otimes \mathbb{C}^n) \cong \mathcal{P}_+^\ell.$$

This rearrangement leads immediately to two important properties of $\varphi_{0,\ell}^*$.

First, we have the following *factorization property* of $\varphi_{0,\ell}^*$ that will play a critical role in the proof of Proposition 8.17. Note that as a special case of Equation (8.7.2), the map $\varphi_{0,1}^* : V_+ \otimes \mathbb{C}^n \to \mathcal{P}_+$ satifies the equation

$$\varphi_{0,1}^*(v_\alpha \otimes e_j) = z_{\alpha,j}.$$

We see then that $\varphi_{0,\ell}^* : T^\ell(V_+ \otimes \mathbb{C}^n) \to \mathcal{P}_+^\ell$ may be factored as follows. Given a decomposable ℓ-tensor $\mathbf{x} \otimes \mathbf{z} = (x_1 \otimes \cdots \otimes x_\ell) \otimes (z_1 \otimes \cdots \otimes z_\ell) \in T^\ell(V_+) \otimes T^\ell(\mathbb{C}^n)$, rearrange the tensor factors to obtain $(x_1 \otimes z_1) \otimes \cdots \otimes (x_\ell \otimes z_\ell) \in T^\ell(V_+ \otimes \mathbb{C}^n)$. Then we have

$$\varphi_{0,\ell}^*(\mathbf{x} \otimes \mathbf{z}) = \varphi_{0,1}^*(x_1 \otimes z_1)\varphi_{0,1}^*(x_2 \otimes z_2) \cdots \varphi_{0,1}^*(x_\ell \otimes z_\ell).$$

From this the following *multiplicative property* is clear

8.8. **Lemma.** *Let* $\mathbf{z}_1 \in T^a(\mathbb{C}^n), \mathbf{z}_2 \in T^b(\mathbb{C}^n), \mathbf{x}_1 \in T^a(V_+), \mathbf{x}_2 \in T^b(V_+)$ *with* $a + b = \ell$. *Then*

$$\varphi_{0,\ell}^*((\mathbf{x}_1 \otimes \mathbf{x}_2) \otimes (\mathbf{z}_1 \otimes \mathbf{z}_2)) = \varphi_{0,a}^*(\mathbf{x}_1 \otimes \mathbf{z}_1) \cdot \varphi_{0,b}^*(\mathbf{x}_2 \otimes \mathbf{z}_2).$$

The second property we will need is that $\varphi_{0,\ell}^*$ is (up to identifications) simply the projection from the ℓ-th graded summand of the tensor algebra on $V_+ \otimes \mathbb{C}^n$ to the corresponding summand of the symmetric algebra. Hence, the map $\varphi_{0,\ell}^*$ descends to give a map

$$\varphi_{0,\ell}^* : \mathrm{Sym}^\ell(V_+ \otimes \mathbb{C}^n) \to \mathrm{Sym}^\ell(V_+ \otimes \mathbb{C}^n) \tag{8.8.1}$$

which is clearly the identity map. The following lemma is then clear.

8.9. **Lemma.** $\varphi_{0,\ell}^*$ *carries harmonic tensors to harmonic tensors.*

From now on we will abuse notation and abbreviate $\varphi_{0,\ell}^*$ to $\varphi_{0,\ell}$ for the rest of this subsection.

8.10. **The Vogan-Zuckerman K-types associated to the special Schwartz forms $\varphi_{nq,[\lambda]}$.** For the rest of this section the symbols V, V_+ and V_- will mean the complexifications of the the real vector spaces formerly denoted by these symbols. Let $\mathfrak{q}$ be a θ-stable parabolic algebra of $\mathfrak{g}$ with associated Levi subgroup $\mathrm{SO}(p-2n)$ times a compact group.

We recall that the Vogan-Zuckerman K-type $\mu(\mathfrak{q})$ is the lowest K-type of $A_\mathfrak{q}$. It may be realized by the K-invariant subspace $V(\mathfrak{q}) \subset \wedge^R(\mathfrak{p})$ generated by the highest weight vector $e(\mathfrak{q}) \in \wedge^R(\mathfrak{u} \cap \mathfrak{p}) \subset \wedge^R(\mathfrak{p})$. The K-type $\mu(\mathfrak{q}, \lambda)$ is the lowest K-type of $A_\mathfrak{q}(\lambda)$. It may be realized by the K-invariant subspace $V(\mathfrak{q}, \lambda) \subset \wedge^R(\mathfrak{p}) \otimes S_{[\lambda]}(V)^*$ which is the Cartan product of $V(\mathfrak{q})$ and $S_{[\lambda]}(V)^*$.

Our goal in this section is to prove the following

8.11. **Proposition.**

$$\varphi_{nq,[\lambda]}(S_\lambda(\mathbb{C}^n)[m/2] \otimes V(\mathfrak{q}, \lambda)) \subset \mathcal{H}_+.$$

Remark. We remind the reader that we modified $\varphi_{0,\ell}$ to $\varphi^*_{0,\ell}$ in subsection 8.7.2. This results in a modification of $\varphi_{0,[\lambda]}$ and hence of $\varphi_{nq,[\lambda]}$. Thus we should have written $\varphi^*_{nq,[\lambda]}$ instead of $\varphi_{nq,[\lambda]}$ in the above theorem and in all that follows. Since this amounts to rearranging a tensor product we will continue to make this abuse of notation in what follows.

The key to proving the Proposition will be to explicitly compute $\mu(\mathfrak{q}), V(\mathfrak{q})$ and $e(\mathfrak{q})$ and the harmonic Schur functors $S_{[\lambda]}$ for the case in hand in terms of the multilinear algebra of V_+ and the form $(\ ,\ )$. We will define a totally isotropic subspace $E_n \subset V_+ \otimes \mathbb{C}$ of dimension n and $\mathfrak{q}$ will be the stabilizer of a fixed flag in E_n. Anticipating this we change the notation from $V(\mathfrak{q}, \lambda)$ to $V(n, \lambda)$. Also we take $R = nq$ because for all the parabolics we construct below we will have

$$\dim\ (\mathfrak{u} \cap \mathfrak{p}) = nq.$$

For special orthogonal groups associated to an even dimensional vector space parabolic subalgebras are not in one-to-one correspondence with isotropic flags but rather with isotropic oriflammes. This will not be a problem here. The reason for this comes from the following considerations. First all parabolics we consider here will come from flags of isotropic subspaces in E_n. Second, there is no difference betweeen oriflammes and flags if all the isotropic subspaces considered are in dimension strictly less than the middle dimension minus 1. Finally, note that in the even case we have $n < m_0 - 1$. See [28, Chapter 11, p. 158] for details.

Furthermore, again in the even case, since our highest weight λ of $S_{[\lambda]}(V)^*$ has at most n nonzero entries and $n < m_0 - 1 < m_0$ the irreducible representation of $\mathrm{SO}(V)$ with highest weight λ will extend to an irreducible representation of $\mathrm{O}(V)$. This extension will be unique up to tensoring with the determinant representation. In other words for us there will be no difference (up to tensoring by the determinant representation) between the representation theory of $\mathrm{SO}(V)$ and $\mathrm{O}(V)$.

8.11.1. *The proof of Proposition 8.11 for the case of trivial coefficients.* We will first prove Proposition 8.11 for the case of trivial coefficients.

If we are interested only in obtaining a representation $A_\mathfrak{q}$ which will give cohomology *with trivial coefficients* in degree nq we may take $\mathfrak{q}$ to be a maximal parabolic, hence to be the stabilizer of a totally isotropic subspace $E' \subset V$. We remind the reader that throughout this section V, V_+ and V_- are the complexifications of the corresponding real subspaces which we have denoted V, V_+ and V_- in the rest of the paper. In order for $\mathfrak{q}$ to be θ-stable it is necessary and sufficient that E' splits compatibly with the splitting $V = V_+ \oplus V_-$. The simplest way to arrange this is to choose $E' \subset V_+$. Hence, we choose E' to be the n-dimensional totally isotropic subspace $E' = E'_n \subset V_+$ given by

$$E'_n = \mathrm{span}\{u'_1, u'_2, \cdots, u'_n\}.$$

Now let E''_n be the dual n dimensional subspace of V_+ given by

$$E''_n = \mathrm{span}\{u''_1, u''_2, \cdots, u''_n\}.$$

Then E''_n is a totally isotropic subspace of V_+ of dimension n with $E'_n \cap E''_n = 0$ such that the restriction of $(,)$ to $E'_n + E''_n$ is nondegenerate (so E'_n and E''_n are dually-paired by $(,)$). Let $U = (E'_n + E''_n)^\perp$. We will abbreviate E'_n and E''_n to E' and E'' henceforth. We obtain

$$V = E' \oplus U \oplus E'' \oplus V_-. \tag{8.11.1}$$

In what follows we will identify the Lie algebra $\mathfrak{so}(V)$ with $\wedge^2(V)$ by $\rho : \wedge^2(V) \to \mathfrak{so}(V)$ given by

$$\rho(u \wedge v)(w) = (u, w)v - (v, w)u.$$

The reader will verify that under this identification the Cartan splitting of $\mathfrak{so}(V)$ corresponds to

$$\mathfrak{so}(V) = \mathfrak{k} \oplus \mathfrak{p} = (\wedge^2(V_+) \oplus \wedge^2(V_-)) \oplus (V_+ \otimes V_-).$$

Equation (8.11.1) then induces the following splitting of $\mathfrak{so}(V) \cong \wedge^2(V)$:

$$\wedge^2(V) = (E' \otimes E'') \oplus (E' \otimes U) \oplus (E' \otimes V_-) \oplus (E'' \otimes U) \oplus (E'' \otimes V_-) \\ \oplus (U \otimes V_-) \oplus \wedge^2(E') \oplus \wedge^2(U) \oplus \wedge^2(E'') \oplus \wedge^2(V_-).$$

The reader will then verify the following lemma concerning the Levi splitting of $\mathfrak{q}$ and its relation with the above Cartan splitting of $\mathfrak{so}(V)$. Recall that $\mathfrak{q}$ is the stabilizer of E'. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the Levi decomposition.

8.12. **Lemma.** (1) $\mathfrak{q} = [(E' \otimes E'') \oplus (U \otimes V_-) \oplus \wedge^2(U)] \oplus [(E' \otimes U) \oplus (E' \otimes V_-)) \oplus \wedge^2(E')]$
(2) $\mathfrak{l} = (E' \otimes E'') \oplus (U \otimes V_-) \oplus \wedge^2(U)$
(3) $\mathfrak{u} = (E' \otimes U) \oplus (E' \otimes V_-)) \oplus \wedge^2(E')$
Hence, we have

$$\mathfrak{u} \cap \mathfrak{p} = (E' \otimes V_-) = \operatorname{span}(\{u'_j \wedge v_{p+k} : 1 \le j \le n, 1 \le k \le q\})$$

whence

$$e(\mathfrak{q}) = [(u'_1 \wedge v_{p+1}) \wedge \cdots \wedge (u'_1 \wedge v_{p+q})] \wedge \cdots \wedge [(u'_n \wedge v_{p+1}) \wedge \cdots \wedge (u'_n \wedge v_{p+q})].$$

Next we describe the Vogan-Zuckerman subspace $V(n) \subset \wedge^{nq}\mathfrak{p} \cong \wedge^{nq}(V_+ \otimes V_-)$ (underlying the realization of the Vogan-Zuckerman special K-type in $\wedge^{nq}\mathfrak{p}$) using the standard formula for the decomposition of the exterior power of a tensor product, see equation (1.12.1) or Equation (19), or the formula on page 80 of [24]. Here $n \times q$ denotes the partition of nq given by q repeated n times.

8.13. **Lemma.** *The subspace $V(n)$ of $\wedge^{nq}(V_+ \otimes V_-)$ is given by*

$$V(n) = S_{[n\times q]}(V_+) \boxtimes S_{[q\times n]}(V_-) = S_{[n\times q]}(V_+) \boxtimes \mathbb{C}.$$

8.14. We denote by $V(n, \lambda)$ the *Cartan product* of $V(n) \otimes S_{[\lambda]}(V)^*$ i.e. the highest K-type of the tensor product $V(n) \otimes S_{[\lambda]}(V)^* \cong V(n) \otimes S_{[\lambda]}(V)$.

We can now prove Proposition 8.11 for the case of trivial coefficients. We refer the reader to subsection 7.19 for the definition of the p_0 by n matrix $W''(\mathbf{x})$. We recall that $\Delta_n(\mathbf{x})$ is the determinant of the leading principal n by n minor of $W''(\mathbf{x})$.

8.15. **Proposition.** *We have*

$$\varphi_{nq}(e(\mathfrak{q}))(\mathbf{x}) = \Delta_n(\mathbf{x})^q \in \mathcal{H}''(V_+^n).$$

and consequently

$$\varphi_{nq}(V(n)) \subset \mathcal{H}_+.$$

Proof. It follows from [8, Lemma 3.16] that seen as an element of $\wedge^{nq,0}\mathfrak{p}_U$ the vector $e(\mathfrak{q})$ is a Vogan-Zuckerman vector for the theta stable parabolic $\mathfrak{q}_{n,0}$ of $\mathfrak{u}(p, q)$. The cocycle φ_{nq} being the restriction of ψ_{nq} the Proposition follows from [8, Proposition 5.24]. ∎

8.16. **A derivation of the formulas for the simultaneous highest weight harmonic polynomials in Case (1).** In this section we will prove the general case of Proposition 8.11, in other words, we will prove that $\varphi_{nq,[\lambda]}(V(n,\lambda))$ takes values in $\mathcal{H}_+$. We will do this by giving an *explicit* formula for $\varphi_{0,[\lambda]}(e_\lambda \otimes v^*_{[\lambda]})$, where $v^*_{[\lambda]}$ is the highest weight vector of $S_{[\lambda]}(V)^*$ and e_λ is a highest weight vector of $S_\lambda(\mathbb{C}^n)$, which will obviously be in $\mathcal{H}''(V_+^n)$. Our computation will give a new derivation of the formulas of Kashiwara and Vergne for the simultaneous highest weight vectors in $\mathcal{H}_+$ for their Case (1), Propositions (6.6) and (6.11) of [41]. This derivation will be an immediate consequence of the multiplicative property, Lemma 8.8, of $\varphi_{0,\ell}$ and standard facts in representation theory.

First we need to make an observation. Note that $V(n,\lambda)$ is the lowest K-type of $A_{\mathfrak{q}}(\lambda)$, with $\mathfrak{u}\cap\mathfrak{p}$ as above. But if not all the entries of λ are equal then we can no longer take $\mathfrak{q}$ to be a *maximal* parabolic and we will have to replace the totally isotropic subspace E_n by a flag in E_n. For example, if all the entries of λ are different then we must take a full flag in E_n. However for all parabolics $\mathfrak{q}$ obtained we obtain the same formula for $\mathfrak{u}\cap\mathfrak{p}$ and the formula above for $e(\mathfrak{q})$ remains valid. In fact (because induction in stages is satisfied for derived functor induction) one can always take $\mathfrak{q}$ to be the stabilizer of a full flag in E_n and hence will have the property that the Levi subgroup L_0 of Q intersected with G will be $\mathrm{U}(1)^n \times \mathrm{SO}(p-2n,q)$. This we will do for the rest of the paper.

Next, we recall that since $V(n,\lambda)$ is embedded in $\wedge^{nq}(\mathfrak{p})\otimes S_{[\lambda]}(V)^*$ as the Cartan product, the highest weight vector $e(\mathfrak{q},\lambda)$ of $V(n,\lambda)$ is given by

$$e(\mathfrak{q},\lambda) = e(\mathfrak{q}) \otimes v^*_{[\lambda]}. \tag{8.16.1}$$

Also since $\varphi_{nq,[\lambda]} = \varphi_{nq,0}\cdot\varphi_{0,[\lambda]}$ we have

$$\varphi_{nq,[\lambda]}(e_\lambda \otimes e(\mathfrak{q},\lambda)) = \varphi_{nq,0}(e(\mathfrak{q}))\cdot\varphi_{0,[\lambda]}(e_\lambda\otimes v^*_{[\lambda]}). \tag{8.16.2}$$

Here and in the formula just above we think of $\varphi_{nq,0}$ as an element of $\mathrm{Hom}_{K\times K'}(\mathbb{C}[m/2]\otimes \wedge^{nq}(\mathfrak{p}),\mathcal{P})$ and $\varphi_{0,[\lambda]}$ as an element of $\mathrm{Hom}_{K\times K'}(S_\lambda(\mathbb{C}^n)\otimes S_{[\lambda]}(V)^*,\mathcal{P})$. Since $\varphi_{nq,0}$ takes values in the polynomials (actually half-forms) and $\varphi_{0,[\lambda]}$ tkes values in the polynomials $\mathcal{P}$ we can multiply those values. The resulting product is what is used in the equation (8.16.2) and induces the product in the previous equation. Since $\mathcal{H}''(V_+^n)$ is closed under multiplication if we can prove that both factors of the right-hand side of equation (8.16.2) are contained in the ring $\mathcal{H}''(V_+^n)$ we can conclude that

$$\varphi_{nq,[\lambda]}(e_\lambda\otimes e(\mathfrak{q},\lambda)) \in \mathcal{H}''(V_+^n)\subset\mathcal{H}_+.$$

Accordingly, since $V(n,\lambda)$ is an irreducible $K\times K'$-module, the action of $K\times K'$ preserves $\mathcal{H}_+$, and $\varphi_{nq,[\lambda]}$ is a $K\times K'$ homomorphism, Proposition 8.11 will follow.

In fact we now prove a stronger statement than the required $\varphi_{0,[\lambda]}(e_\lambda\otimes v^*_{[\lambda]}) \in \mathcal{H}''(V_+^n)$. Namely we give a new proof of the formulas of [41] in Case (1). This proof makes clear that their formula follows with very little computation. Namely we first realize the representation $V(\lambda)$ with highest weight λ of $\mathrm{O}(V_+)$ in a tensor product of symmetric powers of fundamental representations (exterior powers of the standard representation V_+) corresponding to representing the highest weight λ in terms of the fundamental weights $\varpi_j, 1\le j\le p_0$

$$\lambda = \sum_{i=1}^{p_0} a_i\varpi_i. \tag{8.16.3}$$

We then write down the standard realization of the highest weight vector in this tensor product, see equation (8.18.1) below. The point is that this realization is represented as a product, it is "factored". We then apply the K-homomorphism $\varphi_{0,\ell}$ to this vector using the *multiplicative property*, Lemma 8.8 and obtain the desired realization of it in $\mathcal{P}_{+}^{\ell}$ as a product of powers of leading principal minors of the matrix $W''(\mathbf{x})$. Thus the new feature of the proof is the existence and factorization property of the map $\varphi_{0,\ell}$. We now give the details.

The reader will verify that (after changing from the basis of the dual of the Cartan given by the fundamental weights to the standard basis) that the following formula is the same as those of [41], Proposition (6.6), Case (1) and Proposition (6.11), Case (1).

8.17. **Proposition.** *Write the highest weight λ in terms of the fundamental weights according to $\lambda = \sum_{i=1}^{n} a_i \varpi_i$. Then we have (up to a constant multiple)*

$$\varphi_{0,[\lambda]}(e_\lambda \otimes v^*_{[\lambda]})(\mathbf{x}) = \Delta_1(\mathbf{x})^{a_1}\Delta_2(\mathbf{x})^{a_2}\cdots\Delta_n(\mathbf{x})^{a_n}. \tag{8.17.1}$$

and consequently

$$\varphi_{0,[\lambda]}(S_\lambda(\mathbb{C}^n)\otimes S_{[\lambda]}(V_+)) \subset \mathcal{H}_+ \tag{8.17.2}$$

Combining equation (8.17.1) *with the equation of Proposition 8.15 we obtain*

$$\varphi_{nq,[\lambda]}(e_\lambda \otimes e(\mathfrak{q}) \otimes v^*_{[\lambda]})(\mathbf{x}) = \Delta_1(\mathbf{x})^{a_1}\Delta_2(\mathbf{x})^{a_2}\cdots\Delta_n(\mathbf{x})^{a_n+q}. \tag{8.17.3}$$

8.18. **Corollary.**

$$\varphi_{nq,[\lambda]}(e_\lambda \otimes e(\mathfrak{q}) \otimes v^*_{[\lambda]}) \in \mathcal{H}''(V_+^n) \subset \mathcal{H}_+.$$

Proof. First, as stated, we give the standard realization of the highest weight vectors in the tensor product $T^\ell(\mathbb{C}^n)\otimes T^\ell(V^*)$, namely we have the formula

$$\begin{aligned} e_\lambda \otimes v^*_{[\lambda]} = {} & [e_1^{\otimes a_1} \otimes (e_1\wedge e_2)^{\otimes a_2} \otimes \cdots \otimes (e_1 \wedge e_2 \wedge \cdots \wedge e_n)^{\otimes a_n}] \\ & \otimes [{u'_1}^{\otimes a_1} \otimes (u'_1 \wedge u'_2)^{\otimes a_2} \otimes \cdots \otimes (u'_1\wedge u'_2 \wedge \cdots \wedge u'_n)^{\otimes a_n}]. \end{aligned} \tag{8.18.1}$$

Indeed, note that the above tensor is annihilated by the nilradicals of both Borels. It is obviously annihilated by the vectors $u'_i \wedge u'_j$, $1 \le i,j \le p_0$. Since the rest of the nilradical of the Borel subalgebra $\mathfrak{b}$ for $\mathfrak{so}(V_+)$ is spanned by the root vectors $u'_j \wedge u''_i, 1 \le j < i \le p_0$, that map u'_i to u'_j with $j < i$ (and u'_1 to 0), the claim follows for $\mathfrak{so}(V_+)$. Similarly the Borel subalgebra for $\mathfrak{gl}(n,\mathbb{C})$ is spanned by the elements $E_{ij}, 1 \le j < i \le n$, that map e_i to e_j with $j < i$ (and e_1 to 0) and are zero on all basis vectors other than e_i. Note also that the u'_i's are orthogonal isotropic vectors hence the above tensor in the u'_i's is a harmonic tensor in $T^\ell(V_+)$. Lastly the above vector has weight λ by construction. Note that with the above realization of $e_\lambda \otimes v^*_{[\lambda]}$ in $T^\ell(\mathbb{C}^n \otimes V_+^*)$ we have

$$\varphi_{0,[\lambda]}(e_\lambda \otimes v^*_{[\lambda]}) = \varphi_{0,\ell}(e_\lambda \otimes v^*_{[\lambda]})$$

where on the right-hand side we consider $e_\lambda \otimes v^*_{[\lambda]} \in T^\ell(\mathbb{C}^n \otimes V_+^*)$. We now apply the factorization property of $\varphi_{0,\ell}$, see Lemma 8.8.

