DEFORMATION SPACES ASSOCIATED TO COMPACT HYPERBOLIC MANIFOLDS

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In this paper we take a first step toward understanding representations of cocompact lattices in SO(n,1) into arbitrary Lie groups by studying the deformations of rational representations - see Proposition 5.1 for a rather general existence result. This proposition has a number of algebraic applications. For example, we remark that such deformations show that the Margulis Super-Rigidity Theorem, see [30], cannot be extended to the rank 1 case. We remark also that if $\Gamma \subset SO(n,1)$ is one of the standard arithmetic examples described in Section 7 then Γ has a faithful representation ρ' in SO(n+1), the Galois conjugate of the uniformization representation, and Proposition 5.1 may be used to deform the direct sum of ρ' and the trivial representation in SO(n+2) thereby constructing non-trivial families of irreducible orthogonal representations of Γ. However, most of this paper is devoted to studying certain spaces of representations which are of interest in differential geometry in a sense which we now explain.

Recently, there has been considerable interest in spaces of locally homogeneous (or geometric) structures on smooth manifolds, see for example, Thurston [25]. The spaces of conformal, projective and hyperbolic structures are of particular interest. If M is a smooth manifold we will denote the corresponding spaces of marked structures, see Lok [13], page 7, by $\mathcal{C}(M)$, $\mathcal{P}(M)$ and $\mathcal{H}(M)$ respectively. Since these spaces are a measure of the complexity of the fundamental group, it makes sense to study them in the case that M is a hyperbolic n-manifold. Of course, if $n \geq 3$ and M is compact, then the celebrated Mostow Rigidity Theorem states that $\mathcal{H}(M)$ is a point. Our main theme is that this is far from true for $\mathcal{C}(M)$ and $\mathcal{P}(M)$. Also

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$\mathcal{H}(M \times \mathbb{R})$ is an interesting space closely related to $\mathcal{C}(M)$. Our first main result is a lower bound for the dimensions of the three previous deformation spaces by r, the largest number of disjoint, non-singular, totally geodesic hypersurfaces contained in M. If M is a hyperbolic surface of genus g then $r = 3g - 3$. From this bound, it is easily shown that the deformation spaces have arbitrarily large dimension as M varies. Our second main result is the existence of non-isolated singularities. In fact, we prove that the deformation spaces are locally homeomorphic to certain singular algebraic varieties; however, it should be possible to prove that the deformation spaces themselves have natural local analytic structures (see the remark at the end of Section 7) preserved by the local homeomorphism hol (see below). We would then have established that $\mathcal{C}(M)$, $\mathcal{P}(M)$ and $\mathcal{H}(M \times \mathbb{R})$ are singular for their natural local analytic structures.

To obtain the above results concerning the spaces of structures it is convenient to replace them with the space of classes of representations of Γ, the fundamental group of M, into the automorphism groups of the model space. This is possible because of the following general result.

Let $\mathcal{S}(M)$ be a space of marked locally homogeneous structures modeled on a homogeneous space $X = G/H$. Given a structure $s \in \mathcal{S}(M)$, by continuing coordinate charts around elements of Γ, see Lok [13], page 6, we obtain the holonomy representation $\rho: \Gamma \to G$ of s and a map:

$$\text{hol}: \mathcal{S}(M) \to \text{Hom}(\Gamma, G)/G$$

defined by $\text{hol}(s) = G\cdot\rho$ where G acts by conjugation. Then Theorem 1.11 of Lok [13] states that hol is an open map which lifts to a local homeomorphism from the space of (G,X)-developments to $\text{Hom}(\Gamma,G)$. We will refer to this result as the "Holonomy Theorem". Unfortunately hol is not necessarily a local homeomorphism. However, if ρ is a stable representation (see Section 1) then there exist neighborhoods U of s in $\mathcal{S}(M)$ and V of ρ in $\text{Hom}(\Gamma,G)/G$, finite groups H_1 and H_2 with $H_1 \subset H_2$ (the isotropy subgroups of s and ρ) and finite quotient mappings $U = \tilde{U}/H_1$, $V = \tilde{V}/H_2$ such that hol lifts to a homeomorphism $\tilde{U}$ to $\tilde{V}$. In particular if ρ is good (see Section 1) then hol is a homeomorphism from a neighborhood of ρ to a neighborhood of ρ. We see then that if ρ is stable then local information around ρ gives us information around s.