Indeed, factoring the right-hand side of equation (8.18.1) into n factors (not counting the powers a_i) we obtain

$$\varphi_{0,\ell}(e_\lambda \otimes v_{[\lambda]}^*) = \varphi_{0,1}(e_1 \otimes u_1')^{a_1} \varphi_{0,2}([e_1 \wedge e_2] \otimes [u_1' \wedge u_2'])^{a_2} \cdots \cdots \varphi_{0,n}([e_1 \wedge \cdots e_n] \otimes [u_1' \wedge \cdots \wedge u_n'])^{a_n}.$$

But then observe that

$$\varphi_{0,k}([e_1 \wedge e_2 \wedge \cdots \wedge e_k] \otimes [u_1' \wedge u_2' \wedge \cdots \wedge u_k']) = \Delta_k(\mathbf{x}).$$

The Proposition follows. □

8.19. **The derivation of the correspondence of representations on the harmonics.** In this subsection we will see how the map $\varphi_{0,\ell} : \mathrm{Sym}^\ell(\mathbb{C}^n \otimes V_+) \to \mathrm{Pol}^\ell(\mathrm{M}_{p\times n}(\mathbb{C}))$ induces the decomposition formula for the dual pair $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{O}(V_+)$ acting on $\mathcal{H}^\ell(\mathbb{C}^n \otimes V_+)$. In what follows let $P(\ell, n)$ be the set of ordered partitions of ℓ into less than or equal to n parts (counting repetitions). We will assume the known result that $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{O}(V_+)$ *acting on* $\mathcal{H}^\ell(\mathbb{C}^n \otimes V_+)$ *forms a dual pair*, see [35], and compute what the resulting correspondence is using $\varphi_{0,\ell}$.

8.20. **Proposition.** *Under the assumption* $n \leq [p/2]$ *the map* $\varphi_{0,\ell}$ *induces an isomorphism of* $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{O}(V_+)$*-modules*

$$\varphi_{0,\ell} : \sum_{\lambda \in P(\ell,n)} S_\lambda(\mathbb{C}^n) \boxtimes S_{[\lambda]}(V_+) \to \mathcal{H}^\ell(\mathbb{C}^n \otimes V_+).$$

As a consequence the correspondence between $\mathrm{GL}(n,\mathbb{C})$*-modules and* $\mathrm{O}(V_+)$ *modules induced by the action of the dual pair on the harmonics is* $S_\lambda(\mathbb{C}^n) \leftrightarrow S_{[\lambda]}(V_+)$ *(so loosely put "take the same partition").*

Proof. We first recall the decomposition of the ℓ-th symmetric power of a tensor product, see [24], page 80,

$$\mathrm{Sym}^\ell(\mathbb{C}^n \otimes V_+) = \bigoplus_{\lambda \in P(\ell,n)} S_\lambda(\mathbb{C}^n) \otimes S_\lambda(V_+).$$

Hence we obtain an isomorphism of $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(V_+)$-modules

$$\varphi_{0,\ell} : \bigoplus_{\lambda \in P(\ell,n)} S_\lambda(\mathbb{C}^n) \otimes S_\lambda(V_+) \to \mathrm{Sym}^\ell(\mathbb{C}^n \otimes V_+) \cong \mathrm{Pol}^\ell((\mathbb{C}^n)^* \otimes V_+^*).$$

But by equation (8.17.2) of Proposition 8.17 we have (under the assumption $n \leq [p/2]$),

$$\varphi_{0,\ell}(S_\lambda(\mathbb{C}^n) \otimes S_{[\lambda]}(V_+)) \subset \mathcal{H}^\ell(\mathbb{C}^n \otimes V_+).$$

The map is obviously an injection.

To prove the map is a surjection let $\lambda \in P(\ell, n)$. Then, from the assumption preceding the statement of the theorem, the $\mathrm{O}(V_+)$-isotypic subspace

$$\mathrm{Hom}_{\mathrm{GL}(n,\mathbb{C})}(S_\lambda(\mathbb{C}^n), \mathcal{H}^\ell(\mathbb{C}^n \otimes V_+))$$

is an irreducible representation for $\mathrm{O}(V_+)$. But we have just seen it contains the subspace $S_{[\lambda]}(V_+)$. Hence it coincides with $S_{[\lambda]}(V_+)$. From this the proposition follows. □

8.21. **The relation with the work of Kashiwara and Vergne.** The previous results are closely related to the work of Kashiwara and Vergne [41] studying the action of $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{O}(k)$ on the harmonic polynomials on $M(n,k)$. We first note that Propositions 8.15 and 8.17 do not follow from the results of [41] since we do not know a priori that the cocycles φ_{nq} resp. $\varphi_{nq,[\lambda]}$ take harmonic values on the highest weight vectors $e(\mathfrak{q})$ resp. $e(\mathfrak{q},\lambda)$ (and this and the results of Howe are the key tools underlying the proof of Theorem 8.23).

For the benefit of the reader in comparing the results of Section 8.16, Proposition 8.17 and Section 8.19, Proposition 8.20 with the corresponding results of [41] we provide a dictionary between the notations of our paper and theirs. We are studying the action of the dual pair $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{O}(V_+)$ on the harmonic polynomials $\mathcal{H}_+$. Thus our p corresponds to their k and their n coincides with our n. Their ℓ is the rank of $\mathrm{O}(V_+)$ which we have denoted p_0. Kashiwara and Vergne take for the Fock module the polynomials on $\mathrm{M}_{n\times p}(\mathbb{C}) \cong V_+^* \otimes \mathbb{C}^n$, that is the $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{O}(V_+)$-module $\mathrm{Sym}(V_+ \otimes (\mathbb{C}^n)^*)$ whereas we take the polynomials on $V_+^* \otimes (\mathbb{C}^n)^* \cong V_+ \otimes (\mathbb{C}^n)^* \cong \mathrm{M}_{p\times n}(\mathbb{C})$ that is $\mathrm{Sym}(V_+ \otimes \mathbb{C}^n)$ for our Fock model. In the two correspondences between $\mathrm{GL}(n,\mathbb{C})$ modules and $\mathrm{O}(V_+)$ modules the $*$ on the second factor causes their $\mathrm{GL}(n,\mathbb{C})$-modules to be the contragredients of ours.

There are two results in [41] that are reproved here using $\varphi_{0,\ell}$ (in "Case 1", $n = \mathrm{rank}(\mathrm{GL}(n,\mathbb{C}) \leq p_0 = \mathrm{rank}(\mathrm{O}(V_+))$. First, we give a new derivation of their formula for the simultaneous $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{O}(V_+)$-highest weight vectors in the space of harmonic polynomials $\mathcal{H}_+$ in Proposition 8.17. Second, we give a new proof of the correspondence between $\mathrm{GL}(n,\mathbb{C})$ modules and $\mathrm{O}(V_+)$ modules in Proposition 8.20. Both of our proofs here are based on the properties of the element $\varphi_{0,\ell}$. As noted above, the correspondence between irreducible representations of $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{O}(V_+)$ is different from that of Kashiwara and Vergne (the representations of $\mathrm{GL}(n,\mathbb{C})$ they obtain are the contragredients of ours).

8.22. **The computation of** $\mathrm{Hom}_K(V(n,\lambda),\mathcal{P})$ **as a** $U(\mathfrak{g}')$ **module.** In this subsection we will prove the following theorem by combining Proposition 8.11 with a result of Howe. Restricting the elements of $\varphi_{nq,[\lambda]}(S_\lambda(\mathbb{C}^n)[m/2])$ to $V(n,\lambda)$ we get a subspace of $\mathrm{Hom}_K(V(n,\lambda),\mathcal{P})$. In the following theorem we abusively denote by $\varphi_{nq,[\lambda]}$ any non-zero element of this subspace of $\mathrm{Hom}_K(V(n,\lambda),\mathcal{P})$, e.g. the image by $\varphi_{nq,[\lambda]}$ of a dominant weight vector of S_λ tensored with a generator of $\mathbb{C}_{\frac{m}{2}}$.

8.23. **Theorem.** *As a* $(\mathfrak{g}',K')$*-module,* $\mathrm{Hom}_K(V(n,\lambda),\mathcal{P})$ *is generated by the restriction of* $\varphi_{nq,[\lambda]}$ *to* $V(n,\lambda)$*, i.e.*

$$\mathrm{Hom}_K(V(n,\lambda),\mathcal{P}) = U(\mathfrak{g}')\varphi_{nq,[\lambda]}.$$

Moreover: there exists a $(\mathfrak{g}\times\mathfrak{g}',K\times K')$*-quotient* $\mathcal{P}/\mathcal{N}$ *of* $\mathcal{P}$ *such that the* $(\mathfrak{g}',K')$*-module* $\mathrm{Hom}_K(V(n,\lambda),\mathcal{P}/\mathcal{N})$ *is irreducible, generated by the image of* $\varphi_{nq,[\lambda]}|_{V(n,\lambda)}$ *and isomorphic to the underlying* $(\mathfrak{g}',K')$*-module of the (holomorphic) unitary discrete series representation with lowest* K'*-type (having highest weight)* $S_\lambda(\mathbb{C}^n)\otimes\mathbb{C}_{\frac{m}{2}}$.

Here $U(\mathfrak{g}')$ denotes the universal enveloping algebra of $\mathfrak{g}'$.

The theorem will be a consequence of general results of Howe and the results obtained above combined with the following lemmas.

8.24. We consider the decomposition of $\mathcal{P}$ into K-isotypical components:

$$\mathcal{P} = \bigoplus_{\sigma \in \mathcal{R}(K,\omega)} \mathcal{J}_\sigma,$$

see [36, §3].

Remark. However, we have to be careful about two group-theoretic points concerning our maximal compact subgroups K and K'. First we address K'. The action of $\mathrm{GL}(n,\mathbb{C})$ on the Fock model $\mathcal{P}$ is the standard action on polynomial functions *twisted by a character*. Recall we identify $\mathcal{P} = \mathcal{P}(\mathbb{C}^{mn})$ with the space $\mathcal{P}(\mathrm{M}_{m,n}(\mathbb{C}))$ of polynomials on m by n complex matrices, $\mathrm{M}_{m,n}(\mathbb{C})$. Then the action of the restriction of the Weil representation ω to $\mathrm{GL}(n,\mathbb{C})$ on $\mathcal{P}(\mathrm{M}_{m,n}(\mathbb{C}))$ is given by the following formula. Let $Z \in \mathrm{M}_{m,n}(\mathbb{C})$ and $P \in \mathcal{P}(\mathrm{M}_{m,n}(\mathbb{C}))$. Then we have for $g \in \mathrm{GL}(n,\mathbb{C})$

$$\omega(g)P(Z) = \det(g)^{\frac{p-q}{2}} P(Zg). \tag{8.24.1}$$

To be precise, we do not get an action of the general linear group but rather of its connected two-fold cover $\widetilde{\mathrm{GL}}(n,\mathbb{C})$, the metalinear group. We will ignore this point in what follows as we have done with the difference between the symplectic and metaplectic groups. Second we address K. In what follows the theory of dual pairs requires us to use $\mathrm{O}(V_+)$ and $\mathrm{O}(V_-)$ below. The reader will verify that in fact we may replace $\mathrm{O}(V_+)$ and $\mathrm{O}(V_-)$ by $\mathrm{SO}(V_+)$ and $\mathrm{SO}(V_-)$. However we note $\varphi_{nq,[\lambda]}$ transforms by a power of the determinant representation of $\mathrm{O}(V_-)$ which will consequently be ignored. Thus in what follows K will denote the product $\mathrm{SO}(V_+) \times \mathrm{SO}(V_-)$. The main point that allows us to make this restriction to the connected group $K = \mathrm{SO}(V_+) \times \mathrm{SO}(V_-)$ is that the restriction of the representation $V(n,\lambda)$ of $\mathrm{O}(V_+)$ to $\mathrm{SO}(V_+)$ is irreducible.

In what follows the key point will be to compute the $V(n,\lambda)$-isotypic component in $\mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_-$ as a $\mathrm{GL}(n,\mathbb{C})$ module under the action induced by the Weil representation. We will temporarily ignore the twist by $\det^{(p-q)/2}$. Then denoting the above isotypic subspace by $\mathcal{H}_{V(n,\lambda)}$ we have

$$\mathcal{H}_{V(n,\lambda)} = \mathrm{Hom}_K(V(n,\lambda),\mathcal{H}) \otimes V(n,\lambda). \tag{8.24.2}$$

Here the first factor is a $\mathrm{GL}(n,\mathbb{C})$ module where $\mathrm{GL}(n,\mathbb{C})$ acts by post-composition.

In what follows it will be very important that the representation $V(n,\lambda)$ of $\mathrm{SO}(V_+) \times \mathrm{SO}(V_-)$ has trivial restriction to the second factor. To keep track of this, up until the end of the proof of Lemma 8.29, we will denote the restriction of the representation $V(n,\lambda)$ to the first factor $\mathrm{SO}(V_+)$ of K by $V(n,\lambda)_+$. Thus as a representation of the product $\mathrm{SO}(V_+) \times \mathrm{SO}(V_-)$ we have

$$V(n,\lambda) = V(n,\lambda)_+ \boxtimes 1$$

and

$$\begin{aligned} \mathrm{Hom}_K(V(n,\lambda),\mathcal{H}_+ \otimes \mathcal{H}_-) &\\ = \mathrm{Hom}_{\mathrm{SO}(V_+)}(V(n,\lambda)_+,\mathcal{H}_+) &\otimes \mathrm{Hom}_{\mathrm{SO}(V_-)}(1,\mathcal{H}_-) \\ = \mathrm{Hom}_{\mathrm{SO}(V_+)}(V(n,\lambda)_+,\mathcal{H}_+) &\otimes \mathbb{C}. \end{aligned} \tag{8.24.3}$$

Thus it remains to compute the first tensor factor.

8.25. **Lemma.** *The* $\mathrm{GL}(n,\mathbb{C})$*-module* $\mathrm{Hom}_{\mathrm{SO}(V_+)}(V(n,\lambda)_+,\mathcal{H}_+)$ *is the irreducible module with highest weight* $(q+\lambda_1,\dots,q+\lambda_n)$*. Hence we have an isomorphism of* $\mathrm{GL}(n,\mathbb{C})$*-modules*

$$\mathrm{Hom}_{\mathrm{SO}(V_+)}(V(n,\lambda)_+,\mathcal{H}_+)\cong S_\lambda(\mathbb{C}^n)\otimes\mathbb{C}_q.$$

Proof. The lemma is an immediate consequence of Proposition 8.20. Since the irreducible $\mathrm{O}(V_+)$-module $V(n,\lambda)_+$ has dominant weight $\mu=(q+\lambda_1,\dots,q+\lambda_n,0\dots,0,\cdots,0)$ it follows from Proposition 8.20 that the corresponding irreducible module for $\mathrm{GL}(n,\mathbb{C})$ is isomorphic to $S_\lambda(\mathbb{C}^n)\otimes\mathbb{C}_q$ and consequently has highest weight $(q+\lambda_1,\dots,q+\lambda_n)$. . □

Taking into account the twist of the standard $\mathrm{GL}(n,\mathbb{C})$-action by $\det^{(p-q)/2}$ in the action of the Weil representation on $\mathcal{P}$ we find that the final determinant twist is $q+(p-q)/2=m/2$. We conclude:

8.26. **Lemma.** *Under the action coming from the restriction of the Weil representation the* $V(n,\lambda)$ *isotypic subspace of* $\mathcal{H}=\mathcal{H}_+\otimes\mathcal{H}_-$ *decomposes as a* $\mathrm{GL}(n,\mathbb{C})\times[\mathrm{SO}(V_+)\times\mathrm{SO}(V_-)]$ *-module according to :*

$$\mathcal{H}_{V(n,\lambda)}=(S_\lambda(\mathbb{C}^n)\otimes\mathbb{C}_{\frac{m}{2}})\otimes V(n,\lambda). \tag{8.26.1}$$

Recall by Proposition 8.11 we have

$$\varphi_{nq,[\lambda]}\in\mathrm{Hom}_K(V(n,\lambda),\mathcal{H}_+\otimes\mathcal{H}_-)$$

But $\varphi_{nq,[\lambda]}$ takes values in $\mathcal{H}_+$ thus induces an element $\varphi^+_{nq,[\lambda]}\in\mathrm{Hom}_K(V(n,\lambda),\mathcal{H}_+)$ such that

$$\varphi_{nq,[\lambda]}=\varphi^+_{nq,[\lambda]}\otimes 1.$$

Note that $\mathrm{GL}(n)\times\mathrm{O}(V_+)$ acting on $\mathcal{H}_+$ forms a dual pair and $\mathrm{GL}(n)\times\mathrm{O}(V_-)$ acting on $\mathcal{H}_-$ forms a dual pair, $V(n,\lambda)=V(n,\lambda)_+\boxtimes\mathbb{C}$ and $\varphi_{nq,[\lambda]}=\varphi^+_{nq,[\lambda]}\otimes 1$. We have

8.27. **Lemma.** *We have:*

1. $\mathrm{Hom}_{\mathrm{SO}(V_+)}(V(n,\lambda)_+,\mathcal{H}_+)=U(\mathfrak{gl}(n,\mathbb{C}))\cdot\varphi^+_{nq,[\lambda]}$.
2. $\mathrm{Hom}_{\mathrm{SO}(V_-)}(\mathbb{C},\mathcal{H}_-)=U(\mathfrak{gl}(n,\mathbb{C}))\cdot 1=\mathbb{C}$.
3. $\mathrm{Hom}_K(V(n,\lambda),\mathcal{H}_+\otimes\mathcal{H}_-)=[U(\mathfrak{gl}(n,\mathbb{C}))\otimes U(\mathfrak{gl}(n,\mathbb{C}))]\cdot\varphi_{nq,[\lambda]}$.

Note that we have a product of dual pairs $(\mathrm{GL}(n,\mathbb{C})\times\mathrm{O}(V_+))\times(\mathrm{GL}(n,\mathbb{C})\times\mathrm{O}(V_-))$. Since $\varphi_{nq,[\lambda]}$ takes values in $\mathcal{H}_+\otimes\mathbb{C}$ the action of $U(\mathfrak{gl}(n,\mathbb{C}))\otimes U(\mathfrak{gl}(n,\mathbb{C}))$ of (3) of Lemma 8.27 on $\varphi_{nq,[\lambda]}$ coincides with the "*diagonal*" action of $U(\mathfrak{gl}(n,\mathbb{C}))$ (i.e. coming from the diagonal inclusion of $\mathrm{GL}(n,\mathbb{C})$ into the product of the two first factors of the two dual pairs above). It is critical in what follows that this diagonal $\mathfrak{gl}(n,\mathbb{C})$ is the complexification of the Lie algebra of the maximal compact subgroup K' of the metaplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ in our basic dual pair $\mathrm{Sp}_{2n}(\mathbb{R})\times\mathrm{O}(V)$. Hence we obtain the improved version of (3) of the previous lemma that we will need to prove the first statement of Theorem 8.23.

8.28. **Lemma.**

$$\mathrm{Hom}_K(V(n,\lambda),\mathcal{H}_+\otimes\mathcal{H}_-)=U(\mathfrak{gl}(n,\mathbb{C}))\cdot\varphi_{nq,[\lambda]}.$$

The next lemma proves the first assertion of Theorem 8.23.

8.29. **Lemma.**

$$\mathrm{Hom}_K(V(n,\lambda),\mathcal{P}) = U(\mathfrak{g}')\cdot\varphi_{nq,[\lambda]}.$$

Proof. By Howe [36, Proposition 3.1] the space $\mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_-$ generates $\mathcal{P}$ as a $U(\mathfrak{g}')$-module, i.e.

$$\mathcal{P} = U(\mathfrak{g}')(\mathcal{H}_+ \otimes \mathcal{H}_-).$$

We obtain:

$$\begin{aligned}\mathrm{Hom}_K(V(n,\lambda),\mathcal{P})) &= \mathrm{Hom}_K(V(n,\lambda), U(\mathfrak{g}')(\mathcal{H}_+\otimes\mathcal{H}_-))\\ &= U(\mathfrak{g}')(\mathrm{Hom}_K(V(n,\lambda),\mathcal{H}_+\otimes\mathcal{H}_-))\\ &= U(\mathfrak{g}')U(\mathfrak{gl}(n,\mathbb{C}))\cdot\varphi_{nq,[\lambda]} = U(\mathfrak{g}')\cdot\varphi_{nq,[\lambda]}.\end{aligned}$$

□

8.30. We now prove the second assertion of Theorem 8.23.

It follows from Li [53] that the $(\mathfrak{g},K)$-module $A_{\mathfrak{q}}(\lambda)$ occurs in Howe's theta correspondence (see [36, Theorem 2.1]). In particular: there exists a $(\mathfrak{g}\times\mathfrak{g}', K\times K')$ quotient of $\mathcal{P}$ which has the form

$$\mathcal{P}/\mathcal{N} \cong A_{\mathfrak{q}}(\lambda)\otimes\pi',$$

where π' is a finitely generated, admissible, and quasisimple $(\mathfrak{g}',K')$-module. This yields a projection

$$\begin{aligned}\mathrm{Hom}_K(V(n,\lambda),\mathcal{P}) &\to \mathrm{Hom}_K(V(n,\lambda),\mathcal{P}/\mathcal{N})\\ &= \pi'\otimes\mathrm{Hom}_K(V(n,\lambda),A_{\mathfrak{q}}(\lambda)) = \pi'\otimes\mathbb{C}.\end{aligned}$$

But since $\mathrm{Hom}_K(V(n,\lambda),\mathcal{P}) \cong U(\mathfrak{g}')\varphi_{nq,[\lambda]}$, the projection is nonzero, and π' is irreducible, the projection must map the generator $\varphi_{nq,[\lambda]}|_{V(n,\lambda)}$ of the $U(\mathfrak{g}')$-module $\mathrm{Hom}_K(V(n,\lambda),\mathcal{P})$ to a generator of π'. Finally: Li makes Howe's correspondence explicit. In our case π' is the underlying $(\mathfrak{g}',K')$-module of the holomorphic unitary discrete series representation with lowest K'-type $S_\lambda(\mathbb{C}^n)\otimes\mathbb{C}_{\frac{m}{2}}$ which is the K' type generated by $\varphi_{nq,[\lambda]}$.

This concludes the proof of Theorem 8.23.