The representation $\rho:\Gamma \to G$ is necessarily rigid because $\rho(\Gamma)$ is not necessarily a lattice in G. Thus, we circumvent the Mostow Rigidity Theorem by, on the one hand, considering second-order structures such as conformal and projective structures and on the other, by considering non-lattice subgroups. For the three deformations spaces considered above we have $G = SO(n+1,1)$ for $\mathcal{C}(M)$, $G = PGL_{n+1}(\mathbb{R})$ for $\mathcal{P}(M)$ and $G = SO(n+1,1)$ for $\mathcal{H}(M \times \mathbb{R})$. Thus, we can concentrate our efforts on the two families of spaces $\mathrm{Hom}(\Gamma, SO(n+1,1))$ and $\mathrm{Hom}(\Gamma, PGL_{n+1}(\mathbb{R}))$ and their quotients by $SO(n+1,1)$ and $PGL_{n+1}(\mathbb{R})$ respectively. With the exception of Section 8, this paper is entirely concerned with these latter spaces. In addition to proving the previous lower bound for the dimensions of these spaces, we give examples where they are singular at certain representations, including irreducible ones, for their natural algebraic structures.

It seems the first result showing the non-triviality of $\mathrm{Hom}(\Gamma, SO(n+1,1))$ for $n \geq 3$ was Apanasov [1]. The matter was greatly clarified by Thurston's idea of bending a Fuchsian group, see Sullivan [24] or Kourouniotis [27].

There are a number of technical theorems contained in this paper in addition to the main results alluded to above. For the reader's convenience we briefly state them in order of occurrence.

Section 1 defines stable representations, characterizes them in terms of parabolic subgroups and proves they are Zariski open in $\mathrm{Hom}(\Gamma, G)$. A slice theorem is proved for the action of G on the stable representations. A very general notion of quasi-Fuchsian representation is studied and found to be surprisingly restrictive.

Section 2 treats deformations and infinitesimal deformations of representations and the first obstruction to integrating an infinitesimal deformation. We study this obstruction via the dual homology class in later sections.

Section 3 deals with quasi-Fuchsian representations of $\Gamma = \pi_1(M)$ on hyperbolic (n+1)-space and our main theorem in this section shows they are open in $\mathrm{Hom}(\Gamma, SO(n+1,1))$. We prove various theorems concerning the local nature of the space of conjugacy classes of quasi-Fuchsian representations; for example, for n even, this space is an open subset of the real algebraic set $X(\Gamma, SO(n+1,1))$ - see Section 1 for notation. In the odd case, this result is not necessarily correct, there is another component of the real points passing through the uniformization representation ρ of Γ in $SO(n,1)$ corresponding to deformations in the group $SO(n,2)$ of ρ composed with the inclusion of $SO(n,1)$ in $SO(n,2)$.

Section 4 is a technical section dealing with cycles with coefficients and their intersection products. This material is needed to compute the first obstruction class.

Section 5 is one of the main sections of the paper. We introduce an algebraic version of Thurston's bending deformation - see Kourouniotis [27] for a geometric definition justifying the name "bending". Theorem 5.1 identifies the derivative of the bending deformation with the Poincaré dual of a totally geodesic hypersurface with an obvious coefficient from Minkowski space. The rest of the section is concerned with proving that $\dim X(\Gamma, SO(n+1,1))$ and $\dim X(\Gamma, PGL_{n+1}(\mathbb{R}))$ are bounded below by r, the maximum number of disjoint, embedded, totally geodesic hypersurfaces in M. In the classical case of a hyperbolic surface of genus g we have $r = 3g - 3$ and the bound is a weak one. By a simple geometric construction the problem is reduced to deforming a representation in G of the fundamental group of a graph of groups such that all edge groups have non-zero invariants in $\mathfrak{g}$, the Lie algebra of G.

Section 6 gives a criterion for the above spaces to have non-isolated singularities. This criterion involves computing some intersections of cycles with coefficients. It is possible that the space $\mathcal{C}(M)$ is singular for any hyperbolic n-manifold $(n \geq 4)$ admitting two different intersecting, two-sided, non-singular, totally geodesic hypersurfaces. However, we have not been able to prove this.

Section 7 is a technical section proving the existence of nicely intersecting totally geodesic submanifolds in the standard arithmetic examples. For example, we show (Theorem 7.2) that if $p + q \neq n - 1$ there exist totally geodesic submanifolds of codimension p and q respectively intersecting in a single component. The reader may find this section difficult - he is advised to refer to O'Meara [18] for background information on the Strong Approximation Theorem and the spinor norm.

Section 8 is concerned with the interaction of $\mathcal{C}(M)$ with Riemann geometry. We state a theorem suggested to us by Jim Simons and proved by S.Y. Cheng which shows each conformal class of metrics on M contains a unique metric of constant scalar curvature $-n(n-1)$. (This also follows as a special case of the Yamabe problem, recently solved by R. Schoen [29].) This is a generalization of the General Uniformization Theorem for Riemann surfaces. Using this metric we construct an interesting function vol: $\mathcal{C}(M) \to \mathbb{R}_+$ which assigns to a conformal structure the volume of M for the canonical metric belonging to that structure. We prove that vol is not constant if $n \geq 3$ (it is constant if $n = 2$). In case $n = 4$, we prove that vol has an absolute minimum at the hyperbolic structure. For all $n \geq 3$, it has a local minimum (with positive definite Hessian) at the hyperbolic structure. The existence of the canonical metric combined with work of Gasqui and Goldschmidt [10] yields a Riemannian metric on $\mathcal{C}(M)$, generalizing the Petersson-Weil metric.

There are a great many problems concerning $\mathcal{C}(M)$ and $\mathcal{P}(M)$ which remain unanswered - their topological properties for example. It would be very useful to have some examples, for instance for some hyperbolic 3-manifolds. We believe that the most important problem is to decide whether r is always equal to the dimensions of $\mathcal{C}(M)$ and $\mathcal{P}(M)$ or just a lower bound. A closely related problem is to construct more deformations - perhaps by analytic methods.

We would like to thank a number of people who helped us with this paper. Above all, we thank Bill Goldman for suggesting the main lines of Theorem 3.1 and many other conversations. Also we would like to thank John Morgan for suggesting the proof of Lemma 3.4, and for to thank John Morgan for suggesting the proof of Lemma 3.4, and for suggesting the graph of hypersurfaces of Section 5, Larry Lok for providing us with his thesis and an extension of an argument of his thesis (see the proof of Theorem 3.1), Robert Steinberg for providing us with the proof of Lemma 1.1 and S.Y. Cheng for proving Theorem 8.1 and a helpful conversation concerning Theorem 8.3. We should acknowledge a debt to Bill Thurston, for his idea of bending a Fuchsian group is at the heart of this paper. Finally, this paper is dedicated to Dan Mostow on the occasion of his sixtiety birthday (the second author presented it at the conference at Yale marking this occasion). The second author would like to take this occasion to thank Dan Mostow and the Yale mathematics faculty for the hospitality shown him as a visitor in 1983-84, as an assistant professor from 1974-78 and on many other occasions.

After we had finished writing this paper we learned of the thesis of Kourouniotis [27]. Kourouniotis also obtains the lower bound for the dimension of $\mathcal{C}(M)$. His thesis contains a careful description of the geometric version of bending.