8.31. *Remark.* The $(\mathfrak{g},K)$-module $A_{\mathfrak{q}}(\lambda)$ does not occur in Howe's theta correspondence from a symplectic group smaller than $\mathrm{Sp}_{2n}(\mathbb{R})$.

Proof. Let $k < n$. It follows e.g. from [23, Cor. 3 (a)] that as a K-module $\mathcal{P}(\mathbb{C}^{km})_+ \cong \mathrm{Sym}((\mathbb{C}^p)^{\oplus k})$ does not contain $V(n)$. As $V(n)$ occurs as a K-type in $A_{\mathfrak{q}}(\lambda)\otimes S_{[\lambda]}(V)$ it follows from the proof of Theorem 8.23 that $A_{\mathfrak{q}}(\lambda)$ does not occur in Howe's theta correspondence from $\mathrm{Sp}_{2k}(\mathbb{R})$. □

Part 3. Geometry of arithmetic manifolds

9. Cohomology of arithmetic manifolds

9.1. **Notations.** Let F be a totally real field of degree d and $\mathbb{A}$ the ring of adeles of F. Let V be a nondegenerate quadratic space over F with $\dim_F V = m$. We assume that $G = \mathrm{SO}(V)$ is compact at all but one infinite place. We denote by v_0 the infinite place where $\mathrm{SO}(V)$ is non compact and assume that $G(F_{v_0}) = \mathrm{SO}(p,q)$.

Let $\widetilde{G} = \mathrm{GSpin}(V)$ be the set of all invertible elements in the even Clifford algebra such that $gVg^{-1} = V$. There is an exact sequence

$$1 \to F^* \to \widetilde{G} \to G \to 1, \tag{9.1.1}$$

where F^* is the subgroup of the center which acts trivially on V. We denote by $\mathrm{Nspin} : \widetilde{G} \to F^*$ the spinor norm map and let $\widetilde{G}^{\mathrm{der}}$ be its kernel. We finally let

$$D = \mathrm{SO}_0(p,q)/(\mathrm{SO}(p) \times \mathrm{SO}(q)).$$

9.2. **Arithmetic manifolds.** In this paragraph we mainly follow [45, §1]. For any compact open subgroup $K \subset G(\mathbb{A}_f)$, we denote by $\widetilde{K}$ its preimage in $\widetilde{G}(\mathbb{A}_f)$ and let

$$X_K = \widetilde{G}(F)\backslash(\mathrm{SO}(p,q) \times \widetilde{G}(\mathbb{A}_f))/(\mathrm{SO}(p) \times \mathrm{SO}(q))\widetilde{K}.$$

The connected components of X_K can be described as follows. Write

$$\widetilde{G}(\mathbb{A}_f) = \sqcup_j \widetilde{G}(F)_+ g_j \widetilde{K}.$$

Here $\widetilde{G}(F)_+$ consists of those elements whose spinor norm — viewed as an element of F^* — is totally positive, i.e. lies in $F^*_{\infty+} = (\mathbb{R}^*_+)^d$ where d is the degree of $F/\mathbb{Q}$. Then

$$X_K = \sqcup_j \Gamma_{g_j}\backslash D,$$

where Γ_{g_j} is the image in $\mathrm{SO}(p,q)_0$ of

$$\Gamma'_{g_j} = \widetilde{G}(F)_+ \cap g_j \widetilde{K} g_j^{-1}.$$

Since the group $\widetilde{G}^{\mathrm{der}}$ is connected, simply connected[8] and semisimple, the strong approximation theorem implies (see e.g. [61, Thm. 5.17]) that

$$\pi_0(X_K) \cong \widetilde{G}(F)_+\backslash\widetilde{G}(\mathbb{A}_f)/\widetilde{K} \cong \mathbb{A}^*/F^*_c \mathrm{Nspin}(\widetilde{K})$$

where F^*_c denote the closure of $F^* F^*_{\infty+}$ in $\mathbb{A}^*$.

We let $\Gamma_K = \Gamma_1$ and $Y_K = \Gamma_K\backslash D$ be the associated connected component of X_K. These are the arithmetic manifolds we are interested in. Note that the manifolds considered in the Introduction are particular cases of these.

9.3. **Differential forms.** Let (ρ, E) be a finite dimensional irreducible representation of $G_\infty = \mathrm{SO}_0(p,q)$. Let $K_\infty = \mathrm{SO}(p) \times \mathrm{SO}(q)$. The representation $\rho_{|K_\infty}$ on E gives rise to a G_∞-equivariant Hermitian bundle on D, namely, $(E \times G_\infty)/K_\infty$, where the K_∞-action (resp. G_∞-action) is given by $(v,g) \stackrel{k}{\mapsto} (\rho(k)v, gk^{-1})$ (resp. $(v,g) \stackrel{x}{\mapsto} (v, xg)$). There is, up to scaling, one G_∞-invariant Hermitian metric on E. We fix an inner product $(\,,)_E$ in this class. We denote this Hermitian vector bundle also by E. Note that this bundle is G_∞-equivariantly isomorphic to the trivial vector bundle $E \times D$, where the G_∞-action is via $x : (v, gK_\infty) \mapsto (\rho(x)v, xgK_\infty)$. Smooth sections of E are identified with maps $C^\infty(G,E)$ with the property that $f(gk) = \rho(k)f(g)$.

[8] Here we work in the algebraic category: connected means Zariski-connected and a semisimple group G is simply connected if any isogeny $G' \to G$ with G' connected is an isomorphism.

9.4. Now let $\Gamma_K = \Gamma_1$ as above and keep notations as in section 5 with G (resp. K) replaced by G_∞ (resp. K_∞). The bundle of E-valued differential k-forms on $Y_K = \Gamma_K \backslash D$ can be identified with the vector bundle associated with the K_∞-representation $\wedge^k \mathfrak{p}^* \otimes E$. Note that $\wedge^k \mathfrak{p}^* \otimes E$ is naturally endowed with a K_∞-invariant scalar product: the tensor product of $(\, ,)_E$ with the scalar product on $\wedge^k \mathfrak{p}^*$ defined by the Riemannian metric on D. The space of differentiable E-valued k-forms on Y_K, denoted $\Omega^k(Y_K, E)$, is therefore identified with

$$\left(C^\infty(\Gamma_K \backslash G_\infty) \otimes E \otimes \wedge^k \mathfrak{p}^*\right)^{K_\infty} \cong \mathrm{Hom}_{K_\infty}\left(\wedge^k \mathfrak{p}, C^\infty(\Gamma_K \backslash G_\infty) \otimes E\right).$$

A compactly supported element $\varphi \in \Omega^k(Y_K, E)$ defines a smooth map $\Gamma_K \backslash G_\infty \to \wedge^k \mathfrak{p}^* \otimes E$ which satisfies:

$$\varphi(gk) = \wedge^k \mathrm{ad}_{\mathfrak{p}}^*(k) \otimes \rho(k)(\varphi(g)) \quad (g \in G_\infty, \quad k \in K_\infty)$$

so that the norm

$$\varphi \mapsto \int_{Y_K} ||\varphi(xK_\infty)||^2_{\wedge^k \mathfrak{p}^* \otimes E} dx$$

is well defined. The space of square integrable k-forms $\Omega^k_{(2)}(Y_K, E)$ is the completion of the space of compactly supported differentiable E-valued k-forms on Y_K with respect to this latter norm.

9.5. **The de Rham complex.** The de Rham differential

$$d : \Omega^k(Y_K, E) \to \Omega^{k+1}(Y_K, E)$$

turns $\Omega^\bullet(Y_K, E)$ into a complex. Let d^* be the formal adjoint. We refer to $dd^* + d^*d$ as the *Laplacian*. It extends to a self-adjoint non-negative densely defined elliptic operator $\Delta_k^{(2)}$ on $\Omega^k_{(2)}(Y_K, E)$, the form Laplacian. We let

$$\mathcal{H}^k(Y_K, E) = \{\omega \in \Omega^k_{(2)}(Y_K, E) \ : \ \Delta_k^{(2)}\omega = 0\}$$

be the space of harmonic k-forms. Hodge theory shows that $\mathcal{H}^k(Y_K, E)$ is isomorphic to $\overline{H}^k_{(2)}(Y_K, E)$ — the *reduced* L^2-cohomology group — when Y_K is compact the latter group is just $H^k(Y_K, E)$ the k-th cohomology group of the de Rham complex $\Omega^\bullet(Y_K, E)$. We will mainly work with $\mathcal{H}^k(Y_K, E)$.

9.6. Let (π, V_π) be an irreducible $(\mathfrak{g}, K_\infty)$-module and consider the linear map:

$$T_\pi : \mathrm{Hom}_{K_\infty}(\wedge^* \mathfrak{p}, V_\pi \otimes E) \otimes \mathrm{Hom}_{\mathfrak{g}, K_\infty}(V_\pi, L^2(\Gamma_K \backslash G_\infty)) \to \Omega^k_{(2)}(Y_K, E)$$

which maps $\psi \otimes \varphi$ to $\varphi \circ \psi$. The image of T_π is either orthogonal to $\mathcal{H}^k(Y_K, E)$ or $H^k(\mathfrak{g}, K_\infty; V_\pi \otimes E) \neq 0$ i.e. π is cohomological. In the latter case we denote by $\mathcal{H}^k(Y_K, E)_\pi$ (resp. $\overline{H}^k_{(2)}(Y_K, E)_\pi$) the subspace of $\mathcal{H}^k(Y_K, E)$ (resp. $\overline{H}^k_{(2)}(Y_K, E)$) corresponding to the image of T_π.

A global representation $\sigma \in \mathcal{A}^c(\mathrm{SO}(V))$ with K-invariant vectors and such that the restriction of σ_{v_0} to $\mathrm{SO}_0(p, q)$ is isomorphic to π, and σ_v is trivial for every infinite place $v \neq v_0$, contributes to

$$\mathrm{Hom}_{\mathfrak{g}, K_\infty}(V_\pi, L^2(\Gamma_K \backslash G_\infty)).$$

We denote by $H^k_{\mathrm{cusp}}(Y_K, E)_\pi$ the corresponding subspace of $\mathcal{H}^k(Y_K, E)_\pi$ (obtained using the map T_π).

Let $m_K(\pi)$ be the multiplicity with which π occurs as an irreducible cuspidal summand in $L^2(\Gamma_K\backslash G_\infty)$. It follows from Matsushima's formula, see e.g. [13], that

$$H^k_{\text{cusp}}(Y_K, E)_\pi \cong m_K(\pi) H^k(\mathfrak{g}, K_\infty; V_\pi \otimes E).$$

9.7. Since X_K is a finite disjoint union of connected manifolds Y_L we may easily translate the above definitions into $\mathcal{H}^k(X_K, E)$, $\mathcal{H}^k(X_K, E)_\pi$, $H^k_{\text{cusp}}(X_K, E)_\pi$, etc...

We set[9]

$$H^k_{\text{cusp}}(\text{Sh}(G), E)_\pi = \varinjlim_K H^k_{\text{cusp}}(X_K, E)_\pi$$

and

$$H^k_{\text{cusp}}(\text{Sh}^0(G), E)_\pi = \varinjlim_K H^k_{\text{cusp}}(Y_K, E)_\pi.$$

Now the inclusion map $Y_K \to X_K$ yields a surjective map

$$H^k_{\text{cusp}}(X_K, E)_\pi \to H^k_{\text{cusp}}(Y_K, E)_\pi.$$

As these inclusions have been chosen in a compatible way we get a surjective map:

$$H^k_{\text{cusp}}(\text{Sh}(G), E)_\pi \to H^k_{\text{cusp}}(\text{Sh}^0(G), E)_\pi.$$

9.8. **Cohomology classes arising from the θ-correspondence.** Fix (π, V_π) a cohomological irreducible $(\mathfrak{g}, K_\infty)$-module such that

$$H^k(\mathfrak{g}, K_\infty; V_\pi \otimes E) \neq 0.$$

Note that $H^k_{\text{cusp}}(\text{Sh}(G), E)$ is generated by the images of $H^k(\mathfrak{g}, K_\infty; \sigma \otimes E)$ where σ varies among all irreducible cuspidal automorphic representations of $G(\mathbb{A})$ which occur as irreducible subspaces in the space of cuspidal automorphic functions in $L^2(G(F)\backslash G(\mathbb{A}))$ and such that σ_v is the trivial representation for each infinite place $v \neq v_0$. We let

$$H^k_\theta(\text{Sh}(G), E) \subset H^k_{\text{cusp}}(\text{Sh}(G), E)$$

be the subspace generated by those $\sigma \in \mathcal{A}^c(\text{SO}(V))$ that are in the image of the cuspidal ψ-theta correspondence from a smaller group, see §4.1. And we define

$$H^k_\theta(\text{Sh}(G), E)_\pi = H^k_\theta(\text{Sh}(G), E) \cap H^k_{\text{cusp}}(\text{Sh}(G), E)_\pi.$$

Remark. We can fix one choice of nontrivial additive character ψ: Every other nontrivial additive character is of the form $\psi_t : x \mapsto \psi(tx)$ for some $t \in F^*$. Notations being as in §2.4, one then easily checks that

$$\Theta^V_{\psi_t, X}(\pi') = \Theta^V_{\psi, X}(\pi'_t)$$

where the automorphic representation π'_t is obtained from twisting π' by an automorphism of $\text{Mp}(X)$. We may thus drop explicit reference to ψ.

We can now state and prove the main theorem of this section.

9.9. **Theorem.** *Assume π is associated to a Levi subgroup $L = \text{SO}(p-2r, q) \times \text{U}(1)^r$ with $p > 2r$ and $m - 1 > 3r$. Then: the natural map*

$$H^k_\theta(\text{Sh}(G), E)_\pi \to H^k_{\text{cusp}}(\text{Sh}^0(G), E)_\pi$$

is surjective.

---

[9]These are only notations. We won't consider such spaces as $\text{Sh}(G)$ or $\text{Sh}^0(G)$.

Proof. It follows from Corollary 6.10 and Theorem 4.2 that $H^k_{\rm cusp}({\rm Sh}(G),E)_\pi$ is generated by the images of $H^k(\mathfrak{g},K_\infty;(\sigma\otimes\eta)\otimes E)$ where the representations $\sigma\in\mathcal{A}^c({\rm SO}(V))$ are in the image of the cuspidal ψ-theta correspondence from a smaller group and such that the underlying $(\mathfrak{g},K_\infty)$-module of σ_{v_0} (or equivalently of $\sigma_{v_0}\otimes\eta_{v_0}$) is isomorphic to that of π, each σ_v (for $v\neq v_0$ infinite) is the trivial representation, and η varies among all automorphic characters of $G(\mathbb{A})$.

Now let ω be an element of the image of $H^k(\mathfrak{g},K_\infty;(\sigma\otimes\eta)\otimes E)$ in $H^k_{\rm cusp}({\rm Sh}(G),E)_\pi$. Choose $K\subset G(\mathbb{A}_f)$ a compact open subgroup such that $\omega\in H^k_{\rm cusp}(X_K,E)_\pi$ and η is $\widetilde{K}$-invariant. Seeing ω as an element of

$$\lim_{\overrightarrow{K}}\Omega^k(X_K,E)={\rm Hom}_{K_\infty}(\wedge^k\mathfrak{p},C^\infty(\widetilde{G}(F)\backslash({\rm SO}(p,q)\times\widetilde{G}(\mathbb{A}_f))\otimes E)$$

we may form the tensor product $\omega\otimes\eta^{-1}$. It defines an element of the image of $H^k(\mathfrak{g},K_\infty;\sigma\otimes E)$ in $H^k_{\rm cusp}(X_K,E)_\pi$ whose restriction to Y_K is equal to $\omega_{|Y_K}$. We conclude that ω and $\omega\otimes\eta$ have the same image in $H^k_{\rm cusp}({\rm Sh}^0(G),E)_\pi$ and the theorem follows. □

It is not true in general that the automorphic representations of ${\rm SO}(V)$ that are in the image of the cuspidal theta corespondence from a smaller group are cuspidal. This is the reason why in the next theorem we assume that V is anisotropic.

One can therefore deduce from Theorem 9.9 the following:

9.10. **Theorem.** *Assume that V is anisotropic. Let r be a positive integer such that $p>2r$ and $m-1>3r$ and let $\pi=A_{\mathfrak{q}}(\lambda)$ be a cohomological $(\mathfrak{g},K_\infty)$-module whose associated Levi subgroup L is isomorphic to* ${\rm SO}(p-2r,q)\times{\rm U}(1)^r$ *and such that λ has at most r nonzero entries. Then: the global theta correspondence induces an isomorphism between the space of cuspidal holomorphic Siegel modular forms, of weight $S_\lambda(\mathbb{C}^r)^*\otimes\mathbb{C}_{-\frac{m}{2}}$ at v_0 and of weight $\mathbb{C}_{-\frac{m}{2}}$ at all the other infinite places, on the* connected *Shimura variety associated to the symplectic group* ${\rm Sp}_{2r}|_F$ *and the space $H^{rq}_{\rm cusp}({\rm Sh}^0(G),S_{[\lambda]}(V))_\pi$.*

Proof. The surjectivity follows from Theorem 9.9. The injectivity follows from Rallis inner product formula [71]. In our case it is due to Li, see the proof [54, Theorem 1.1]. More precisely: let f_1, f_2 two cuspidal holomorphic Siegel modular forms of weight $(S_\lambda(\mathbb{C}^r)^*\otimes\mathbb{C}_{-\frac{m}{2}})\otimes\mathbb{C}_{-\frac{m}{2}}\otimes\ldots\otimes\mathbb{C}_{-\frac{m}{2}}$ on the connected Shimura variety associated to the symplectic group ${\rm Sp}_{2r}|_F$. These are functions in $L^2({\rm Mp}(X)\backslash{\rm Mp}_{2r}(\mathbb{A}))$ which respectively belong to the spaces of two cuspidal automorphic representations σ_1', $\sigma_2'\in\mathcal{A}^c({\rm Mp}(X))$. And Rallis' inner product formula — as recalled in [54, §2] — implies that if ϕ_1 and ϕ_2 are functions in $\mathcal{S}(V(\mathbb{A})^r)$ then:

$$\langle\theta^{f_1}_{\psi,\phi_1},\theta^{f_2}_{\psi,\phi_2}\rangle=\begin{cases}\int_{{\rm Mp}(\mathbb{A})}\langle\omega_\psi(h)\phi_1,\phi_2\rangle\langle\sigma'(h)f_1,f_2\rangle dh & \text{if } \sigma_1'=\sigma_2'=\sigma',\\ 0 & \text{if } \sigma_1'\neq\sigma_2'.\end{cases}$$

We are thus reduced to the case where $\sigma_1'=\sigma_2'$. But the integral on the right-hand side then decomposes as a product of local integrals. At each unramified finite place these are special (non-vanishing) values of local L-functions, see [54, §5]. It therefore remains to evaluate the remaining local factors. The non-vanishing of these local integral at ramified finite places follows from the fact that we are in the so-called *stable range*, see [54, Theorem 5.4 (a)]. Finally Li proves that the local Archimedean factors are non-zero in our special case where σ_{v_0}' is a holomorphic discrete series of weight $S_\lambda(\mathbb{C}^r) \otimes \mathbb{C}_{\frac{m}{2}}$ and σ_v (v infinite, $v \neq v_0$) is the trivial representation, see [54, Theorem 5.4 (b)]. $\square$

10. Special cycles

10.1. **Notations.** We keep notations as in §9.1 and keep following the adelization [45] of the work of Kudla-Millson. We denote by $(,)$ the quadratic form on V and let n be an integer $0 \leq n \leq p$. Given an n-tuple $\mathbf{x} = (x_1, \dots, x_n) \in V^n$ we let $U = U(\mathbf{x})$ be the F-subspace of V spanned by the components of $\mathbf{x}$. We write $(\mathbf{x}, \mathbf{x})$ for the $n \times n$ symmetric matrix with ijth entry equal to (x_i, x_j). Assume $(\mathbf{x}, \mathbf{x})$ is totally positive semidefinite of rank t. Equivalently: as a sub-quadratic space $U \subset V$ is totally positive definite of dimension t. In particular: $0 \leq t \leq p$ (and $t \leq n$). The constructions of the preceeding section can therefore be made with the space $U^\perp$ in place of V. Set $H = \mathrm{SO}(U^\perp)$. There is a natural morphism $H \to G$. Recall that we can realize D as the set of negative q-planes in V_{v_0}. We then let D_H be the subset of D consisting of those q-planes which lie in $U_{v_0}^\perp$.

10.2. **Special cycles with trivial coefficients.** Let $U = U(\mathbf{x})$ as above. Fix $K \subset G(\mathbb{A}_f)$ a compact open subgroup. As in §9.2 we write

$$\widetilde{G}(\mathbb{A}_f) = \sqcup_j \widetilde{G}(F)_+ g_j \widetilde{K}.$$

Recall that

$$\Gamma'_{g_j} = \widetilde{G}(F)_+ \cap g_j \widetilde{K} g_j^{-1}.$$

We set

$$\Gamma'_{g_j,U} = \widetilde{H}(F)_+ \cap g_j \widetilde{K} g_j^{-1} = \widetilde{H}(F) \cap \Gamma'_{g_j}.$$

Let $\Gamma_{g_j,U}$ be the image of $\Gamma'_{g_j,U}$ in $\mathrm{SO}(p-t,q)_0$. We denote by $c(U, g_j, K)$ the image of the natural map

$$\Gamma_{g_j,U} \backslash D_H \to \Gamma_{g_j} \backslash D, \quad \Gamma_{g_j,U} z \mapsto \Gamma_{g_j} z. \tag{10.2.1}$$

Remark. For K small enough the map (10.2.1) is an embedding. The cycles $C_U := c(U, 1, K)$ are therefore connected totally geodesic codimension tq submanifolds in Y_K. These are the totally geodesic cycles of the introduction for the particular Y_K considered there.

10.3. We now introduce composite cycles as follows. For $\beta \in \mathrm{Sym}_n(F)$ totally positive semidefinite, we set

$$\Omega_\beta = \left\{ \mathbf{x} \in V^n \ : \ \frac{1}{2}(\mathbf{x}, \mathbf{x}) = \beta \text{ and } \dim U(\mathbf{x}) = \mathrm{rank}\beta \right\}.$$

Then Γ'_{g_j} acts on $\Omega_\beta(F)$ with finitely many orbits. Given a K-invariant Schwartz function $\varphi \in \mathcal{S}(V(\mathbb{A}_f)^n)$ we define

$$Z(\beta, \varphi, K) = \sum_j \sum_{\substack{\mathbf{x} \in \Omega_\beta(F) \\ \mathrm{mod}\ \Gamma'_{g_j}}} \varphi(g_j^{-1}\mathbf{x}) c(U(\mathbf{x}), g_j, K). \tag{10.3.1}$$

Let $t = \mathrm{rank}(\beta)$. Suppose first β is nonsingular. Then in Subsection 1.11 we have associated an element of $H^{qt}_{\mathrm{cusp}}(X_K)$ to the class of the cycle $Z(\beta, \varphi, K)$ which we called the cuspidal projection of the class. We now give a new construction of this projection.