1. Character Varieties and Generalized Quasi-Fuchsian Groups.

Let Γ be a finitely generated group and $\underline{G}$ a simple linear algebraic group defined over $\mathbb{R}$. The complex points of $\underline{G}$ will also be denoted $\underline{G}$ and the real points G. The set $\mathrm{Hom}(\Gamma,\underline{G})$ is the set of complex points of an affine variety defined over $\mathbb{R}$ with real points $\mathrm{Hom}(\Gamma,G)$. We will often denote $\mathrm{Hom}(\Gamma,\underline{G})$ by $(\Gamma,\underline{G})$ and $\mathrm{Hom}(\Gamma,G)$ by (Γ,G). The group $\underline{G}$ acts algebraically on $(\Gamma,\underline{G})$ by conjugation. This action will be denoted $g\cdot\rho$ for $g \in \underline{G}$ and $\rho \in R(\Gamma,\underline{G})$. Since $\underline{G}$ is reductive, there is a quotient variety $X(\Gamma,\underline{G})$ for this action, see Newstead [19]. The set of real points of the quotient variety will be denoted $X(\Gamma,G)$. We let $\underline{\pi}:R(\Gamma,\underline{G}) \to X(\Gamma,\underline{G})$ and $\pi:R(\Gamma,G) \to X(\Gamma,G)$ denote the quotient projections.

The quotient variety $X(\Gamma,\underline{G})$ is obtained as follows. Suppose $\{\gamma_1,\gamma_2,\ldots,\gamma_N\}$ is a set of generators for Γ and $\{f_1,f_2,\ldots,f_m\}$ is a set of algebra generators for the algebra of invariant polynomials on $\underline{G}^N$. We may choose the f_i's so that they take real values on G^N. We define a map $F:R(\Gamma,\underline{G})/\underline{G} \to \mathbb{C}^m$ by:

$$F(\underline{G}\cdot\rho) = (f_1(\rho(\gamma_1),\ldots,\rho(\gamma_N)),\ldots,f_m(\rho(\gamma_1),\ldots,\rho(\gamma_N)))$$

We caution the reader that F is not necessarily injective.

The image of F is contained in an affine variety determined by the relations among the generators $\{\gamma_1,\ldots,\gamma_N\}$ and the relations among the invariants $\{f_1,\ldots,f_m\}$. Precisely, $X(\Gamma,\underline{G})$ is the affine variety corresponding to the ring of $\underline{G}$ invariant polynomials on $\mathrm{Hom}(\Gamma,\underline{G})$. Then $X(\Gamma,\underline{G})$ is defined over $\mathbb{R}$. The set of real points $X(\Gamma,G)$ contains the image under F of the classes of representations on which the invariants $\{f_1,f_2,\ldots,f_m\}$ take real values. We note that F is the mapping induced by $\underline{\pi}$ on the orbit space of $R(\Gamma,\underline{G})$ to $X(\Gamma,\underline{G})$.

As we have remarked previously, the variety $X(\Gamma,\underline{G})$ is not isomorphic to the orbit space $R(\Gamma,\underline{G})/\underline{G}$. However, we now define a subset of $R(\Gamma,\underline{G})$, the set $S = S(\Gamma)$ of stable representations. This set has the property that F induces a homeomorphism from $S/\underline{G}$ onto an open subset of $X(\Gamma,\underline{G})$.

Definitions. A representation ρ in $R(\Gamma,\underline{G})$ is said to be stable if the orbit $\underline{G}\cdot\rho$ is closed in $R(\Gamma,\underline{G})$ and if the isotropy subgroup $Z(\rho)$ of ρ in G is finite.

A stable representation is said to be good if $Z(\rho) = Z_{\underline{G}}$, the center of $\underline{G}$.

By Newstead [19], Proposition 3.8, S is Zariski open in $R(\Gamma,\underline{G})$. However, S might be empty.

Let Γ_N be the free group on N generators. Then we have a closed embedding $R(\Gamma,\underline{G}) \subset R(\Gamma_N,\underline{G})$. Our definition of stability then gives:

$$S(\Gamma) = R(\Gamma,\underline{G}) \cap S(\Gamma_N).$$

This equality allows us to reduce many problems for Γ to the corresponding problems for Γ_N. This is helpful because $R(\Gamma_N,\underline{G}) = \underline{G}^N$, a non-singular variety.

We now characterize the stable representations in terms of complex parabolic subgroups. Recall that a parabolic subgroup P of a semi-simple Lie group G is the full normalizer of a parabolic subalgebra - an algebra whose complexification contains a maximal solvable subalgebra - see Varadarajan [26], 279-288.

<u>Theorem 1.1</u>. <u>A representation</u> ρ <u>in</u> $R(\Gamma,G)$ <u>is stable if and only if the image of</u> ρ <u>is not contained in any proper parabolic subgroup of</u> G.

<u>Proof</u>. Assume ρ is stable. If the image of ρ is contained in a proper parabolic subgroup, then by conjugating ρ by a one parameter group in the center of a Levi subgroup we find a representation in the closure of the orbit of ρ which is contained in the Levi subgroup. Since the orbit of ρ is closed, this limit representation is conjugate to ρ. But the limit representation has an infinite centralizer in $\underline{G}$ (the centralizer contains the center of the Levi subgroup) contradicting the stability of ρ.

Now suppose that ρ is not stable. Hence, either the orbit of ρ is not closed or $Z(\rho)$ is not finite. Assume the former. By the Hilbert-Mumford Theorem, see Birkes [4], there is a one-parameter group $\lambda:\mathbb{C}_m \to \underline{\mathbb{C}}$ so that $\lim_{t\to 0}\lambda(t)\cdot\rho$ exists. By Mumford-Fogarty [17], Proposition 2.6, this implies that the image of ρ is contained in the parabolic group $P(\lambda)$ (notation of [17]). Thus, we may assume that $Z(\rho)$ is infinite. Hence, the image of ρ fixes an element x in $\underline{g}$, the Lie algebra of $\underline{G}$. Hence, the image of ρ fixes the semi-simple part x_s of x and the nilpotent part x_n of x. If $x_s \neq 0$, then the centralizer of x_s is the Levi subgroup of a proper parabolic and we are done. If $x_n \neq 0$ then the centralizer of x_n is

contained in a parabolic subgroup by the following lemma and again we are done. With this the theorem is proved. We owe the next lemma to Robert Steinberg.

<u>Lemma 1.1</u>. <u>Let</u> g <u>be a complex semi-simple Lie algebra and</u> $x \in g$ <u>a non-zero nilpotent element. The subgroup of</u> G <u>which fixes</u> x <u>in the adjoint action is contained in a proper parabolic subgroup of</u> G.

<u>Proof</u>. By the Jacobson-Morosov Theorem, see Kostant [12], we may choose $h,y \in g$ so that $\{x,y,h\}$ is the standard basis for the Lie algebra $s\ell_2(\mathbb{C})$. The element h acts semi-simply on g with integer eigenvalues. We may decompose g according to $g = \oplus g_i$ where g_i is the eigenspace of g under h corresponding to the eigenvalue i. We define a parabolic subalgebra $\mathcal{P}$ of g by $\mathcal{P} = \oplus_{i\geq 0} g_i$. Let P be the normalizer of $\mathcal{P}$ of $\underline{G}$. By Kostant [12], Theorem 3.6, the element h is unique up to conjugacy by the Lie group G_x corresponding to the subalgebra $g_x = \ker \operatorname{ad} x \cap \operatorname{im} \operatorname{ad} x$. Any element of $\ker \operatorname{ad} x$ is a sum of highest weight vectors for h and hence $g_x \subset \mathcal{P}$ and $G_x \subset P$. Hence the subgroup P is uniquely determined by x. Hence if g fixes x in the adjoint action then g normalizes P. But a parabolic group is its own normalizer and the lemma is proved.

We now make an assumption that will be satisfied by all pairs $(\Gamma,\underline{G})$ considered in this paper.

<u>Assumption</u>. There exists a stable real representation; that is, there exists a representation $\rho \in R(\Gamma,G)$ such that the image of ρ is not contained in any proper parabolic subgroup of $\underline{G}$.

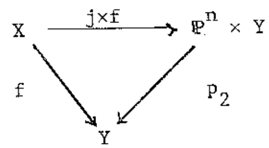
We now recall that a topological group G is said to act properly on a space X if the map $A:G \times X \to X \times X$ given by $A(g,x) = (gx,x)$ is a proper map.

<u>Proposition 1.1</u>. <u>The actions of</u> $\underline{G}$ <u>on</u> S <u>and</u> G <u>on</u> $S \cap R(\Gamma,G)$ <u>are proper</u>.