Let $t = \mathrm{rank}(\beta)$. Because it is rapidly decreasing any cuspidal $q(p-t)$-form can be integrated along $Z(\beta, \varphi, K)$. We claim the canonical pairing between the qt-forms with cuspidal coefficients and the $p(q-t)$-forms with cuspidal coefficients is a perfect pairing. Indeed the forms with cuspidal coefficients are L^2 and they are stable under the restriction of the Hodge star operator so the claim follows. Hence the induced pairing between $H^{qt}_{\mathrm{cusp}}(X_K)$ and $H^{q(p-t)}_{\mathrm{cusp}}(X_K)$ is a perfect pairing. We can thefore associate to $Z(\beta, \varphi, K)$ a class $[\beta, \varphi]^0 \in H^{qt}_{\mathrm{cusp}}(X_K)$. We let

$$[\beta, \varphi] := [\beta, \varphi]^0 \wedge e_q^{n-t} \in H^{qn}_{\mathrm{cusp}}(X_K)$$

where we abusively denote by e_q the Euler form (an invariant q-form) dual to the Euler class of Section 5. Note in particular that, since $e_q = 0$ if q is odd, we have: $[\beta, \varphi] = 0$ if q is odd and $t < n$.

10.4. **Special cycles with nontrivial coefficients.** Following [25] we now promote the cycles (10.3.1) to cycles with coefficients.

Let λ be a dominant weight for G expressed as in §5.11. Assume that λ has at most n nonzero entries and that $n \leq p$. Then λ defines a dominant weight $\lambda_1 \geq \ldots \geq \lambda_n$ of $\mathrm{U}(n)$ and as such a finite dimensional irreducible representation $S_\lambda(\mathbb{C}^n)$ of $\mathrm{U}(n)$ and thus of K'. As above we denote by $S_{[\lambda]}(V)$ the finite dimensional irreducible representation of G with highest weight λ.

Fix a neat level K so that each $\Gamma_{g_j, U(\mathbf{x})}$ in (10.3.1) acts trivially on $U(\mathbf{x})$. The components $x_1, \ldots, x_n$ of each $\mathbf{x}$ are therefore all fixed by $\Gamma_{g_j, U(\mathbf{x})}$. Hence any tensor word in these components will be fixed by $\Gamma_{g_j, U(\mathbf{x})}$. Given a tableau T on λ, see [23],[10] we set

$$c(U(\mathbf{x}), g_j, K)_T = c(U(\mathbf{x}), g_j, K) \otimes \mathbf{x}_T.$$

Here $\mathbf{x}_T \in S_{[\lambda]}(V)$ is the harmonic tensor corresponding to T. We can similarly define $Z(\beta, \varphi, K)_T$ as a cycle with coefficients in $S_{[\lambda]}(V)$. We let $[\beta, \varphi]^0_T$, resp. $[\beta, \varphi]_T$, be the corresponding element in $H^{qt}_{\mathrm{cusp}}(X_K, S_{[\lambda]}(V))$, resp. $H^{qn}_{\mathrm{cusp}}(X_K, S_{[\lambda]}(V))$.

We finally define

$$[\beta, \varphi]_\lambda \in \mathrm{Hom}_{K'_\infty}(S_\lambda(\mathbb{C}^n), H^{qn}_{\mathrm{cusp}}(X_K, S_{[\lambda]}(V)))$$

as the linear map defined by

$$[\beta, \varphi]_\lambda(\epsilon_T) = [\beta, \varphi]_T$$

where $(\epsilon_1, \ldots, \epsilon_n)$ is the canonical basis of $\mathbb{C}^n$ and ϵ_T is the standard basis of $S_\lambda(\mathbb{C}^n)$ parametrized by the tableaux on λ.

10.5. **Back to the forms of Kudla-Millson and Funke-Millson.** For more details on this subsection see Subsection 8.1. Recall from §8 that we have defined a relative Lie algebra nq-cocycle

$$\varphi_{nq,[\lambda]} \in \mathrm{Hom}_{K'_\infty \times K_\infty}(S_\lambda(\mathbb{C}^n)[m/2] \otimes \wedge^{nq}\mathfrak{p}, \mathbf{S}(V(F_{v_0})^n) \otimes S_{[\lambda]}(V))).$$

We consider the quotient space

$$\widehat{D} := G(F_{v_0})/K_\infty \cong \mathrm{SO}(p,q)/(\mathrm{SO}(p) \times \mathrm{SO}(q)).$$

[10] Note that a tableau is called a semistandard filling in [25].

This quotient space is disconnected and is the disjoint union of two copies of D. We let $\Omega^\bullet(\widehat{D}, \mathcal{S}(V(F_{v_0})^n) \otimes S_{[\lambda]}(V))$ denote the complex of smooth $\mathcal{S}(V(F_{v_0})^n) \otimes S_{[\lambda]}(V)$)-valued differentiable forms on $\widehat{D}$:

$$\Omega^\bullet(\widehat{D}, \mathcal{S}(V(F_{v_0})^n) \otimes S_{[\lambda]}(V)) = \left[C^\infty(G(F_{v_0})) \otimes \mathcal{S}(V(F_{v_0})^n) \otimes S_{[\lambda]}(V)) \otimes \wedge^\bullet \mathfrak{p}^*\right]^{K_\infty}.$$

Fixing the base point $z_0 = eK$ in $\widehat{D}$, we have the isomorphism of Lemma 8.3

$$\text{(10.5.1)} \quad \Omega^{nq}(\widehat{D}, \mathcal{S}(V(F_{v_0})^n \otimes S_{[\lambda]}(V))^{G(F_{v_0})} \xrightarrow{\sim} \left[\mathcal{S}(V(F_{v_0})^n) \otimes S_{[\lambda]}(V) \otimes \wedge^{nq}\mathfrak{p}^*\right]^{K_\infty},$$

given by evaluating at z_0. We can therefore consider $\varphi_{nq,[\lambda]}$ as a $G(F_{v_0})$-invariant $(S_\lambda(\mathbb{C}^n)[m/2])^* \otimes \mathcal{S}(V(F_{v_0})^n) \otimes S_{[\lambda]}(V)$-valued differential nq-form (in the orthogonal variable)

$$\varphi_{nq,[\lambda]} \in \mathrm{Hom}_{K'_\infty}\left(S_\lambda(\mathbb{C}^n)[m/2], \left[\Omega^{nq}(\widehat{D}, \mathcal{S}(V(F_{v_0})^n) \otimes S_{[\lambda]}(V))\right]^{G(F_{v_0})}\right).$$

Remark. We did not extend $\varphi_{nq,[\lambda]}$ to a section of the homogeneous vector bundle over $\mathbb{H}_n$ in the symplectic variable in the previous equation.

We note the isomorphism of $\mathrm{MU}_n \times G(F_{v_0})$-modules (which we will use without explicitly mention)

$$\Omega^{nq}\big(\widehat{D}, \mathcal{S}(V(F_{v_0})^n) \otimes S_{[\lambda]}(V)\big) \cong \mathcal{S}(V(F_{v_0})^n) \otimes \Omega^{nq}(\widehat{D}, S_{[\lambda]}(V)).$$

Thus we may consider $\varphi_{nq,[\lambda]}$ as an $\mathrm{MU}_n \times G$-equivariant polynomial mapping, $\mathbf{x} \to \varphi_{nq,[\lambda]}(\mathbf{x})$, of degree nq on $V(F_{v_0})^n$ with values in $(S_\lambda(\mathbb{C}^n)[m/2])^* \otimes \Omega^{nq}(\widehat{D}, S_{[\lambda]}(V))$. This will be important in Subsection 10.7 where we will apply the theta distribution to the input variable $\mathbf{x}$.

10.6. Consider now a positive definite inner product space $V_+, (\ ,\ )_0$ of dimension m over $\mathbb{R}$. We may still consider the Schwartz form (the Gaussian) $\varphi_0 \in \mathcal{S}(V_+^n)$ given by

$$\varphi_0(\mathbf{x}) = \exp(-\pi \mathrm{tr}(\mathbf{x}, \mathbf{x}))_0.$$

Recall that under the equivalence (Bargmann transform) of the the subspace of the Schwartz space given by the span of the Hermite functions with the polynomial Fock space the Gaussian maps to the constant polynomial 1. Hence, by Proposition 7.8 (with $p = m$ and $q = 0$), under the Weil representation ω_+ of $\mathrm{Mp}(n, \mathbb{R})$ associated to V_+, we have

$$\omega_+(k')\varphi_0 = \det(k')^{\frac{m}{2}}\varphi_0.$$

If $\mathbf{x} \in V_+^n$ with $\frac{1}{2}(\mathbf{x}, \mathbf{x}) = \beta \in \mathrm{Sym}_n(\mathbb{R})$, then for $g' \in \mathrm{Mp}_{2n}(\mathbb{R})$ we set

$$\text{(10.6.1)} \qquad W_\beta(g') = \omega_+(g')\varphi_0(\mathbf{x}).$$

Then $W_\beta(g')$ depends only on β not on the particular choice of $\mathbf{x}$, see e.g. [45].

10.7. **Dual forms.** Now we return to the global situation. Let n be an integer with $1 \leq n \leq p$. We fix a level K and a K-invariant Schwartz function $\varphi \in \mathcal{S}(V(\mathbb{A}_f)^n)$. Define

$$\phi = \varphi_{nq,[\lambda]} \otimes \big(\bigotimes_{\substack{v|\infty \\ v \neq v_0}} \varphi_0 \big) \otimes \varphi \tag{10.7.1}$$

in

$$\Big[\mathcal{S}(V(\mathbb{A})^n) \otimes C^\infty(\mathbb{H}_n, S_\lambda(\mathbb{C}^n)^* \otimes \mathbb{C}_{-\frac{m}{2}}) \otimes C^\infty(\mathbb{H}_n, \mathbb{C}_{-\frac{m}{2}})^{\otimes(d-1)} \otimes \Omega^{nq}(\widehat{D}, S_{[\lambda]}(V)) \Big]^{\mathrm{Mp}_{2n}(\mathbb{R})^d \times G(F_{v_0})}.$$

As in §2.3 the global metaplectic group $\mathrm{Mp}_{2p}(\mathbb{A})$ acts in $\mathcal{S}(V(\mathbb{A})^n)$ via the global Weil representation. For $g \in G(\mathbb{A}_f)$ the theta function

$$g' \in \mathrm{Mp}_{2n}(\mathbb{R})^d \subset \mathrm{Mp}_{2n}(\mathbb{A}) \mapsto \theta_{\psi,\phi}(g, g') = \sum_{\xi \in V(F)^n} (\omega_\psi(g')\phi)(g^{-1}\xi)$$

defines an element in $C^\infty(\mathbb{H}_n, S_\lambda(\mathbb{C}^n)^*[m/2]) \otimes C^\infty(\mathbb{H}_n, \mathbb{C}_{-\frac{m}{2}})^{\otimes(d-1)} \otimes \Omega^{nq}(\widehat{D}, S_{[\lambda]}(V))$. As a function of $G(\mathbb{A})$ it therefore defines a $S_{[\lambda]}(V)$-valued closed nq-form on X_K which we abusively denote by $\theta_{nq,\lambda}(g', \varphi)$. Let $[\theta_{nq,\lambda}(g', \varphi)]$ be the corresponding cohomology class in $H^{nq}_{\mathrm{cusp}}(X_K, S_{[\lambda]}(V))$. Now for $g' \in \mathrm{Mp}_{2n}(\mathbb{R})^d \subset \mathrm{Mp}_{2n}(\mathbb{A})$ and for $\beta \in \mathrm{Sym}_n(F)$ with $\beta \geq 0$, set

$$W_\beta(g') = \prod_{v|\infty} W_{\beta_v}(g'_v).$$

The following result is proved by Funke and Millson [25, Theorems 7.6 and 7.7]; it is a generalization to twisted coefficients of the main theorem of [48]. The way we rephrase it here in the adelic language is due to Kudla [45]. Recall that rapidly decreasing $q(p-n)$-forms can be paired with degree nq cohomology classes. We denote by $\langle , \rangle$ this pairing.

10.8. **Proposition.** *As a function of $g' \in \mathrm{Mp}_{2n}(\mathbb{A})$ the cohomology class $[\theta_{nq,\lambda}(g', \varphi)]$ is a holomorphic Siegel modular form of weight $S_\lambda(\mathbb{C}^n)^* \otimes \mathbb{C}_{-\frac{m}{2}}$ with coefficients in $H^{qn}_{\mathrm{cusp}}(X_K, S_{[\lambda]}(V))$. Moreover: for any rapidly decreasing closed $q(p-n)$-form η on X_K with values in $S_{[\lambda]}(V)$, for any element $g' \in \mathrm{Mp}_{2n}(\mathbb{R})^d \subset \mathrm{Mp}_{2n}(\mathbb{A})$ and for any K-invariant function $\varphi \in \mathcal{S}(V(\mathbb{A}_f)^n)$ the Fourier expansion of $\langle [\theta_{nq,\lambda}(g', \varphi)], \eta \rangle$ is given by*

$$\langle [\theta_{nq,\lambda}(g', \varphi)], \eta \rangle = \sum_{\beta \geq 0} \langle [\beta, \varphi]_\lambda, \eta \rangle W_\beta(g').$$

10.9. **Definition.** We let

$$H^{nq}_{\theta_{nq,\lambda}}(\mathrm{Sh}(G), S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}$$

be the subspace of $H^{nq}_\theta(\mathrm{Sh}(G), S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}$ generated by those $\sigma \in \mathcal{A}^c(\mathrm{SO}(V))$ that are in the image of the cuspidal ψ-theta correspondence from $\mathrm{Mp}_{2n}(\mathbb{A})$ where the infinite components φ_v of the global Schwartz function ϕ satisfy

$$\varphi_{v_0} = \varphi_{nq,[\lambda]} \quad \text{and} \quad \varphi_v = \varphi_0, \ v|\infty, \ v \neq v_0.$$

We will call the corresponding map from the space of Siegel modular forms tensored with the Schwartz space of the finite adeles to automorphic forms for

$\mathrm{SO}(V)$ the *special* theta lift and the correspondence between Siegel modular forms and automorphic forms for $\mathrm{SO}(V)$ the special theta correspondence. We will denote the special theta lift evaluated on $f' \otimes \varphi$ by $\theta_{nq,\lambda}(f' \otimes \varphi)$.

11. Main theorem

11.1. **Notations.** We keep notations as in the preceeding paragraphs. In particular we let λ be a dominant weight for G expressed as in §5.11. We assume that λ has at most n nonzero entries and that $n \leq p$. We let $\mathfrak{q}$ be the particular θ-stable parabolic subalgebra of $\mathfrak{g}$ defined in §8.16 and whose associated Levi subgroup is isomorphic to $\mathrm{U}(1)^n \times \mathrm{SO}(p-2n, q)$.

We let $SC^{nq}(\mathrm{Sh}(G), S_{[\lambda]}(V))$ be the subspace of $H^{nq}_{\mathrm{cusp}}(\mathrm{Sh}(G), S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}$ spanned by the projection on the relevant K_∞-type of the images of the classes $[\beta, \varphi]_\lambda$.

11.2. Let K be a compact-open subgroup of $G(\mathbb{A}_f)$. Any K-invariant classes in $SC^{nq}(\mathrm{Sh}(G), S_{[\lambda]}(V))$ defines a class in $H^{nq}_{\mathrm{cusp}}(X_K, S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}$. And it follows from [45, Corollary 5.11] that

$$SC^{nq}(X_K, S_{[\lambda]}(V)) := SC^{nq}(\mathrm{Sh}(G), S_{[\lambda]}(V))^K$$

is precisely the subset of $H^{nq}_{\mathrm{cusp}}(X_K, S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}$ spanned by the projections of the images of the classes $[\beta, \varphi]_\lambda$ for K-invariant functions φ.

Note that if $\mathbf{x} \in V^n$ and $(\mathbf{x}, \mathbf{x})$ is totally positive semidefinite of rank $t \leq n$, the wedge product with e_q^{n-t} of the special cycle with coefficients $c(U(\mathbf{x}), 1, K)_T$ defines a class in $H^{nq}_{\mathrm{cusp}}(Y_K, S_{[\lambda]}(V))$. We let $Z^{nq}(Y_K, S_{[\lambda]}(V))$ be the subspace of $H^{nq}_{\mathrm{cusp}}(Y_K, S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}$ spanned by the projections of these classes. The restriction map $X_K \to Y_K$ (restriction to a connected component) obviously yields a map

$$SC^{nq}(X_K, S_{[\lambda]}(V)) \to Z^{nq}(Y_K, S_{[\lambda]}(V)). \quad (11.2.1)$$

We don't know in general if this map is surjective or not. It will nevertheless follow from Theorem 11.10 that in small degree the map (11.2.1) is indeed surjective.

11.3. **The special theta lift is onto the $A_{\mathfrak{q}}(\lambda)$-isotypic component of the image of the general theta lift.** In this subsection we carry out what was called step 2 in the introduction. This subsection is the analogue for general $\mathrm{SO}(p,q)$ of subsections 6.8 - 6.11 of [33]. In particular we now recall their Lemma 6.9. We need the following definition of the complex linear antiautomorphism $Z \to Z^*$ of $U(\mathfrak{g}')$. For $Z \in U(\mathfrak{g}')$ with $Z = X_1 X_2 \cdots X_n$ we define

$$Z^* = (-1)^n X_n X_{n-1} \cdots X_1.$$

Now for general Schwartz functions φ and Siegel automorphic forms f we have the following

11.4. **Lemma.** *For $Z \in U(\mathfrak{g}')$ we have*

$$\theta(Zf, \varphi) = \theta(f, Z^*\varphi).$$

Proof. See [33, Lemma 6.9]. □

We now show that the projected *special* theta lift $\theta_{nq,\lambda}$ is onto $H^{nq}_\theta(\mathrm{Sh}(G), S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}$ that is we have:

11.5. **Theorem.**

$$H^{nq}_{\theta_{nq,\lambda}}(\mathrm{Sh}(G), S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)} = H^{nq}_\theta(\mathrm{Sh}(G), S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}.$$

Proof. In what follows we will use the following simple observation to produce the commutative diagram (11.5.3) below. Suppose H is a group and we have H-modules A, B, U, V. Suppose further that we have H-module homomorphisms $\Phi : U \to V$ and $\Psi : B \to A$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_H(A,U) & \xrightarrow{\Phi_*} & \mathrm{Hom}_H(A,V) \\ {\scriptstyle \Psi^*}\downarrow & & \downarrow{\scriptstyle \Psi^*} \\ \mathrm{Hom}_H(B,U) & \xrightarrow{\Phi_*} & \mathrm{Hom}_H(B,V) \end{array} \tag{11.5.1}$$

Here Φ_* is postcomposition with Φ and Ψ^* is precomposition with Ψ.

In what follows H will be the group K_∞. We now define K_∞-module homomorphisms Φ and Ψ that will concern us here. We begin with the $(\mathfrak{g}, K_\infty)$-module homomorphism Φ. Let $H_{A_\mathfrak{q}(\lambda)}$ be the subspace of smooth vectors in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ which is the sum of the spaces H_σ of those representations $\sigma \in \mathcal{A}^c(\mathrm{SO}(V))$ such that

- $\sigma_{v_0}|_{\mathrm{SO}_0(p,q)} = A_\mathfrak{q}(\lambda)$.
- σ_v is the trivial representation for all the infinite places $v \neq v_0$ (note that at such places v we have $G(F_v) \cong \mathrm{SO}(p+q)$).
- σ is in the image of the cuspidal ψ-theta correspondence from $\mathrm{Mp}(X)$.

As explained just above Corollary 6.10, Remark 8.31 forces the dimension of the symplectic space X to be exactly $2n$. We now realize $H_{A_\mathfrak{q}(\lambda)}$ as a subspace in

$$L^2(\widetilde{G}(F)\backslash \mathrm{SO}_0(p,q) \times \widetilde{G}(\mathbb{A}_f)).$$

As explained in §9.6 we have:

$$H^{nq}(\mathrm{Sh}(G), S_{[\lambda]}(V))_{A_\mathfrak{q}(\lambda)} \cong H^{nq}(\mathfrak{g}, K_\infty, H_{A_\mathfrak{q}(\lambda)} \otimes S_{[\lambda]}(V)).$$

But by Proposition 5.4 of [80] we have

$$H^{nq}(\mathfrak{g}, K_\infty, H_{A_\mathfrak{q}(\lambda)} \otimes S_{[\lambda]}(V)) \cong \mathrm{Hom}_{K_\infty}(V(n,\lambda), H_{A_\mathfrak{q}(\lambda)}). \tag{11.5.2}$$

Now let π_0' (resp. π') be the holomorphic discrete series representation of $\mathrm{Mp}(2n,\mathbb{R})$ with lowest K-type (having highest weight) $S_\lambda(\mathbb{C}^n) \otimes \mathbb{C}_{\frac{m}{2}}$ (resp. $\mathbb{C}_{\frac{m}{2}}$). As recalled in §4.1 the lowest K_∞-type $V(n,\lambda)$ has a canonical lift $\widetilde{V(n,\lambda)}$ to $\mathrm{O}(p)\times\mathrm{O}(q)$. We let $\widetilde{A_\mathfrak{q}(\lambda)}$ be the unique irreducible unitary representation of $\mathrm{O}(p,q)$ with lowest K-type $\widetilde{V(n,\lambda)}$ and the same infinitesimal character as $A_\mathfrak{q}(\lambda)$, see [32, §6.1] for more details. It follows from [53] that π_0' corresponds to $\widetilde{A_\mathfrak{q}(\lambda)}$ under the local theta correspondence $\mathrm{Mp}(2n,\mathbb{R})\times\mathrm{O}(p,q)$ and that π' corresponds to the trivial representation of $\mathrm{O}(p+q)$ under the local theta correspondence $\mathrm{Mp}(2n,\mathbb{R})\times\mathrm{O}(p+q)$. Let $H'_{\pi_0'}$ be the subspace of $L^2(\mathrm{Mp}(2n,\mathbb{Q})\backslash\mathrm{Mp}(2n,\mathbb{A}))$ which is the sum of the subspaces of smooth vectors $H'_{\sigma'}$ of those representations $\sigma' \in \mathcal{A}^c(\mathrm{Mp}(X))$ such that

- $\sigma'_{v_0} = \pi'_0$.
- $\sigma'_v = \pi'$ for all the infinite places $v \neq v_0$ (note that at such places v we have $G(F_v) \cong \mathrm{SO}(p+q)$).