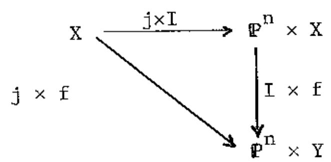
The proposition follows from a result in geometric invariant theory. In order to apply this result we have to relate the algebraic geometry definition of properness to the usual one. To this end, we define a morphism of finite type $f:X \to Y$ of affine varieties X and Y to be Zariski proper if it is Zariski universally closed; that is if for every variety Z the morphism $f \times id:X \times Z \to Y \times Z$ is closed in the Zariski topologies of $X \times Z$ and $Y \times Z$. We now prove a lemma for f as above.

Lemma 1.2. f is proper if and only if f is Zariski proper.

Proof. We leave the implication that proper implies Zariski proper to the reader. Assume f is Zariski proper and let j be an embedding of X in $\mathbb{P}^n$ (the image of X will not be closed in $\mathbb{P}^n$). We have a diagram:



We claim that f Zariski proper implies $j \times f$ is Zariski proper. Certainly $I \times f : \mathbb{P}^n \times \mathbb{X} \to \mathbb{P}^n \times \mathbb{Y}$ is Zariski proper. But from the diagram:



we see that it is enough to prove that $j \times I$ is Zariski proper. But this map is (up to an exchange of factors) the graph map Γ_j of j given by $\Gamma_j(x) = (x, j(x))$. But the graph map of any morphism $h:M \to N$ is proper for it is a closed immersion – the image of Γ_h is the subset of $M \times N$ defined by the equations $h \circ p_1 = p_2$. With this the claim is established. Hence $(j \times f)(X)$ is Zariski closed and hence strongly closed in $\mathbb{P}^n \times Y$. The lemma now follows.

We may now deduce the proposition from the results in Mumford-Fogarty [17], Chapter 2, as follows. By Proposition 2.4, it is sufficient to prove that the action of every one-parameter subgroup $\lambda:\mathbb{C}_m \to G$ on S is proper. Suppose this is false. Then there is a sequence $\{\rho_n\}$ contained in a bounded subset of S, a sequence $\{a_n\}$ in $\mathbb{C}^*$ such that $\lim_{n\to\infty} a_n = \infty$ and a one-parameter group μ such that $\lim_{n\to\infty} \mathrm{Ad}\,\mu(a_n)\cdot\rho_n = \rho$ with $\rho \in S$. But the argument in [17], Proposition 2.6, shows that the image of ρ is contained in a proper parabolic subgroup of G, a contradiction. We note that since G is closed in G and $S \cap R(\Gamma,G)$ is closed in S the first statement of the proposition implies the second. With this the proposition is proved.

We now prove a technical theorem which will be of use later. See Borel-Wallach [6], page 277, for the definition of a slice to a

group action through a point.

Theorem 1.2. The actions of G on S and G on $S \cap R(\Gamma,G)$ admit analytic slices through any ρ.

Proof. It is sufficient to prove the existence of slices on the stable representations in $R(\Gamma_N,G)$ and $R(\Gamma_N,G)$ since a slice in $R(\Gamma_N,G)$ intersects $R(\Gamma,G)$ in a slice. But $S(\Gamma_N)$ and $S(\Gamma_N) \cap R(\Gamma_N,G)$ are manifolds upon which G acts properly with finite isotropy groups. The theorem now follows from Palais [20].

Corollary. If x_t is a germ of a curve through $\pi(\rho)$ in $X(\Gamma,G)$ or $X(\Gamma,G)$ and ρ is good then there is a germ of a curve ρ_t through ρ in $R(\Gamma,G)$ or $R(\Gamma,G)$ with image x_t.

Remark. If ρ is good the quotient map π induces an analytic equivalence between a neighborhood of ρ in a slice through ρ and a neighborhood of $\pi(\rho)$ in $X(\Gamma,G)$.

We prove another general result concerning $R(\Gamma,G)$. Let $S^* = \{\rho \in S : Z(\rho) = Z_G\}$ where Z_G is the center of G. S^* is the set of good representations.

Proposition 1.3. S^* is Zariski open in $R(\Gamma,G)$.

Again it is sufficient to prove the proposition for $\Gamma = \Gamma_N$, the free group on N generators. In this case $R(\Gamma,G)$ is irreducible and consequently S is irreducible.