We realize the oscillator representation as a $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module in the subspace

$$\mathbf{S}(V(F_{v_0})^n) \times \mathcal{S}(V(\mathbb{A}_f)^n) \subset \mathbf{S}(V(\mathbb{A})^n).$$

Here the inclusion maps an element $(\varphi_\infty, \varphi)$ of the right-hand side to

$$\phi = \varphi_\infty \otimes \big(\bigotimes_{\substack{v|\infty \\ v\neq v_0}} \varphi_0 \big) \otimes \varphi$$

where the factors φ_0 at the infinite places v not equal to v_0 are Gaussians, the unique element (up to scalar multiples) of the Fock space $\mathcal{P}(V(F_v)^n)$ which is fixed by the compact group $\mathrm{SO}(V(F_v))$. We abusively write elements of $\mathbf{S}(V(F_{v_0})^n)\times\mathcal{S}(V(\mathbb{A}_f)^n)$ as $\varphi_\infty \otimes \varphi$.

From now on we abreviate $H = H_{A_{\mathfrak{q}}(\lambda)}$, $H' = H'_{\pi'_0}$ and

$$\mathbf{S} = \mathbf{S}(V(F_{v_0})^n) \times \mathcal{S}(V(\mathbb{A}_f)^n).$$

It follows from the definition of the global theta lift (see §2.4) that for any $f \in H'$ and $\phi \in \mathbf{S}$ the map $f \otimes \phi \mapsto \theta^f_{\psi,\phi}$ is a $(\mathfrak{g}, K_\infty)$-module homomorphism from $H' \otimes \mathbf{S}$ to the space H. We will drop the dependence of ψ henceforth and abbreviate this map to θ whence $f\otimes\phi \mapsto \theta(f\otimes\phi)$. Then in the diagram (11.5.1) we take $U = H'\otimes\mathbf{S}$ and $V = H$ and $\Phi = \theta$.

We now define the map Ψ. We will take A as above to be the vector space $\wedge^{nq}(\mathfrak{p}) \otimes S_{[\lambda]}(V)^*$, the vector space B to be the submodule $V(n,\lambda)$ and Ψ to be the inclusion $i_{V(n,\lambda)} : V(n,\lambda) \to \wedge^{nq}(\mathfrak{p}) \otimes S_{[\lambda]}(V)^*$ (note that there is a unique embedding up to scalars and the scalars are not important here).

From the general diagram (11.5.1) we obtain the desired commutative diagram (11.5.3)

$$\begin{array}{ccc} H' \otimes \mathrm{Hom}_{K_\infty}(\wedge^{nq}(\mathfrak{p}) \otimes S_{[\lambda]}(V)^*, \mathbf{S}) & \xrightarrow{\theta_*} & \mathrm{Hom}_{K_\infty}(\wedge^{nq}(\mathfrak{p}) \otimes S_{[\lambda]}(V)^*, H) \\ \Big\downarrow {\scriptstyle i^*_{V(n,\lambda)}} & & \Big\downarrow {\scriptstyle i^*_{V(n,\lambda)}} \\ H' \otimes \mathrm{Hom}_{K_\infty}(V(n,\lambda), \mathbf{S}) & \xrightarrow{\theta_*} & \mathrm{Hom}_{K_\infty}(V(n,\lambda), H) = H^{nq}(\mathrm{Sh}(G), S_{[\lambda]}(V)) \end{array}$$

Then

$$H^{nq}_\theta(\mathrm{Sh}(G), S_{[\lambda]}(V)) = \mathrm{Image}(i^*_{V(n,\lambda)} \circ \theta_*).$$

Now we examine the diagram. Since $V(n,\lambda)$ is a summand, the map on the left is *onto*. Also by Corollary 5.6 the map on the right is an isomorphism.

We now define $U_{\varphi_{nq,[\lambda]}}$ to be the subspace of $\mathrm{Hom}_{K_\infty}(\wedge^{nq}(\mathfrak{p}) \otimes S_{[\lambda]}(V)^*, \mathbf{S})$ defined by $\varphi_\infty = \varphi_{nq,[\lambda]}$.

The theorem is then equivalent to the equation

$$i^*_{V(n,\lambda)} \circ \theta_*(H' \otimes U_{\varphi_{nq,[\lambda]}}) = \mathrm{Image}(i^*_{V(n,\lambda)} \circ \theta_*). \tag{11.5.4}$$

Since the above diagram is commutative, equation (11.5.4) holds if and only if we have

(11.5.5)

$$\theta_* \circ i^*_{V(n,\lambda)}(H' \otimes U_{\varphi_{nq,[\lambda]}}) = \mathrm{Image}(i^*_{V(n,\lambda)} \circ \theta_*) = H^{nq}_\theta(\mathrm{Sh}(G), S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}.$$

Put $\overline{U}_{\varphi_{nq,[\lambda]}} = i^*_{V(n,\lambda)}(U_{\varphi_{nq,[\lambda]}})$. Since the left-hand vertical arrow $i^*_{V(n,\lambda)}$ is onto, equation (11.5.5) holds if and only if

$$\theta_*(H' \otimes \overline{U}_{\varphi_{nq,[\lambda]}}) = \theta_*(H' \otimes \mathrm{Hom}_{K_\infty}(V(n,\lambda), \mathbf{S})). \tag{11.5.6}$$

We now prove equation (11.5.6).

To this end let $\xi \in \theta_*(H' \otimes \mathrm{Hom}_{K_\infty}(V(n,\lambda),\mathbf{S}))$. It is enough to consider the case where ξ is the image by θ_* of a pure tensor. Hence, by definition, there exists $\phi = \varphi_\infty \otimes \varphi \in \mathrm{Hom}_{K_\infty}(V(n,\lambda),\mathbf{S})$ and $f \in H'$ such that

$$\theta_*(f \otimes \phi) = \xi. \tag{11.5.7}$$

We claim that in equation (11.5.7) (up to replacing the component f_{v_0}) we may replace the factor φ_∞ of ϕ by $\varphi_{nq,[\lambda]}$ without changing the right-hand side ξ of equation (11.5.7). Indeed by Theorem 8.23 there exists $Z \in U(\mathfrak{sp}_{2n})$ such that

$$\varphi_{v_0} = Z\varphi_{nq,[\lambda]}. \tag{11.5.8}$$

Now by Lemma 11.4 (with slightly changed notation) we have

$$\theta_*(f \otimes Z\phi) = \theta_*(Z^* f \otimes \phi). \tag{11.5.9}$$

Hence setting $f' = Z^* f$ we obtain, for all $f \in H'$,

$$\begin{aligned} \xi &= \theta_*(f \otimes (\varphi_\infty \otimes \varphi)) \\ &= \theta_*(f \otimes (Z\varphi_{nq,[\lambda]} \otimes \varphi)) \\ &= \theta_*(Z^* f \otimes (\varphi_{nq,[\lambda]} \otimes \varphi)) \\ &= \theta_*(f' \otimes (\varphi_{nq,[\lambda]} \otimes \varphi)). \end{aligned} \tag{11.5.10}$$

We conclude that the image of the space $H' \otimes U_{\varphi_{nq,[\lambda]}}$ under θ_* coincides with the image of $H' \otimes \mathrm{Hom}_{K_\infty}(V(n,\lambda),\mathbf{S})$ as required.

□

Remark. The reader will observe that equation (11.5.2) plays a key role in the paper. Roughly speaking, it converts problems concerning the functor

$$H^{nq}(\mathfrak{g}, K_\infty, \bullet \otimes S_{[\lambda]}(V))$$

on $(\mathfrak{g}, K_\infty)$-modules to problems concerning the functor $\mathrm{Hom}_{K_\infty}(V(n,\lambda),\bullet)$. The first functor is not exact whereas the second is.

11.6. **Special cycles span.** In this section we will prove that the special cycles span at any fixed level. First as a consequence of Theorem 11.5 we have:

11.7. **Proposition.** *We have an inclusion*

$$H^{nq}_\theta(\mathrm{Sh}(G), S_{[\lambda]}(V))_{A_\mathfrak{q}(\lambda)} \subset SC^{nq}(\mathrm{Sh}(G), S_{[\lambda]}(V)).$$

Proof. By Theorem 11.5, in the statement of Proposition 11.7 we may replace

$$H^{nq}_\theta(\mathrm{Sh}(G), S_{[\lambda]}(V))_{A_\mathfrak{q}(\lambda)}$$

by

$$H^{nq}_{\theta_{nq,\lambda}}(\mathrm{Sh}(G), S_{[\lambda]}(V))_{A_\mathfrak{q}(\lambda)}.$$

We then use Proposition 10.8. Since cusp forms are rapidly decreasing we can pair classes in $H^{nq}_{\mathrm{cusp}}(\mathrm{Sh}(G), S_{[\lambda]}(V))$ with classes in $H^{(p-n)q}_{\mathrm{cusp}}(\mathrm{Sh}(G), S_{[\lambda]}(V)^*)$. We denote by $\langle , \rangle$ this pairing. It is a perfect pairing. Hence letting $SC^{nq}(\mathrm{Sh}(G), S_{[\lambda]}(V))^\perp$ and $H^{nq}_\theta(\mathrm{Sh}(G), S_{[\lambda]}(V))^\perp_{A_\mathfrak{q}(\lambda)}$ denote the respective annihilators in $H^{(p-n)q}_{\mathrm{cusp}}(\mathrm{Sh}(G), S_{[\lambda]}(V)^*)$ it suffices to prove

$$SC^{nq}(\mathrm{Sh}(G), S_{[\lambda]}(V))^\perp \subset H^{nq}_\theta(\mathrm{Sh}(G), S_{[\lambda]}(V))^\perp_{A_\mathfrak{q}(\lambda)}. \tag{11.7.1}$$

To this end let $\eta \in SC^{nq}(\mathrm{Sh}(G), S_{[\lambda]}(V))^{\perp} \subset H_{\mathrm{cusp}}^{(p-n)q}(\mathrm{Sh}(G), S_{[\lambda]}(V)^*)$. Assume η is K-invariant for some level K. It then follows from proposition 10.8 (see [25, Theorem 7.7] in the classical setting) that for $g' \in \mathrm{Mp}_{2n}(\mathbb{R})^d \subset \mathrm{Mp}_{2n}(\mathbb{A})$ the Fourier expansion of the Siegel modular form

$$\theta_{\varphi}(\eta) := \langle [\theta_{nq,\lambda}(g', \varphi)], \eta \rangle (= \int_{X_K} \theta_{nq,\lambda}(g', \varphi) \wedge \eta)$$

is given by

$$\theta_{\varphi}(\eta) = \sum_{\beta \geq 0} \langle [\beta, \varphi]_{\lambda}, \eta \rangle W_{\beta}(g').$$

In particular: since η is orthogonal to the subspace

$$SC^{nq}(\mathrm{Sh}(G), S_{[\lambda]}(V)) \subset H_{\mathrm{cusp}}^{nq}(\mathrm{Sh}(G), S_{[\lambda]}(V))$$

generated by the projections of the forms in the image of $[\beta, \varphi]_{\lambda}$ then all the Fourier coefficients of the Siegel modular form $\theta_{\varphi}(\eta)$ vanish and therefore $\theta_{\varphi}(\eta) = 0$. But then for any $f \in H_{\sigma'}$ ($\sigma' \in \mathcal{A}^c(\mathrm{Mp}(X))$), we have:

$$\int_{X_K} \eta \wedge \theta_{nq,[\lambda]}(f \otimes \varphi) = \int_{\mathrm{Mp}(X)\backslash \mathrm{Mp}_{2n}(\mathbb{A})} \theta_{\varphi}(\eta) f(g') dg' = 0.$$

This forces η to be orthogonal to all forms $\theta_{nq,[\lambda]}(f \otimes \varphi)$ hence we have verified equation (11.7.1) and hence proved the proposition. □

11.8. *Remark.* We do not know if the reverse inclusion holds in proposition 11.7. It obviously follows from theorem 9.9 that it holds when $p > 2n$ and $m-1 > 3n$ but the relation between cycles and θ-lifts is less transparent near the middle degree, we address this problem in conjecture 16.11.

11.9. Now Proposition 11.7 and Theorem 9.9 imply[11] that if $p > 2n$ and $m-1 > 3n$ the natural map

$$SC^{nq}(\mathrm{Sh}(G), S_{[\lambda]}(V)) \to H_{\mathrm{cusp}}^{nq}(\mathrm{Sh}^0(G), S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}$$

is surjective.

Taking invariants under a compact open subgroup $K \subset G(\mathbb{A}_f)$ we get the following:

11.10. **Theorem.** *Suppose $p > 2n$ and $m - 1 > 3n$. Then, for any compact open subgroup $K \subset G(\mathbb{A}_f)$, the natural map*

$$SC^{nq}(X_K, S_{[\lambda]}(V)) \to H_{\mathrm{cusp}}^{nq}(Y_K, S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}$$

is surjective. In particular:

$$Z^{nq}(Y_K, S_{[\lambda]}(V)) = H_{\mathrm{cusp}}^{nq}(Y_K, S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}.$$

11.11. *Remark.* As motivated in §6.12 we believe that the conditions $p > 2n$ and $m - 1 > 3n$ are necessary. The decomposition of $L^2(G(F)\backslash G(\mathbb{A}))$ into irreducible automorphic representations yields a decomposition of $H^{\bullet}(\mathrm{Sh}^0(G), S_{[\lambda]}(V))$ and, when $m - 1 \leq 3n$, one may try to classify which automorphic representations contribute to the part of the cohomology generated by special cycles. This is adressed in Theorem 16.10 and Conjecture 16.11.

[11] Note that the $(\mathfrak{g}, K_{\infty})$-module $A_{\mathfrak{q}}(\lambda)$ is associated to the Levi subgroup $L = \mathrm{SO}(p-2n, q) \times \mathrm{U}(1)^n$.

Part 4. **Applications**

12. Hyperbolic manifolds

12.1. **Notations.** Let notations be as in the preceeding section. Assume moreover that G is anisotropic over F and that $q = 1$. For any compact open subgroup $K \subset G(\mathbb{A}_f)$, the connected component Y_K is therefore a closed congruence hyperbolic p-manifold. These are called "of simple type" in the introduction. We point out that in that case the Euler form is trivial (q is odd) so that special cycles are totally geodesic cycles.

12.2. **Proof of Theorem 1.5.** First note that if $n < \frac{p}{3}$ then $2n < p$ and $3n < p = m - 1$ so that Theorem 11.10 applies. We consider the case where $\lambda = 0$.

Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. Since Y_K is closed the cohomology of Y_K is the same as its cuspidal cohomology. And it follows from the first example of §5.11 that $A_{\mathfrak{q}}(0)$ — with $\mathfrak{q}$ as in §8.10 — is the unique cohomological module with trivial coefficients which occurs in degree n. The natural projection map

$$H^n(Y_K, \mathbb{C}) \to H^n_{\mathrm{cusp}}(Y_K, \mathbb{C})_{A_{\mathfrak{q}}(0)}$$

is therefore an isomorphism and we conclude from Theorem 11.10 that $H^n(Y_K, \mathbb{C})$ is spanned by the classes of the special (totally geodesic) cycles. As these define *rational* cohomology classes Theorem 1.5 follows.

Remark. As motivated in §6.12, the bound $n < \frac{p}{3}$ is certainly optimal in general as cohomology classes of 3-dimensional hyperbolic manifolds are not generated by classes of special classes in general, see Proposition 16.15 for an explicit counterexample when $p = 3$ and $n = 1$.

12.3. **Proof of Theorem 1.7.** First note that if $p \geq 4$ then $m - 1 > 3$ so that Theorem 11.10 applies. We consider the case where $\lambda = (1, 0, \dots, 0)$ or $(2, 0, \dots, 0)$. In the first case $S_{[\lambda]}(V) \cong \mathbb{C}^{p+1}$ and in the second case $S_{[\lambda]}(V)$ is isomorphic to the complexification of $\mathcal{H}^2(\mathbb{R}^{p+1})$ — the space of harmonic (for the Minkowski metric) degree two polynomials on $\mathbb{R}^{p+1}$.

Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. Since Y_K is closed the cohomology of Y_K is the same as its cuspidal cohomology. And it follows from the second example of §5.11 that $A_{\mathfrak{q}}(\lambda)$ — with $\mathfrak{q}$ as in §8.10 — is the unique cohomological module with coefficients in $S_{[\lambda]}(V)$ which occurs in degree 1. The natural projection map

$$H^1(Y_K, S_{[\lambda]}(V)) \to H^1_{\mathrm{cusp}}(Y_K, S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}$$

is therefore an isomorphism and we conclude from Theorem 11.10 that $H^1(Y_K, S_{[\lambda]}(V))$ is spanned by the classes of the special cycles. As these define *real* cohomology classes Theorem 1.7 follows.

13. Shimura varieties associated to $\mathrm{O}(p, 2)$

13.1. **Notations.** Let notations be as in section 11. Assume moreover that G is anisotropic over F and that $q = 2$. For any compact open subgroup $K \subset G(\mathbb{A}_f)$, the connected component Y_K is therefore a closed projective complex manifold of (complex) dimension p. These are the connected Shimura varieties associated to $\mathrm{O}(p, 2)$ in the introduction. We consider the case where $\lambda = 0$.

13.2. **The Lefschetz class.** In that setting D is a bounded symmetric domain in $\mathbb{C}^p$. Let $k(z_1, z_2)$ be its Bergmann kernel function and $\Omega(z) = \partial\overline{\partial} \log k(z, z)$ the associated Kähler form. For each compact open subgroup $K \subset G(\mathbb{A}_f)$, $\frac{1}{2i\pi}\Omega$ induces a $(1,1)$-form on Y_K which is the Chern form of the canonical bundle $\mathcal{K}_{Y_K}$ of Y_K. As such it defines a *rational* cohomology class dual to (a possible rational multiple of) a complex subvariety: given a projective embedding this is the class of (a possible rational multiple of) a hyperplane section. Moreover: the cup product with $\frac{1}{2i\pi}\Omega$ induces the Lefschetz operator

$$L : H^k(Y_K, \mathbb{Q}) \to H^{k+2}(Y_K, \mathbb{Q})$$

on cohomology. This is the same operator as the multiplication by the Euler form e_2.

13.3. According to the Hard Lefschetz theorem the map

$$L^k : H^{p-k}(Y_K, \mathbb{C}) \to H^{p+k}(Y_K, \mathbb{C})$$

is an isomorphism; and if we define the *primitive* cohomology

$$H^{p-k}_{\mathrm{prim}}(Y_K, \mathbb{C}) = \ker(L^{k+1} : H^{p-k}(Y_K, \mathbb{C}) \to H^{p+k+2}(Y_K, \mathbb{C}))$$

then we have the *Lefschetz decomposition*

$$H^m(Y_K, \mathbb{C}) = \oplus_k L^k H^{m-2k}_{\mathrm{prim}}(Y_K, \mathbb{C}).$$

The Lefschetz decomposition is compatible with the Hodge decomposition. In particular we set:

$$H^{n,n}_{\mathrm{prim}}(Y_K, \mathbb{C}) = H^{n,n}(Y_K, \mathbb{C}) \cap H^{2n}_{\mathrm{prim}}(Y_K, \mathbb{C}).$$

And it follows from the third example of §5.11 that

$$H^{n,n}_{\mathrm{prim}}(Y_K, \mathbb{C}) = H^{2n}(Y_K, \mathbb{C})_{A_{n,n}}.$$

13.4. **Proof of Theorem 1.9.** First note that if $n < \frac{p+1}{3}$ then $2n < p$ and $3n < p + 1 = m - 1$ so that Theorem 11.10 applies. Note moreover that $A_{\mathfrak{q}}(0) = A_{n,n}$.

Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. Since Y_K is closed the cohomology of Y_K is the same as its cuspidal cohomology. The natural projection map

$$H^{2n}(Y_K, \mathbb{C}) \to H^{2n}_{\mathrm{cusp}}(Y_K, \mathbb{C})_{A_{\mathfrak{q}}(0)}$$

is nothing else but the projection onto $H^{n,n}_{\mathrm{prim}}(Y_K, \mathbb{C})$ and we conclude from Theorem 11.10 that $H^{n,n}_{\mathrm{prim}}(Y_K, \mathbb{C})$ is spanned by the projection of the classes of the special cycles. Those are complex subvarieties of Y_K.

To conclude the proof we recall that

$$H^{n,n}(Y_K, \mathbb{C}) = \oplus_k L^k H^{n-k,n-k}_{\mathrm{prim}}(Y_K, \mathbb{C}).$$

And since $n-k \leq n$ the preceeding paragraph applies to show that $H^{n-k,n-k}_{\mathrm{prim}}(Y_K, \mathbb{C})$ is spanned by the projection of classes of complex subvarieties of Y_K. As wedging with L^k amounts to take the intersection with another complex subvariety we conclude that the whole $H^{n,n}(Y_K, \mathbb{C})$ is spanned by the classes of complex subvarieties of Y_K. These are rational classes we can therefore conclude that every rational cohomology class of type (n, n) on Y_K is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of Y_K.

14. Arithmetic manifolds associated to SO(p,q)

14.1. In the general case where $G = \mathrm{SO}(p,q)$ we can proceed as in the preceeding section to deduce Corollary 1.16 from Theorem 1.15 and Proposition 5.17. Note however that in this general case we have to use both totally geodesic cycles associated to $\mathrm{SO}(p-n,q)$ (related to cohomological representations associated to a Levi of the form $L = C \times \mathrm{SO}_0(p-2n,q)$) and totally geodesic cycles associated to $\mathrm{SO}(p,q-n)$ (related to cohomological representations associated to a Levi of the form $L = C \times \mathrm{SO}_0(p,q-2n)$).

14.2. **Proof of Theorem 1.22.** Recall that the arithmetic manifold $Y = Y_K$ is associated to a non-degenerate quadratic space $(V,(\,,\,))$ over a totally real field F. Consider a totally imaginary quadratic extension L/F and let τ be the corresponding Galois involution. Then

$$h(x,y) = (x, \tau(y)) \quad (x, y \in V \otimes_F L)$$

defines a non-degenerate hermitian form. We let H be the corresponding special unitary group $\mathrm{SU}(h)$. Then H is defined over F and $H(F \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathrm{SU}(p,q) \times \mathrm{SU}(m)^{d-1}$ where d is the degree of F.

We define $\mathrm{Sh}(H)$ as in the orthogonal case and consider $S_\lambda(V)$ as finite H-module (here V is complexified). The group $\mathrm{U}(h)$ is a member of a reductive dual pair

$$\mathrm{U}(W) \times \mathrm{U}(h) \subset \mathrm{Sp}(V \otimes W)$$

where $\mathrm{U}(W)$ is the centralizer of some positive definite complex structure on W defined over F. We can suppose that our choice of a positive definite complex structure J on W is precisely this one. In this way we can define the (ψ-)theta correspondence from $\mathrm{U}(W)$ to $\mathrm{U}(h)$ where the test functions ϕ still vary in $\mathbf{S}(V(\mathbb{A})^n)$.

At the place v_0 we have introduced the cocycle ψ_q, see §8.5. We set

$$\psi_{nq,\ell} = (\underbrace{\psi_q \wedge \ldots \wedge \psi_q}_{n \text{ times}}) \cdot \varphi_{0,\ell}$$

and

$$\psi_{nq,\lambda} = (1 \otimes \pi_\lambda) \circ \psi_{nq,\ell}(\bullet) \circ \iota_\lambda,$$

where $\pi_\lambda : V^{\otimes \ell} \to S_\lambda(V)$ is the natural projection. We may therefore define

$$H^{nq,0}_{\theta_{nq,\lambda}}(\mathrm{Sh}(H), S_\lambda(V))$$

as the subspace of $H^{nq,0}(\mathrm{Sh}(H), S_\lambda(V))$ generated by those $\sigma \in \mathcal{A}^c(\mathrm{SU}(h))$ that are in the image of the cuspidal ψ-theta correspondence from $\mathrm{U}(W)$ where the infinite components φ_v of the global Schwartz function ϕ satisfy

$$\varphi_{v_0} = \psi_{nq,\lambda} \quad \text{and} \quad \varphi_v = \varphi_0, \; v|\infty, \; v \neq v_0.$$

We consider the natural map

$$H^{nq,0}_{\theta_{nq,\lambda}}(\mathrm{Sh}(H), S_\lambda(V)) \to H^{nq}_{\theta_{nq,\lambda}}(\mathrm{Sh}(G), S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}. \tag{14.2.1}$$

obtained by composing the restriction map with the projection $\pi_{[\lambda]}$ onto the harmonic tensors.

14.3. **Lemma.** *The map* (14.2.1) *is onto.*

Proof. This follows from two facts: First the reductive dual pairs $(\mathrm{U}(W), \mathrm{U}(h))$ and $(\mathrm{Mp}(W), \mathrm{SO}(V))$ form a see-saw pair, in the terminology of Kudla [44]. Secondly, the restriction of the holomorphic form $\psi_{nq,\lambda}$ composed with the projection $\pi_{[\lambda]}$ onto the harmonic tensors is equal to $\varphi_{nq,[\lambda]}$, see §8.5.

Here are some more details: first note that the image of the map (14.2.1) is invariant under Hecke operators (on $\mathrm{Sh}(G)$). It is therefore enough to prove that the map (14.2.1) is surjective up to a Hecke translation. Now suppose that there is a form α representing a class in $H^{nq}_{\theta_{nq,\lambda}}(\mathrm{Sh}(G), S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}$ that is orthogonal to the image of the restriction map (14.2.1). Then, the integral of α against the restriction of any theta lift from $\mathrm{U}(W)$ vanishes, and by the see-saw identity, we conclude that α is a lift of a Siegel modular form f on $\mathrm{Mp}(W)$ whose integral against any form on $\mathrm{U}(W)$ vanishes. Applying Hecke operators to α we even conclude that the integral of any Hecke translate of f against any form on $\mathrm{U}(W)$ vanishes. This forces f to vanish on a dense subset and therefore to vanish everywhere. We conclude that α is trivial. □

It therefore follows from Theorem 11.5 and Theorem 9.9 that if $2n < p$ and $3n < m-1$, the natural map:

$$H^{nq,0}(\mathrm{Sh}(U), S_\lambda(V)) \to H^{nq}_{\mathrm{cusp}}(\mathrm{Sh}^0(G), S_{[\lambda]}(V))_{A_{\mathfrak{q}}(\lambda)}$$

is surjective. Taking invariants under a compact open subgroup $L \subset U(\mathbb{A}_f)$ such that $L \cap G(F) = K$ we get Theorem 1.22 with

$$Y^{\mathbb{C}} = \Lambda_L \backslash D^{\mathbb{C}},$$

where

$$D^{\mathbb{C}} = \mathrm{SU}(p,q)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$$

and Λ_L is the image of $U(F) \cap L$ inside $\mathrm{SU}(p,q)$.

15. Growth of Betti numbers

15.1. **Notations.** Let F be a totally real field and $\mathcal{O}$ its ring of integer. Let V be a nondegenerate quadratic space over F with $\dim_F V = m$. We assume that $G = \mathrm{SO}(V)$ is anisotropic over F and compact at all but one infinite place. We denote by v_0 the infinite place where $\mathrm{SO}(V)$ is non compact and assume that $G(F_{v_0}) = \mathrm{SO}(p,1)$. We fix L an integral lattice in V and let $\Phi = G(\mathcal{O})$ be the subgroup of $G(F)$ consisting of those elements that take L into itself. We let $\mathfrak{b}$ be an ideal in $\mathcal{O}$ and let $\Gamma = \Phi(\mathfrak{b})$ be the congruence subgroup of Φ of level $\mathfrak{b}$ (that is, the elements of Φ that are congruent to the identity modulo $\mathfrak{b}$). We let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ which we assume to be prime to $\mathfrak{b}$. Set $\Gamma(\mathfrak{p}^k) = \Phi(\mathfrak{b}\mathfrak{p}^k)$. The quotients $Y_{\Gamma(\mathfrak{p}^k)}$ are real hyperbolic p-manifolds and, as first explained in [18], the combination of Theorem 9.9 and works of Cossutta [17, Theorem 2.16] and Cossutta-Marshall [18, Theorem 1] implies the following strengthening of a conjecture of Sarnak and Xue [73] in this case.

15.2. **Theorem.** *Suppose $i < \frac{p}{3}$. Then:*

$$b_i(\Gamma(\mathfrak{p}^k)) \ll \mathrm{vol}(Y_{\Gamma(\mathfrak{p}^k)})^{\frac{2i}{p}}.$$

Proof. Cossutta and Marshall prove this inequality with $b_i(\Gamma(\mathfrak{p}^k))$ replaced by the dimension of the part of $H^i(\Gamma(\mathfrak{p}^k))$ which come from θ-lift. But it follows from the first example of §5.11 that only one cohomological module can contribute to

$H^i(\Gamma(\mathfrak{p}^k))$ and since $i < \frac{p}{3}$ it follows from Theorem 9.9 that the classes which come from θ-lifts generate $H^i(\Gamma(\mathfrak{p}^k))$. Theorem 15.2 follows. □

15.3. *Remark.* 1. Raising the level $\mathfrak{b}$ one can prove that the upper bound given by Theorem 15.2 is sharp when p is even, see [18, Theorem 1].

2. The main theorem of Cossutta and Marshall is not limited to hyperbolic manifolds. In conjunction with Theorem 9.9 one may get similar asymptotic results for the multiplicity of cohomological automorphic forms in orthogonal groups. In fact Theorem 9.10 relates the multiplicities of certain cohomological automorphic forms to the multiplicities of certain Siegel modular forms. The latter are much easier to deal with using limit formulas as in [16, 74]. The main issue then is to control the level; this is exactly what Cossutta and Marshall manage to do.

16. Periods of automorphic forms

16.1. **Notations.** We keep notations as in §9.1 and 10.1. In particular we let F be a totally real field of degree d and denote by $\mathbb{A}$ its ring of adeles of F. Let V be a nondegenerate quadratic space over F with $\dim_F V = m$. We assume that $G = \mathrm{SO}(V)$ is compact at all but one infinite place. We denote by v_0 the infinite place where $\mathrm{SO}(V)$ is non compact and assume that $G(F_{v_0}) = \mathrm{SO}(p,q)$. Hereafter U will always denote a totally positive definite subquadratic space of dimension $n \leq p$ in V. And we denote by H the group $\mathrm{SO}(U^\perp)$.

We furthermore let (π, V_π) be an irreducible $(\mathfrak{g}, K_\infty)$-module such that

$$H^k(\mathfrak{g}, K_\infty; V_\pi \otimes E) \neq 0$$

for some finite dimensional irreducible representation (ρ, E) of $\mathrm{SO}_0(p,q)$ of dominant weight λ with at most n nonzero entries. Recall that $K_\infty = \mathrm{SO}(p) \times \mathrm{SO}(q) \subset \mathrm{SO}_0(p,q)$.

For any cusp form f in $L^2(\widetilde{G}(F)\backslash(\mathrm{SO}(p,q) \times \widetilde{G}(\mathbb{A}_f))$ and any character χ of $\widehat{\pi}_0 = \mathbb{A}^*/F_c^*$, we define the period integral

$$P(f,U,\chi) = \int_{\widetilde{H}(F)\backslash(\mathrm{SO}(p-n,q)\times\widetilde{H}(\mathbb{A}_f))} f(h)\chi(\mathrm{Nspin}_{U^\perp}(h))dh$$

where $dh = \otimes_v dh_v$ is a fixed Haar measure on $\widetilde{H}(F)\backslash(\mathrm{SO}(p-n,q) \times \widetilde{H}(\mathbb{A}_f))$.

16.2. **Distinguished representations.** Let (σ, V_σ) be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ which occurs as an irreducible subspace V_σ in the space of cuspidal automorphic functions in $L^2(G(F)\backslash G(\mathbb{A}))$ and such that σ_v is trivial for any infinite place $v \neq v_0$.

Note that a function $f \in V_\sigma$ lifts, in a canonical way, as a function in

$$L^2(\widetilde{G}(F)\backslash(\mathrm{SO}(p,q) \times \widetilde{G}(\mathbb{A}_f))).$$

By a slight abuse of notation we still denote by f this function. We call $P(f,U,\chi)$ the (χ,U)-period of f. We write $P(\sigma,U,\chi) \neq 0$ if there exists $f \in V_\sigma$ such that $P(f,U,\chi)$ is nonzero and $P(\sigma,U,\chi) = 0$ otherwise. Let us say that σ is χ-*distinguished* if $P(\sigma,U,\chi) \neq 0$ for some U.

16.3. Given a place v of F we denote by $V(\sigma_v)$ the space of the representation σ_v. We furthermore fix an isomorphism

$$\tau : \otimes'_v V(\sigma_v) \to V_\sigma \subset L^2(G(F)\backslash G(\mathbb{A})).$$

We finally write $\sigma_f = \otimes'_{v \not| \infty} \sigma_v$ and $V(\sigma_f) = \otimes'_{v \not| \infty} V(\sigma_v)$.

We will only be concerned with the very special automorphic representations (σ, V_σ) such that the restriction of σ_{v_0} to $\mathrm{SO}_0(p,q)$ is isomorphic to π. We abusively write "$\sigma_{v_0} \cong \pi$" for "the restriction of σ_{v_0} to $\mathrm{SO}_0(p,q)$ is isomorphic to π".

Classes in the cuspidal cohomology $H^\bullet_{\mathrm{cusp}}(\mathrm{Sh}(G), E)$ are represented by cuspidal automorphic forms. We may therefore decompose $H^\bullet_{\mathrm{cusp}}(\mathrm{Sh}(G), E)_\pi$ as a sum

$$H^\bullet_{\mathrm{cusp}}(\mathrm{Sh}(G), E)_\pi = \bigoplus_{(\sigma, V_\sigma)\,:\,\sigma_{v_0} \cong \pi} H^\bullet_{\mathrm{cusp}}(\mathrm{Sh}(G), E)(\sigma)$$

where we sum over irreducible cuspidal automorphic representations (σ, V_σ) of $G(\mathbb{A})$ which occurs as an irreducible subspace V_σ in the space of cuspidal automorphic functions in $L^2(G(F)\backslash G(\mathbb{A}))$.

16.4. Recall from §10.2 and 10.4 that for K small enough we have associated to a subspace U and a tableau T some connected cycles-with-coefficient $c(U, g_j, K)_T = c(U, g_j, K) \otimes \epsilon_T$ where $\epsilon_T \in E$. We restrict to the case where the g_j's belong to $H(\mathbb{A}_f)$ so that the $c(U, g_j, K)$ are the images of the connected components of

$$\widetilde{H}(F)\backslash(\mathrm{SO}(p-n,q) \times \widetilde{H}(\mathbb{A}_f))/(SO(p-n) \times SO(q))\widetilde{K \cap H}. \tag{16.4.1}$$

Any character χ of finite order of $\mathbb{A}^*/F_c^*$ defines a locally constant function on (16.4.1). Let $Z^\chi_{U,T} = \sum_j \chi(\mathrm{Nspin}_U(g_j)) c(U, g_j, K)_T$ and $[Z^\chi_{U,T}]$ be the corresponding cohomology class in $H^{qn}_{\mathrm{cusp}}(X_K, E)$. We let $Z^{nq}(\mathrm{Sh}(G), E)(\sigma)$ be the subspace of $H^{nq}_{\mathrm{cusp}}(\mathrm{Sh}(G), E)(\sigma)$ spanned by the projection of the classes $[Z^\chi_{U,T}]$.

16.5. **Proposition.** *Suppose that π is associated to a Levi subgroup $L = \mathrm{SO}(p - 2n, q) \times \mathrm{U}(1)^n$ with $p - 2n \geq 0$. If the space $Z^{nq}(\mathrm{Sh}(G), E)(\sigma)$ is non-trivial then σ is χ-distinguished for some finite character χ.*

Proof. This proposition is part of [12, Theorem 6] and we recall the proof for the reader's convenience.

Recall that σ is an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ which occurs as an irreducible subspace $V_\sigma \subset L^2(G(F)\backslash G(\mathbb{A}))$. We identify V_σ as a subspace

$$V_\sigma \subset L^2(\widetilde{G}(F)\backslash(\mathrm{SO}(p,q) \times \widetilde{G}(\mathbb{A}_f))).$$

We first relate the period integral to the pairing of a cohomology class with a special cycle, as done in [12, Theorem 6].

16.6. Using the above identification we have:

$$H^{nq}_{\mathrm{cusp}}(\mathrm{Sh}(G), E)(\sigma) = \mathrm{Hom}_{K_\infty}(\wedge^{nq}\mathfrak{p}, V_\sigma \otimes E). \tag{16.6.1}$$

Recall from §5.4 that the K_∞-module $V(n)$ occurs with multiplicity one in $\wedge^{nq}\mathfrak{p}$ and that any non-zero element in the right-hand side of (16.6.1) factorizes through the isotypical component $V(n)$. We furthermore note that $V(n)$ occurs with multiplicity one in $V(\pi) \otimes E$. Since $V(\pi) = V(\sigma_{v_0})$ this leads to a canonical (up to multiples) non-zero element $\omega \in \mathrm{Hom}_{K_\infty}(V(n), V(\sigma_{v_0}) \otimes E)$.

Let $\{v_j\}$ and $\{l_j\}$ be dual bases of $V(n)$ and $V(n)^*$, respectively. Fix a neat level $K \subset G(\mathbb{A}_f)$ such that σ has K-invariant vectors. For x_f in the space $V(\sigma_f)$ of σ_f, the element

$$\omega \otimes x_f \in \mathrm{Hom}_{K_\infty}(V(n), V(\sigma_{v_0}) \otimes E) \otimes V(\sigma_f)^K$$

corresponds to a E-valued harmonic form Ω_{x_f} on

$$X_K = \widetilde{G}(F) \backslash (\mathrm{SO}(p,q) \times \widetilde{G}(\mathbb{A}_f))/K_\infty \widetilde{K}$$

which decomposes as:

$$\Omega_{x_f} = \sum_j f_{j,x_f} \otimes e_j l_j$$

where $f_{j,x_f} \in V_\sigma \subset L^2(\widetilde{G}(F)\backslash(\mathrm{SO}(p,q) \times \widetilde{G}(\mathbb{A}_f)))$ is a cusp form, $e_j \in E$ and the tensor product $f_{j,x_f} \otimes e_j$ is the image of the vector $\omega(v_j) \otimes x_f$ under the map τ. The form Ω_{x_f} depends only on x_f and ω but not on the choice of bases $\{v_j\}$ and $\{l_j\}$.

The Hodge $*$-operator establishes a one-to-one K_∞-equivariant correspondence between $\wedge^{nq}\mathfrak{p}$ and $\wedge^{(p-n)q}\mathfrak{p}$ as well as on their dual spaces. It maps Ω_{x_f} onto the E^*-valued $(p-n)q$-harmonic form $*\Omega_{x_f} = \sum_j f_{j,x_f} \otimes e_j * l_j$.

Recall from §9.3 that we have fixed a $\mathrm{SO}_0(p,q)$-invariant inner product $(,)_E$ on E and that this induces a natural inner product $\langle , \rangle$ on E-valued differential forms.

16.7. Let ω_H be a fixed non-zero element in the dual space of $\wedge^{(p-n)q}\mathfrak{p}_H$ where $\mathfrak{p}_H = \mathfrak{p} \cap \mathfrak{h}$ and $\mathfrak{h}$ is the complexified Lie algebra of $\mathrm{SO}(p-n,q)$. Since $\dim \wedge^{(p-n)q}\mathfrak{p}_H = 1$, ω_H is unique up to multiples. Each $*l_j$ may be represented as an element in the dual space of $\wedge^{(p-n)q}\mathfrak{p}$. The restriction of $*\Omega_{x_f}$ to a connected cycle $c(U,g,K)_T$ is therefore of the form $(\sum_j c_j f_{j,x_f} \otimes e_j)\omega_H$ where the c_j are complex constants. Let

$$f_{x_f,T} = \sum_j c_j (e_j, e_T)_{E^*} f_{j,x_f}.$$

Then for a suitable normalization of Haar measure dh, we have:

$$P(f_{x_f,T}, U, \chi) = \langle [Z^\chi_{U,T}], \Omega_{x_f} \rangle \tag{16.7.1}$$

where $\langle , \rangle$ denotes the inner product on differential forms. This remains true even if G is not anisotropic: the projection of $[Z^\chi_{U,T}]$ in the cuspidal part of the cohomology belongs to the L^2-cohomology.

If $Z^{nq}(\mathrm{Sh}(G), E)(\sigma) \neq \{0\}$ equation (16.7.1) therefore implies that

$$P(f_{x_f,T}, U, \chi) \neq 0$$

and σ is χ-distinguished. □

16.8. **Question.** Does the converse to Proposition 16.5 also hold ?

Answering this question seems to lie beyond the tools of this paper. As a corollary of Proposition 16.5 we nevertheless get the following:

16.9. **Theorem.** *Let σ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ which occurs as an irreducible subspace in $L^2(G(F)\backslash G(\mathbb{A}))$. Suppose that the restriction of σ_{v_0} to $\mathrm{SO}_0(p,q)$ is a cohomological representation $\pi = A_{\mathfrak{q}}(\lambda)$ whose associated Levi subgroup L is isomorphic to $\mathrm{SO}(p-2r,q) \times \mathrm{U}(1)^r$ with $p > 2r$ and $m-1 > 3r$ and such that λ has at most r nonzero entries. And suppose that for all infinite places $v \neq v_0$, the representation σ_v is trivial. Then: σ is χ-distinguished for some finite character χ.*

Proof. Applying Theorem 11.10 — with $r = n$ — to each component of X_K we conclude that the projections of the classes $[Z^{\chi}_{U,T}]$ for varying U, T and χ generate $H^{qn}_{\rm cusp}(X_K, S_{[\lambda]}(V))_{\pi}$. (Here we should note that classes obtained by wedging a cycle class with a power of the Euler form are not primitive and therefore project trivially into $H^{qn}_{\rm cusp}(X_K, S_{[\lambda]}(V))_{\pi}$.) Therefore: if $H^{nq}_{\rm cusp}(X_K, S_{[\lambda]}(V))(\sigma)$ is non trivial, then $Z^{nq}(X_K, S_{[\lambda]}(V))(\sigma) \neq \{0\}$. And Theorem 16.9 follows from Proposition 16.5. □

Remark. We believe that the conditions $p > 2r$ and $m - 1 > 3r$ are necessary in general but even assuming the results of §6.12 we would still have to answer positively to Question 16.8.

Note that [29, Theorem 1.1] and the proof of Theorem 16.9 imply the following:

16.10. **Theorem.** *Let σ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ which occurs as an irreducible subspace in $L^2(G(F)\backslash G(\mathbb{A}))$. Suppose that the restriction of σ_{v_0} to $\mathrm{SO}_0(p,q)$ is a cohomological representation $\pi = A_{\mathfrak{q}}(\lambda)$ whose associated Levi subgroup L is isomorphic to $\mathrm{SO}(p-2r, q) \times \mathrm{U}(1)^r$ with $p \geq 2r$, $q \geq 1$ and such that λ has at most r nonzero entries. And suppose that for all infinite places $v \neq v_0$, the representation σ_v is trivial. Assume moreover that there exists a quadratic character η of $F^*\backslash\mathbb{A}^*$ such that the partial L-function $L^S(s, \sigma \times \eta)$ has a pole at $s_0 = \frac{m}{2} - r$ and is holomorphic for $\mathrm{Re}(s) > s_0$. Then: σ is χ-distinguished for some finite character χ.*

We don't believe that belonging in the image of the theta correspondence — which is a subtle property away from stable range — can be characterized only by the existence of a pole for a partial L-function. We propose the following:

16.11. **Conjecture.** *Let σ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ which occurs as an irreducible subspace in $L^2(G(F)\backslash G(\mathbb{A}))$. Suppose that the restriction of σ_{v_0} to $\mathrm{SO}_0(p,q)$ is a cohomological representation $\pi = A_{\mathfrak{q}}(\lambda)$ whose associated Levi subgroup L is isomorphic to $\mathrm{SO}(p-2r, q) \times \mathrm{U}(1)^r$ with $p \geq 2r$, $q \geq 1$ and such that λ has at most r nonzero entries. And suppose that for all infinite places $v \neq v_0$, the representation σ_v is trivial. Then there exists an automorphic character χ such that $\sigma \otimes \chi$ is in the image of the cuspidal ψ-theta correspondence from a smaller group associated to a symplectic space of dimension $2r$ if and only if σ is η-distinguished for some finite quadratic character η and the global (Arthur) L-function $L(s, \sigma^{\mathrm{GL}} \times \eta)$ has a pole at $s_0 = \frac{m}{2} - r$ and is holomorphic for $\mathrm{Re}(s) > s_0$.*

16.12. **A 3-dimensional example.** Let $F = \mathbb{Q}$ and let V be the 4-dimensional $\mathbb{Q}$-vector space with basis e_1, e_2, e_3, e_4 and quadratic form

$$q(x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4) = x_1^2 - x_2^2 - x_3^2 - dx_4^2.$$

Assume that d is positive and not a square in $\mathbb{Q}$ and let

$$D = \begin{cases} 4d & \text{if } d \equiv 1, 2 \pmod 4, \\ d & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

Consider the group $G = \mathrm{Res}_{\mathbb{Q}(\sqrt{-d})/\mathbb{Q}}\mathrm{GL}(2)$. Recall — e.g. from [19, Proposition 3.14] — that G is isomorphic to $\mathrm{GSpin}(V)$ over $\mathbb{Q}$. The corresponding arithmetic manifolds are therefore the same; these are the 3-dimensional hyperbolic manifolds obtained as quotient of the hyperbolic 3-space by Bianchi groups.

Distinguished representations of G have already attracted a lot of attention: Consider periods associated to totally positive vectors in V. The corresponding groups

H are inner forms of $\mathrm{GL}(2)_{|\mathbb{Q}}$ and the corresponding cycles are totally geodesic surfaces.

16.13. **Proposition.** *Suppose that π is an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. Then if π is χ-distinguished for some finite character χ then some twist of π is the base-change lift of a cuspidal automorphic representation of* $\mathrm{GL}(2,\mathbb{A})$.

Proof. This follows from the proof of [21, §5]. □

Finis, Grunewald and Tirao [20] compare two classical ways of constructing cuspidal cohomology classes in $H^1(\mathrm{SL}(2,\mathcal{O}_{-d}))$ where $\mathcal{O}_{-d}$ is the ring of integers of $\mathbb{Q}(\sqrt{-d})$. The first is the base-change construction where the corresponding cuspidal automorphic representations are obtained as twists of base-change lifts of cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$. The second construction is via automorphic induction from Hecke characters of quadratic extensions of $\mathbb{Q}(\sqrt{-d})$. In many cases, the part of the cuspidal cohomology thus obtained is already contained in the part obtained from the base-change construction. But it is not always the case — see [20, Corollary 4.16] for a precise criterion — and they in particular prove the following:

16.14. **Proposition.** *Suppose there exists a real quadratic field L such that*

$$\mathbb{Q}(\sqrt{-d})L/\mathbb{Q}(\sqrt{-d})$$

is unramified and the narrow class number h_L^+ is strictly bigger than the corresponding number of genera $g_L^+ = 2^{\mathcal{R}(L)|-1}$, where $\mathcal{R}(L)$ denotes the set of primes ramified in L. Then: there exists a non base-change cohomology class in $H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}_{-d}))$.

The smallest discriminant of a real quadratic field L such that $h_L^+ > g_L^+$ is $d_L = 136$. Note that $\mathbb{Q}(\sqrt{-d})L/\mathbb{Q}(\sqrt{-d})$ is unramified if and only if d_L divides D and d_L and D/d_L are coprime.

As a corollary (using propositions 16.5 and 16.13) we get the following:

16.15. **Proposition.** *There exists a cohomology class in $H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}_{-34}))$ which is not a linear combination of classes of totally geodesic cycles.*

Appendix

In this appendix we prove Proposition 6.9. The strategy — as suggested by Lemma 6.7 — relies on the general principle stated by Clozel according to which *in an Arthur packet we find the representations belonging to the Langlands packet and more tempered representations.* We address this question through the study of exponents.

17.1. Here again we let p and q two non-negative integers with $p+q=m$ and let $G=\mathrm{SO}(p,q)$. We set $\ell=[m/2]$ and $N=2\ell$.

The goal of this appendix is to prove the following.

17.2. **Proposition.** *Let π be an irreducible unitary cohomological representation of G associated to a Levi subgroup $L=\mathrm{SO}(p-2r,q)\times \mathrm{U}(1)^r$. Assume that π is the local component of an automorphic representation with associated (global) Arthur parameter Ψ. Assume that $3(m-2r-1)>m-1$. Then the parameter Ψ contains a factor $\eta\boxtimes R_a$ where η is a quadratic character and $a\geq m-2r-1$.*

Remark. The inequality $3(m-2r-1)>m-1$ is equivalent to $m-1>3r$. Proposition 6.9 therefore indeed follows from Proposition 17.2.

Before entering into the proof we first recall some basic facts about the local classification of representations of $\mathrm{GL}(N,\mathbb{R})$ and G.

17.3. **Discrete series of the linear groups.** Let k be a positive integer. If $k\geq 2$ we let $\delta(k)$ be the tempered irreducible representation of $\mathrm{GL}(2,\mathbb{R})$ obtained as the unique irreducible subspace of the induced representation

$$\mathrm{ind}(|\cdot|^{\frac{k}{2}}\otimes|\cdot|^{-\frac{k}{2}})$$

(normalized induction from the Borel). It is the unique irreducible representation with trivial central character which restricted to the subgroup $\mathrm{SL}^{\pm}(2,\mathbb{R})$ of elements g such that $|\det(g)|=1$ is isomorphic to $\mathrm{ind}_{\mathrm{SL}(2,\mathbb{R})}^{\mathrm{SL}(2,\mathbb{R})^{\pm}}(D_k)$ where D_k is the (more standard) discrete series representation of $\mathrm{SL}(2,\mathbb{R})$ as considered in e.g. [42, Chapter II, §5]. Recall that by definition the parameter of $\delta(k)$ is the half integer $(k-1)/2$

If $k=1$ we denote by $\delta(k)$ the trivial character of $\mathbb{R}^*=\mathrm{GL}(1,\mathbb{R})$.

We note that if μ is a tempered irreducible representation of $\mathrm{GL}(d,\mathbb{R})$ that is square integrable modulo the center then $d=1$ or 2 and μ is obtained by tensoring some $\delta(k)$ with a unitary character ν of $\mathbb{R}^*$. More precisely:

- If $d=1$ either $\nu=1\otimes|\cdot|^{it}$ or $\nu=\mathrm{sgn}\otimes|\cdot|^{it}$. Here 1 denotes the trivial representation and sgn the sign character of $\mathbb{R}^*$ and $t\in\mathbb{R}$. Then $\mu=\nu$.
- If $d=2$, $\nu=|\det(\cdot)|^{it}$ $(t\in\mathbb{R})$. Then $\mu=\delta(k)\otimes|\det(\cdot)|^{it}$.

Hereafter we denote by $\mu(k,\nu)$ the representation obtained by tensoring $\delta(k)$ with ν. And we simply denote by $\mu(k,\nu)|\cdot|^s$ $(k\geq 1,\ s\in\mathbb{C})$ the representation $\mu(k,\nu)\otimes |\det(\cdot)|^s$.

17.4. **Admissible representations of $\mathrm{GL}(N,\mathbb{R})$.** Let r be a positive integer and, for each $i=1,\dots,r$, fix k_i a positive integer and ν_i a unitary character of $\mathbb{R}^*$. We let $d_i=1$ if $k_i=1$ and let $d_i=2$ if $k_i\geq 2$. We assume that $N=d_1+\ldots+d_r$.

Now let $\mathbf{x} = (x_1, \dots, x_r) \in \mathbb{R}^r$ be such that $x_1 \geq \dots \geq x_r$. Consider the induced representation of $\mathrm{GL}(N, \mathbb{R})$:

$$\mathrm{I}((k_i, \nu_i, x_i)_{i=1,\dots,r}) = \mathrm{ind}(\mu(k_1, \nu_1)| \cdot |^{x_1}, \dots, \mu(k_r, \nu_r)| \cdot |^{x_r})$$

(normalized induction from the standard parabolic of type $(d_1, \dots, d_r)$). We call such an induced representation a *standard module*. These generate[12] the Grothendieck group of the smooth admissible representation of $\mathrm{GL}(N, \mathbb{R})$.

According to Langlands [51] $\mathrm{I}((k_i, \nu_i, x_i)_{i=1,\dots,r})$ has a unique irreducible quotient. We note (see e.g. [10, Chap. 3]) that — when restricted to $\mathbb{C}^*$ — the L-parameter of this representation is conjugate into the diagonal torus of $\mathrm{GL}(N, \mathbb{C})$ and each (k_j, ν_j, x_j) contributes in the following way:

- if $k_j = 1$ and $\nu_j = 1 \otimes | \cdot |^{it_j}$ or $\nu_j = \mathrm{sgn} \otimes | \cdot |^{it_j}$, it contributes by $z \mapsto (z\overline{z})^{\frac{x_j + it_j}{2}}$, and
- if $k_j \geq 2$ and $\nu_j = |\det(\cdot)|^{it_j}$, it contributes by

$$z \mapsto \begin{pmatrix} z^{x_j + it_j + \frac{k_j}{2}} \overline{z}^{x_j + it_j - \frac{k_j}{2}} & \\ & z^{x_j + it_j - \frac{k_j}{2}} \overline{z}^{x_j + it_j + \frac{k_j}{2}} \end{pmatrix}.$$

17.5. **Arthur parameters.** Consider the outer automorphism:

$$\theta : x \mapsto J^t x^{-1} J \quad (x \in \mathrm{GL}(N, \mathbb{R})).$$

Recall that a local Arthur parameter Ψ is a formal sum of formal tensor products $\mu_j \boxtimes R_j$ where each μ_j is a (unitary) representation of say $\mathrm{GL}(a_j, \mathbb{R})$ that is square integrable modulo the center, R_j is an irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension b_j and $N = \sum_j a_j b_j$. We furthermore request that $\Psi^\theta = \Psi$.

Note that we shall only be interested in local Arthur parameters which have the same (regular) infinitesimal character as a finite dimensional representation. This implies in particular that each μ_j is a discrete series or a quadratic character.

As is now standard, for j as above, we denote by $\mathrm{Speh}(\mu_j, b_j)$ the (unique) irreductible quotient of the standard module obtained by inducing the representation

$$\mu_j | \cdot |^{\frac{1}{2}(b_j - 1)} \otimes \mu_j | \cdot |^{\frac{1}{2}(b_j - 3)} \otimes \dots \otimes \mu_j | \cdot |^{\frac{1}{2}(1 - b_j)}$$

as in §3.5. Recall that the representation of $\mathrm{GL}(N, \mathbb{R})$ associated to Ψ is the induced representation of $\otimes_j \mathrm{Speh}(\mu_j, b_j)$; it is an irreducible and unitary representation.

We note that each μ_j is isomorphic to some $\mu(k, \nu)$ ($k \geq 1$, ν unitary character of $\mathbb{R}^*$). By Langlands' classification Π_Ψ can then be realized as the unique irreducible quotient of some standard module $\mathrm{I}((k_i, \nu_i, x_i)_{i=1,\dots,r})$ where $(k_i, \nu_i, x_i)_{i=1,\dots,r}$ are obtained as follows: We write $b_1 \geq b_2 \geq \dots$. Then

$$| \cdot |^{x_1} \mu(k_1, \nu_1) = | \cdot |^{\frac{b_1 - 1}{2}} \mu_1, | \cdot |^{x_2} \mu(k_2, \nu_2) = | \cdot |^{\frac{b_1 - 1}{2}} \mu_2 \text{ (if } b_2 = b_1)$$
$$\dots \ | \cdot |^{x_j} \mu(k_j, \nu_j) = | \cdot |^{\frac{b_1 - 1}{2}} \mu_j \text{ (if } b_j = b_1).$$

We then put the characters of smaller absolute value, and so on. As Π_Ψ is θ-stable, we may furthermore arrange the (k_i, ν_i, x_i) so that there exists a integer $r_+ \in [0, r/2]$ such that:

- For each $i = 1, \dots, r_+$, we have $k_i = k_{r-i+1}$, $\nu_i = \nu_{r-i+1}^{-1}$ and $x_i = -x_{r-i+1}$.
- For any $j = r_+ + 1, \dots, r - r_+$, ν_j is a quadratic character (trivial if $k_j > 1$ with our convention) and $x_j = 0$.

[12]To have a basis we still have to take care of possible permutations of the indices $\{1, \dots, r\}$.

- For any $i, j \in \{r_+ + 1, \dots, r - r_+\}$ with $i \neq j$, we have either $k_i \neq k_j$ or $\nu_i \neq \nu_j$.

A standard module satisfying the above conditions is called a *θ-stable standard module.*[13]

To ease notations we will denote by $I(\lambda)$ a general standard module of $\mathrm{GL}(N, \mathbb{R})$; then λ has to be understood as $(k_i, \nu_i, x_i)_{i=1,\dots,r}$.

17.6. **Twisted traces.** Let Π be a θ-stable irreducible admissible representation of $\mathrm{GL}(N, \mathbb{R})$. We fix an action of θ on the space of Π that is: an operator A_θ ($A_\theta^2 = 1$) intertwining Π and Π^θ. For any test function $f \in C_c^\infty(\mathrm{GL}(N, \mathbb{R}))$ we can then form the θ-trace $\mathrm{trace}_\theta \Pi(f) = \mathrm{trace}(\Pi(f)A_\theta)$. As $A_\theta^2 = 1$ the action of θ is well defined up to a sign. The θ-trace of Π is thus well defined — independently of the choice of A_θ — but only up to a sign. We shall fix the sign following Arthur's normalization (the so called Whittaker's normalization): Fix a character of $\mathbb{R}$. It defines a character of the upper diagonal unipotent subgroup of $\mathrm{GL}(N, \mathbb{R})$. Recall that a Whittaker model is a map (continuous in a certain sense) between a smooth representation and the continuous functions on $\mathrm{GL}(N, \mathbb{R})$ which transform on the left under the unipotent subgroup through the character we have just defined. Any standard module has a unique Whittaker model (this is due to Shalika and Hashizume). And any standard module whose parameter is invariant under θ has a unique action of θ which stabilizes the Whittaker model and varies holomorphicaly in the parameter, see [5, §2.2] for more details. When Π is θ invariant, its standard module has a Whittaker model and an action of θ. This action restricts to an action of θ on the space of Π which gives our choice of normalization for the operator A_θ. Using this normalization $\mathrm{trace}_\theta(\Pi)$ is now well defined.

More generally, a standard module is an induced representation from a tempered modulo center representation from a Levi subgroup. If $I(\lambda)$ is a θ-stable standard module, we shall denote by $\mathrm{trace}_\theta I(\lambda)$ its twisted trace normalized, as above, using the theory of Whittaker models. We are mainly interested in the case of the real places and have assumed that the infinitesimal character is regular. So the standard modules relative to this situation are induced representations from discrete series modulo center. The Langlands subquotient of a standard module is defined and occurs with multiplicity one. And it is the unique irreducible quotient (or submodule) if the exponant are in the positive (or negative) Weyl chamber, which is a way to define it without positivity or negativity assumption.

17.7. **Stabilization of the trace formula.** We now turn to the global study. Let F be a number field and v_0 a real place of F. We denote by G an orthogonal group defined over F such that $G(F_{v_0}) = \mathrm{SO}(p, q)$, our local G above. We also fix ν an infinitesimal character of $\mathrm{SO}(p, q)$ which is the character of a finite dimensional representation of $\mathrm{SO}(p, q)$.[14] We denote by $R^G_{\mathrm{disc},\nu}$ the subspace of the square integrable funtions on $G(F)\backslash G(\mathbb{A})$ with infinitesimal character ν at the place v_0. Let V be a finite set of places of F big enough so that it contains all the Archimedean places and for $v \notin V$ the group $G(F_v)$ is quasi-split and splits over a finite unramified extension of F_v; in particular $G(F_v)$ contains a hyperspecial compact subgroup K_v, see [78, 1.10.2]. As usual we shall denote by F^V, resp. F_V, the restricted

[13]Note that standard modules are generally not irreducible θ-stability has thus to be defined.

[14]This will be the infinitesimal character of the cohomological representation π of the proposition.

product of all completions F_v for $v \notin V$, resp. $v \in V$. Similarly we shall use the corresponding standard notations $G(F_V)$, $G(F^V)$ and $K^V = \prod_{v \notin V} K_v$.

Denote by $R^{G,V}_{\mathrm{disc},\nu}$ the subspace of $R^G_{\mathrm{disc},\nu}$ which consists of the representations of G that are unramified outside V. Let c^V be a character of the spherical Hecke algebra of $G(F^V)$ and denote by $R^G_{\mathrm{disc},\nu,c^V}$ the subspace of $R^{G,V}_{\mathrm{disc},\nu}$ where this spherical algebra acts through the character c^V.

It is a consequence of Arthur's work that $R^G_{\mathrm{disc},\nu,c^V}$ is a representation of finite length. Given a test function f_V on $G(F_V)$ the product $f_V 1_{K^V}$ defines a test function on $G(\mathbb{A})$. We shall denote by $I^G_{\mathrm{disc},\nu,c^V}$ the distribution

$$f_V \mapsto \mathrm{trace}\ R^G_{\mathrm{disc},\nu,c^V}(f_V 1_{K^V})$$

on $G(F_V)$.

Denote by G^* the quasi-split inner form of G. Arthur defines a distribution $S^{G^*}_{\mathrm{disc},\nu,c^V}$ (supported on character) of $G^*(F_V)$ inductively by the following formula: for any f_V a test function on $G^*(F_V)$

$$S^{G^*}_{\mathrm{disc},\nu,c^V}(f_V) := I^{G^*}_{\mathrm{disc},\nu,c^V}(f_V 1_{K^V}) - \sum_H \iota(G^*,H) S^H_{\mathrm{disc},\nu,c^V}(f_V^H),$$

where H runs through the set of all elliptic endoscopic data of G^* different of G^* itself and the coefficients $\iota(G^*,H)$ are certain positive rational numbers. We shall not give details, the only thing that matters to us is that such an H is a product of two non trivial special orthogonal group and is quasi-split. Then the function f_V^H is the Langlands-Shelstad transfer of f_V. We also have to explain the meaning of ν and c^V for H: these are respectively the sums over the ν''s and the $(c')^V$'s, resp. an infinitesimal character of $H(F_{v_0})$ and a character of the spherical algebra of $H(F^V)$ which transfer to ν and c^V through the Langlands functoriality.

We can now state the *stabilization of the trace formula*:

(1) the distribution $S^{G^*}_{\mathrm{disc},\nu,c^V}$ is stable;
(2) for any test function f_V on $G(F_V)$, we have

$$I^G_{\mathrm{disc},\nu,c^V}(f_V) = \sum_H \iota(G,H) S^H_{\mathrm{disc},\nu,c^V}(f_V^H), \tag{17.7.1}$$

where now H runs through the set of *all* (including G^*) elliptic endoscopic data of G and the coefficients $\iota(G,H)$ are again positive rational numbers.[15]

17.8. **Stabilization of the twisted trace formula.** Keep notations as above. Arthur has given a way to compute the distribution $S^{G^*}_{\mathrm{disc},\nu,c^V}$. First recall that such a distribution only depends on the image of f_V modulo functions whose stable orbital integrals are zero. Thanks to the good property of the twisted transfer, such a f_V has the same image in the quotient as the twisted transfer to $G^*(F_V)$ of a function f_V^{GL} and Arthur (see [5, §3.4]) has proved: there exists a (global) Arthur parameter Ψ such that

$$\mathrm{trace}_\theta(\Pi_\Psi)(f_V^{\mathrm{GL}} 1_{\tilde{K}^V}) = S^{G^*}_{\mathrm{disc},\nu,c^V}(f_V); \tag{17.8.1}$$

[15]Note that (17.7.1) — applied to G^* rather than to H, and inductively to its endoscopic subgroup — uniquely defines the distributions $S^H_{\mathrm{disc},\nu,c^V}$.

here $\tilde{K}^V$ is the product of the hyperspecial compact subgroups $\tilde{K}_v$ of $\mathrm{GL}(N, F_v)$. In particular: outside V, the representation Π_Ψ is unramified and the character of the twisted spherical algebra is obtained from c^V by functoriality.

17.9. **Two invariants and the statement.** Let π_0 be a cohomological representation of $G(F_{v_0}) \cong \mathrm{SO}(p,q)$ associated to the Levi subgroup

$$L = \mathrm{U}(p_1, q_1) \times \ldots \times \mathrm{U}(p_r, q_r) \times \mathrm{SO}(p_0, q_0)$$

with $p_0 + 2\sum_j p_j = p$ and $q_0 + 2\sum_j q_j = q$. Set

$$m_0 (= m_0(\pi_0)) = p_0 + q_0.$$

Note that this only depends on π_0 and not on a particular choice of L.

We now define $m^{\mathrm{GL}}(\pi_0)$ in the following way. Let π be a square integrable representation of $G(\mathbb{A})$ which localize in π_0 at the place v_0. The global representation π determines two characters ν and c^V as in §17.7. It then follows from §17.8 that these two characters determine some global Arthur parameter Ψ. Next we localize Ψ at the place v_0; in the subspace where $W_{\mathbb{R}}$ acts through characters in this localization, we look at the action of $\mathrm{SL}(2,\mathbb{C})$ and denote by $m(\Psi)$ the biggest dimension for an irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ in this space. Finally we set:

$$m^{\mathrm{GL}}(\pi_0) := \min_{\pi} m(\Psi).$$

17.10. **Proposition.** *We have the inequality:* $m_0(\pi_0) - 1 \leq m^{\mathrm{GL}}(\pi_0)$.

In fact the inequality is an equality but we do not need it here.

17.11. **Proposition 17.10 implies Proposition 17.2.** Indeed take $\pi_0 = \pi$ as in Proposition 17.2 so that $m_0 = p+q-2r = m-2r$. Note that the representation π_0 being cohomological its infinitesimal character is regular. So if π_0 is the local component of an automorphic representation with associated *global* Arthur parameter Ψ, then Ψ is very particular: writing

$$\Psi = \mu_1 \boxtimes R_1 \boxplus \ldots \boxplus \mu_r \boxtimes R_r$$

the regularity of the infinitesimal character has the following consequences.

(1) If m is odd, then there is at most one j such that μ_j is a quadratic character. If there exists such a j we enumerate so that $j = 1$.
(2) If m is even there is no such j or there is two such j. In the later case, we enumerate so that these two j's are 1 and 2 and R_1 is maximal. We then have $R_2 = 1$.

In particular, $m(\Psi) = n_1$ and we conclude from Proposition 17.10 that

$$n_1 \geq m_0 - 1 = m - 2r - 1.$$

Finally if m is odd, $3n_1 \geq 3(m-2r-1) > m-1 = N$, and if m is even $3n_1 + 1 \geq 3(m-2r-1)+1 > m = N$, it therefore follows from 1. and 2. above that the local character μ_1 can only occur as the localization of a global character. This concludes the proof of Proposition 17.10.

We shall prove Proposition 17.10 by decomposing the traces and twisted traces as sums of traces and twisted traces of standard modules, at the place considered. We shall therefore decompose cohomological representations in the Grothendieck group using standard modules.

17.12. **Standard modules and their exponents.** The only representations of $\mathrm{SO}(p,q)$ we will be interested in have real infinitesimal character. The relevant standard modules $I_P(\sigma)$ of $\mathrm{SO}(p,q)$ are induced from a standard parabolic subgroup $P = MN$ of $\mathrm{SO}(p,q)$ with

$$M \cong \mathrm{GL}(d_1,\mathbb{R}) \times \cdots \times \mathrm{GL}(d_t,\mathbb{R}) \times \mathrm{SO}(p-n_0, q-n_0),$$

where $t \in \mathbb{N}$, $d_i = 1$ or 2, and $n_0 = d_1 + \ldots + d_t$, and

$$\sigma = \delta_1 |\cdot|^{x_1} \otimes \ldots \otimes \delta_t |\cdot|^{x_t} \otimes \pi_0.$$

Here each δ_i is either a discrete series of $\mathrm{GL}(2,\mathbb{R})$, if $d_i = 2$, or a quadratic character of $\mathbb{R}^*$, if $d_i = 1$, the x_i's are positive real numbers with $x_1 \geq x_2 \geq \cdots \geq x_t$, and π_0 is a tempered representation of the group $\mathrm{SO}(p-n_0, q-n_0)$.

The set of *exponents* of such a standard module is the set $\{\pm x_1, \ldots, \pm x_t\}$. We define the set of *character exponents* as

$$\mathrm{CarExp}(I_P(\sigma)) = \{\pm x_i \ : \ i \in [1,t] \text{ and } d_i = 1\}.$$

17.13. **Lemma.** *Let π_0 be a cohomological representation of $\mathrm{SO}(p,q)$. There exists a finite set $\mathcal{E}$ of data (P,σ) such that the corresponding standard modules $I_P(\sigma)$ satisfy*

$$\mathrm{CarExp}(I_P(\sigma)) \subset \left[-\frac{m_0-2}{2}, \frac{m_0-2}{2} \right]$$

and, in the Grothendieck group,

$$\pi = \sum_{(P,\sigma)\in\mathcal{E}} m(P,\sigma) I_P(\sigma), \quad (m(P,\sigma) \in \mathbb{Z}),$$

with $m(P,\sigma) = 1$ if π is the Langlands subquotient of the standard module $I_P(\sigma)$.

Proof. Since π_0 is cohomological, it is obtained by cohomological induction of a character of the associated Levi subgroup L. To decompose π_0 in the Grothendieck group using standard modules, we take the resolution of the character of L with standard modules and cohomologically induce it; see Johnson's thesis [40]. The part coming from the unitary group gives only exponent of the form $\delta|\cdot|^x$ where δ is a discrete series of $\mathrm{GL}(2,\mathbb{R})$. And the exponent coming form the $\mathrm{SO}(p_0,q_0)$-part are of the form $|\cdot|^{-x}$ with $0 < x \leq \frac{1}{2}(m_0-2)$. □

17.14. **The twisted case.** We look only at very particular local Arthur parameters $\Psi = \Psi_{v_0}$ since the infinitesimal character of the corresponding representation is regular. Writing

$$\Psi = \mu_1 \boxtimes R_1 \boxplus \ldots \boxplus \mu_r \boxtimes R_r$$

recall that:

1. If m is odd, then there is at most one j such that μ_j is a quadratic character. If there exists such a j we numerate so that $j = 1$.
2. If m is even there is no such j or there is two such j's. In the latter case, we numerate so that these two j's are 1 and 2 and R_1 is maximal. We then have $R_2 = 1$.

Set $m(\Psi) = n_1$ (the dimension of R_1).

Standard modules $I(\lambda)$ for $\mathrm{GL}(N,\mathbb{R})$, as for $\mathrm{SO}(p,q)$, are representations induced from a tempered representation modulo a character of a Levi subgroup. The tempered representation contains twist of discrete series and twist of quadratic character (since we only consider representation with real infinitesimal character).

We similarly denote by $\mathrm{CarExp}(I(\lambda))$ the set of real number $\pm x_i$ such that $x_i > 0$ and x_i is the absolute value of a character occuring in the definition of $I(\lambda)$:

$$\mathrm{CarExp}(I((k_i, \nu_i, x_i)_{i=1,\dots,r})) = \{\pm x_i \ : \ i \in [1, r], \ x_i \neq 0 \text{ and } k_i = 1\}.$$

Recall that we denote by Π_Ψ the representation of $\mathrm{GL}(n, \mathbb{R})$ associated to Ψ. We shall prove the following analogue of Lemma 17.13.

17.15. **Lemma.** *There exists a finite set $\mathcal{E}_\theta$ of data λ such that the standard modules $I(\lambda)$ are θ-stable and*

$$\mathrm{CarExp} I(\lambda) \subset [-(m(\Psi) - 1)/2, (m(\Psi) - 1)/2]$$

and

$$\mathrm{trace}_\theta \ \Pi_\Psi = \sum_{\lambda \in \mathcal{E}_\theta} m(\pi, \lambda) \mathrm{trace}_\theta \ I(\lambda), \quad (m(\pi, \lambda) \in \mathbb{Z}).$$

The proof requires a bit of preparation. We first write each module $\mathrm{Speh}(\mu, b)$ in the Grothendieck group using the basis made by standard module:

$$\mathrm{Speh}(\mu, b) = \sum_\lambda m(\lambda) I(\lambda). \tag{17.15.1}$$

We assume here that μ is a discrete series of $\mathrm{GL}(2, \mathbb{R})$. Let k be the positive integer such that $\mu = \delta(k)$, this means that the infinitesimal character of μ is $(k-1)/2, -(k-1)/2$, so in fact $k > 1$. The condition that the infinitesimal character of $\mathrm{Speh}(\mu, b)$ is regular is equivalent to the fact that $k > b$: we must have $(k-1)/2-(b-1)/2 > -(k-1)/2+(b-1)/2$, that is $k > b$ and with this inequality the regularity is clear.

17.16. **Lemma.** *We assume that $k > b$. In* (17.15.1)*, if $m(\lambda) \neq 0$ then* $\mathrm{CarExp} I(\lambda) = \emptyset$.

Proof. The module $\mathrm{Speh}(\mu, b)$ is cohomologicaly induced by a unitary character $(z/\overline{z})^{(k-1)/2}$ of $\mathrm{GL}(b, \mathbb{C})$. To obtain (17.15.1) we take the analogous resolution of the trivial character of $\mathrm{GL}(b, \mathbb{C})$ tensor it by the previous character and then make the cohomological induction. In our situation cohomological induction is an exact functor which sends irreducible representations to irreducible representations because the infinitesimal character is regular; we have assume that $k > b$ to have the regularity of the infinitesimal character. Moreover this functor sends a standard module to a standard module. So it's enough to prove the analogous lemma for the trivial character of $\mathrm{GL}(b, \mathbb{C})$. But the resolution in this simple situation is perfectly known: the analogous of (17.15.1) is (up to a sign)

$$\sum_{\sigma \in \mathfrak{S}_b} (-1)^{\ell(\sigma)} I(\lambda, \sigma(\lambda)), \tag{17.16.1}$$

where $\lambda = ((b-1)/2, \cdots, -(b-1)/2)$ is seen as a set of b elements $(\lambda_1, \cdots, \lambda_b)$ and $I(\lambda, \sigma(\lambda))$ is the principal series of $\mathrm{GL}(b, \mathbb{C})$ induced by the character $\otimes_{i \in [1,b]} z^{\lambda_i} \overline{z}^{\sigma(\lambda_i)}$ of the diagonal torus. The length $\ell(\sigma)$ is the ordinary length in $\mathfrak{S}_b$. The exponents of $I(\lambda, \sigma(\lambda))$ are obtained as products of a unitary character with an absolute value $(z\overline{z})^y$ with

$$y \in \{(\lambda_i + \sigma(\lambda_i))/2 \ : \ i \in [1, b]\} \quad (\text{and } |y| \leq (b-1)/2).$$

The unitary part of the character is of the form $(z/\overline{z})^c$ with $c = (\lambda_i - \sigma(\lambda_i))/2$. In particular $c > -(k-1)/2$. Recall that we have to cohomologically induce this character tensored with the character $(z/\overline{z})^{(k-1)/2}$. This is necessarily a non trivial unitary character and the induced representation is a discrete series. $\square$

We need a more technical result which is a corollary of the proof:

17.17. **Lemma.** *Let ρ be an irreducible subquotient of any standard module appearing in* (17.15.1) *with non-zero coefficient. Then there exists some λ such that $I(\lambda)$ appears in* (17.15.1) *with $m(\lambda) \neq 0$ and ρ is the Langlands quotient of it.*

Proof. We go back to the previous proof. Let ρ be as in the statement of the lemma. By the previous proof, ρ is cohomologicaly induced by an irreducible subquotient $\rho_{\mathbb{C}}$ appearing as subquotient of one of the principal series of (17.16.1). But in (17.16.1) all possible principal series appear (the infinitesimal character is fixed, of course). So $\rho_{\mathbb{C}}$ is a Langlands quotient of such a principal series and ρ is the Langlands quotient of the standard module obtained from it by cohomological induction. $\square$

Now fix $b_0 \in \mathbb{N}$ and denote by ϵ_{b_0} either the trivial representation of $\mathrm{GL}(b_0, \mathbb{R})$ or the sign representation of this group. Denote by π_0 either ϵ_{b_0} or the representation of $\mathrm{GL}(b_0+1, \mathbb{R})$ obtained by inducing the tensor product of the representation ϵ_{b_0} of $\mathrm{GL}(b_0, \mathbb{R})$ and either the trivial representation or the sign character of $\mathbb{R}^\times$ from the maximal parabolic corresponding to the partition $(b_0, 1)$ of $b_0 + 1$. In any case π_0 is a representation of $\mathrm{GL}(b_0 + \eta, \mathbb{R})$ where $\eta = 0$ or 1 (the second case occurs in the case of an even orthogonal group when a character of $W_{\mathbb{R}}$ occurs in the local parametrization of the Arthur's packet). Write the analogue of (17.15.1) for π_0:

$$\pi_0 = \sum_{\lambda} m(\lambda) I(\lambda). \tag{17.17.1}$$

17.18. **Lemma.** *In* (17.17.1)*, the exponents of λ are of absolute value less than or equal to* $(b_0 - 1)/2$ *and this is also true for any irreducible subquotient ρ of the $I(\lambda)$ appearing in* (17.17.1)*. This last property is true for any irreducible representation of* $\mathrm{GL}(b_0 + \eta, \mathbb{R})$ *with the same infinitesimal character as π_0.*

Proof. It is not obvious to write down explicitly (17.17.1) but the second assertion of the lemma is more general than the first one and we do not need to know (17.17.1) to prove it. We have to prove that any representation of a Levi subgroup of $\mathrm{GL}(b_0 + \eta, \mathbb{R})$, tempered modulo center, with the same infinitesimal character as π_0 satisfies the lemma. Such a representation can be written as a tensor product $\otimes \delta(k_j)|\cdot|^{x_j}$ with notations as above. Then the condition on the infinitesimal character implies that

$$k_j - 1 + x_j \in [(b_0-1)/2, -(b_0-1)/2] \quad \text{and} \quad -k_j + 1 + x_j \in [(b_0-1)/2, -(b_0-1)/2].$$

In particular $k_j - 1 + |x_j| \in [(b_0 - 1)/2, -(b_0 - 1)/2]$ and $|x_j| \leq (b_0 - 1)/2$. This proves the lemma. $\square$

We now come back to the local Arthur parameter Ψ and the corresponding representation Π_Ψ. Recall that Π_Ψ is the induced representation of $\otimes_j \mathrm{Speh}(\mu_j, b_j)$. Since Ψ satisfies (1) and (2) of §17.14 we may rewrite Π_Ψ as the induced representation of

$$\mathrm{Speh}(\delta(k_1), b_1) \otimes \ldots \otimes \mathrm{Speh}(\delta(k_t), b_t) \otimes \pi_0$$

with $k_j > 1$, for $j = 1, \ldots, t$, and either π_0 is as above or does not appear. In the latter case we will put $b_0 = 0$. Moreover the infinitesimal character of Π_Ψ is "almost" regular, meaning that it is regular if π_0 is a character and that, if

π_0 is not a character, the infinitesimal character of the representation induced from $\otimes_j \mathrm{Speh}(\delta(k_j), b_j) \otimes \epsilon_{b_0}$ is regular. Note that $N = 2(b_1 + \ldots + b_t) + b_0$ or $N = 2(b_1 + \ldots + b_t) + b_0 + 1$ according to the parity of b_0 (or equivalently the parity of m) and $b_0 = m(\Psi)$.

Now we decompose each representation $\mathrm{Speh}(\delta(k_j), b_j)$ as in (17.15.1). And for any $j \in \{1, \ldots, t\}$ we let ρ_j be a subquotient in one of the standard modules appearing non-trivially, i.e. with non-zero coefficient $m(\lambda)$, in this decomposition. Finally we let ρ_0 be an irreducible representation with the same infinitesimal character as π_0.

17.19. **Lemma.** *The representation of* $\mathrm{GL}(N, \mathbb{R})$ *induced from* $\rho_1 \otimes \ldots \otimes \rho_t \otimes \rho_0$ *is irreducible.*

Proof. This lemma will appear in the thesis of N. Arancibia but, for the ease of the reader we briefly include a proof. It follows from the properties of the infinitesimal character and (now classical) results of Speh on the irreducibility of induced representations of GL : suppose that $\delta(k)| \cdot |^x$ appears in one of the standard modules for ρ_j and $\delta'(k')| \cdot |^{x'}$ appears in a standard module for $\rho_{j'}$ with $j \neq j'$. Then we have

$$\frac{k-1}{2} + x \in \left[\frac{k_j + b_j - 2}{2}, \frac{|b_j - k_j|}{2} \right]$$

and

$$-\frac{k-1}{2} + x \in \left[-\frac{|b_j - k_j|}{2}, -\frac{b_j + k_j - 2}{2} \right];$$

and similarly with $'$. But the two sets for j are symmetric to 0 and the corresponding set for j' have the same property and are disjoint from the sets for j. After Speh (see also [62, Lemma 1.7]) this is a enough to conclude that the induced representation is irreducible. □

Denote by $\mathcal{F}$ the set of irreducible representations σ as in the previous lemma. This set has a length, Vogan's length. The tempered representations are of length 0. Let $\sigma \in \mathcal{F}$ be a self-dual representation and denote by $I(\lambda_\sigma)$ the standard module of σ. Whittaker normalization provides choices of actions on both σ and $I(\lambda_\sigma)$ that are compatible; the twisted characters $\mathrm{trace}_\theta(\sigma)$ and $\mathrm{trace}_\theta I(\lambda_\sigma)$ are taken with respect to these actions.

17.20. **Lemma.** *Let σ be as above. Then there exists a finite set subset $\mathcal{F}_\sigma$ of $\mathcal{F}$ containing only self-dual representations of length strictly less than the length of σ such that for suitable $m(\tau, \sigma) \in \mathbb{Z} - \{0\}$*

$$\mathrm{trace}_\theta(\sigma) - \mathrm{trace}_\theta I(\lambda_\sigma) = \sum_{\tau \in \mathcal{F}_\sigma} m(\tau, \sigma) \mathrm{trace}_\theta(\tau).$$

Proof. If σ is tempered we can take $\mathcal{F}_\sigma = \emptyset$. In general we prove the lemma by induction. We shall first prove that any irreducible subquotient of $I(\lambda_\sigma)$ is in $\mathcal{F}$. By definition σ is an induced representation of the ρ_j. So $I(\lambda_\sigma)$ is the standard module induced from the standard modules of the ρ_j. Let τ be a subquotient of $I(\lambda_\sigma)$. For any j there exists an irreducible subquotient τ_j of the standard module of ρ_j such that τ is a subquotient of the induced representation of the τ_j. But we have seen that such an induced representation is irreducible and has to coincide with τ. This proves that $\tau \in \mathcal{F}$. If the length of τ equals the length of σ then $\tau = \sigma$ and otherwise the length of τ is strictly less than the length of σ. This proves that

$\mathrm{trace}_\theta(\sigma) - \mathrm{trace}_\theta I(\lambda_\sigma)$ is a linear combination of the $\mathrm{trace}_\theta \tau$ for those τ which are self-dual (up to a sign which depends of the choices). □

17.21. **Proof of Lemma 17.15.** We apply Lemma 17.20 to $\sigma = \Pi_\Psi$. The twisted trace of σ can be written as a linear combination of twisted trace of standard modules $\mathrm{trace}_\theta(I_{\lambda_\tau})$ where τ runs in a subset of self-dual representations in $\mathcal{F}$ of length smaller or equal than that of σ. Moreover, it follows from the description of the exponent for the representation inducing the element in $\mathcal{F}_\sigma$ (see Lemma 17.16, 17.17 and 17.18) that if one of the representation τ has an exponent which is a character with absolute value x then $b_0 \geq 2|x| + 1$. Since $b_0 = m(\Psi)$ we are done. □

17.22. **Transfer, local version.** We now come back to the setting (and notations) of §17.7 and §17.8. The distribution $S^{G^*}_{\mathrm{disc},\nu,c^V}$ on $G^*(F_V)$ is a product of local stable distributions with a global coefficient: fix $v \in V$, there exists a finite set $\prod_v = \prod(\Psi_v)$ of representations π_v of $G^*(F_v)$ and some multiplicities $m(\pi_v) > 0$ and signs $\varepsilon(\pi_v)$ ($\pi_v \in \prod_v$) such that such that

$$S^{G^*}_{\mathrm{disc},\nu,c^V}(f_V) = x(c^V) \prod_{v \in V} \left(\sum_{\pi_v \in \prod_v} \varepsilon(\pi_v) m(\pi_v) \mathrm{trace}\ \pi_v(f_v) \right),$$

where $x(c^V)$ is a global constant. And the local packets $\prod_v$ (with the corresponding multiplicities and signs) are determined (see Proposition 3.7) by:

$$\sum_{\pi_v \in \prod_v} \varepsilon(\pi_v) m(\pi_v) \mathrm{trace}\ \pi_v(f_v) = \mathrm{trace}_\theta\ \Pi_{\Psi_v}(\tilde{f}_v), \tag{17.22.1}$$

where $\tilde{f}_v$ is a function on $\mathrm{GL}(N, F_v)$ whose twisted transfer to $G(F_v)$ is f_v modulo unstable functions. We will denote by $\mathrm{trace}(\prod(\Psi_v))$ the (local stable) distribution on the left hand side. Similarly we denote by $\mathrm{trace}(\prod(\Psi_v)^H)$ the local stable distribution on $H(F_v)$ associated to $S^H_{\mathrm{disc},\nu,c^V}(f_V)$.

Now we turn to the stabilization of the trace formula for G. The left hand side of (17.7.1) is (the character of) a linear combination with positive coefficients of representations of $G(F_V)$. Because the right hand side is of finite length as a representation of endoscopic groups, we conclude that this also holds for the left hand side.

Assume now that π_0 is a cohomological representation occuring as the local component of at least one of these representations. We first fix f_V outside of v_0. For each choice, $f_V^{v_0}$, we have a distribution

$$f_{v_0} \mapsto I^G_{\mathrm{disc},\nu,c^V}(f_{v_0} f_V^{v_0} 1_{K^V});$$

it is a finite linear combination of traces of representations of $G(F_{v_0}) = \mathrm{SO}(p,q)$. Because of the positivity of the coefficients in the decomposition of $R^G_{\mathrm{disc},\nu,c^V}$ and the linear independence of characters, we may find a test function $f_V^{v_0}$ such that this linear combination contains the trace of π_0.

In this way we define a finite set $\mathcal{F}$ of representations of $G(F_{v_0})$ containing π_0 such that for any test function f_{v_0} on $G(F_{v_0})$ we have:

$$\sum_{\pi \in \mathcal{F}} c(\pi) \mathrm{trace}\ \pi(f_{v_0}) = \sum_H x(H) \mathrm{trace}(\prod(\Psi_{v_0})^H)(f^H_{v_0}), \tag{17.22.2}$$

for suitable coefficients $c(\pi)$ and $x(H)$. What is important for us is that $c(\pi_0) \neq 0$. We recall that any $\pi \in \mathcal{F}$ is a unitary representation with infinitesimal character ν, this implies (see [72]) that any $\pi \in \mathcal{F}$ is cohomological.

17.23. **End of the proof of Proposition 17.10.** We first decompose the left hand side of (17.22.2) in terms of standard modules. It follows from Lemma 17.13 that there is certainly one standard module $I_P(\sigma)$ of $G(F_{v_0}) = \mathrm{SO}(p,q)$ that contributes non trivially to the left hand side of (17.22.2) and whose associated set $\mathrm{CarExp}(I_P(\sigma))$ contains a term $(m-2)/2$ with $m \geq m_0(\pi_0)$; the fact that we have $m \geq m_0(\pi_0)$, and not just an equality, takes into account the fact that the standard module from which π_0 is the Langlands quotient can be canceled by another standard module coming from a π with $m_0(\pi) \geq m_0(\pi_0)$. Now we decompose the right hand side of (17.22.2) in terms of "stable" standard modules (instead of inducing a tempered representation modulo the center, we induce a stable tempered representation modulo the center (see [4] and [83, §3.3 and §3.4]). At least one of these stable standard modules is such that its associated set CarExp contains a term $(m-2)/2$ in CarExp with the same m as above but with a suitable H. To ease the understanding we assume that $H = G^*$ (otherwise we have to introduce product of orthogonal groups and a decomposition of Ψ into a product). We now use (17.22.1): by the results of Mezo [59] a stable standard module for G^* has a twisted transfer to $\mathrm{GL}(N,\mathbb{R})$ which is a θ-stable standard module with the predicted Langlands parameter. In particular the left hand side of (17.22.1) contains a θ-stable standard module with a term $(m-2)/2$ in CarExp. It then follows from Lemma 17.15 that $m-1 \leq m(\Psi)$ and Proposition 17.10 follows.

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