Lemma 1.3. S^* is open in the strong topology on $R(\Gamma,G)$.

Proof. This is an immediate consequence of the existence of local slices since all identifications on a slice through ρ are made by $Z(\rho)$.

Lemma 1.4. S^* is constructible.

Proof. Let Δ be the diagonal in $S \times S$ and $p_2 : G \times S \to S$ the projection. Then $S - S^* = p_2(A^{-1}(\Delta) - Z_G \times S)$ is a constructible set.

Since S^* is constructible and strongly open it is Zariski open and the proposition is proved.

Now let Γ be a torsion-free group, H a classical simple algebraic group defined over $\mathbb{R}$ with real points H and $\rho_0 : \Gamma \to H$ an embedding of Γ into H as a uniform discrete subgroup. If H is not locally isomorphic to $PSL_2(\mathbb{R})$ then Γ is rigid by the Mostow Rigidity Theorem. However, we now suppose that H is represented in another algebraic group $\underline{G}$ also defined over $\mathbb{R}$ so that the representation $\underline{H} \to \underline{G}$ is defined over $\mathbb{R}$. We assume the image of $\underline{H}$ is not contained in a proper parabolic subgroup of $\underline{G}$. Then ρ_0 composed with the representation $H \to G$ of real points embeds Γ into G as a discrete subgroup (usually no longer a lattice). We abuse notation and denote the resulting embedding by ρ_0. We now discuss two problems motivated by the classical theory of quasi-Fuchsian groups.

The first problem is to study the deformation space $\mathrm{Hom}(\Gamma,G)/G$. The second is to realize this space as the target of the holonomy mapping of a space of locally homogeneous structures on the original compact locally symmetric space $M = \rho_0(\Gamma)\backslash H/K$ where K is a maximal compact subgroup of H. In the classical case of quasi-Fuchsian groups we have $H = PSL_2(\mathbb{R})$ and $G = PSL_2(\mathbb{C})$. In this case M is a compact surface and the deformation space is the space of holonomy representations of flat conformal structures on M.

Unfortunately, in the general case, the possibilities are severely limited. Recall that a representation $\rho \in \mathrm{Hom}(\Gamma,G)$ is said to be locally rigid if the orbit of ρ in $\mathrm{Hom}(\Gamma,G)$ is open in $\mathrm{Hom}(\Gamma,G)$. We recall that two Lie groups are said to be locally isomorphic if they have isomorphic Lie algebras.

Theorem 1.3. ρ_0 is locally rigid in G unless H is locally isomorphic to $SO(n,1)$ or $SU(n,1)$.

Proof. Assume H is not one of the above groups. Let $\underline{h}$ be the complexification of the Lie algebra of H and $\underline{g}$ the complexification of the Lie algebra of G. By a theorem of Weil, see Raghunathan [21], Theorem 6.7, it is sufficient to prove $H^1(\Gamma,\underline{g}) = 0$. Now, decompose $\underline{g}$ into a sum of irreducible $\underline{H}$ modules according to $\underline{g} = \oplus_{j=1}^m V_j$. Then since $\Gamma \subset H$ we have $H^1(\Gamma,\underline{g}) = \oplus_{j=1}^m H^1(\Gamma,V_j)$. Now $\rho_0(\Gamma)$ is a uniform discrete subgroup of H and V_j is an irreducible representation of H. Hence, by Raghunathan [22], we have $H^1(\Gamma,V_j) = 0$ unless H is as above. With this the theorem is proved.

In fact, Raghunathan's theorem tells us that ρ_0 is locally rigid unless V_j is a symmetric power of the standard representation in case $G = SU(n,1)$ or a space of (Minkowski) spherical harmonics in case $G = SO(n,1)$. We give three examples of representations $H \to G$ for which local rigidity does not follow from Raghunathan's theorem. In each case, let V denote the standard representation of H. We assume $n > 2$ for $SO(n,1)$ and $n > 1$ for $SU(n,1)$ so we

have $H^1(\Gamma,\underline{h}) = 0$ by a theorem of Weil, see Raghunathan [21], Chapter VII, section 5. In the orthogonal case, we let S_0^2V denote the (Minkowski) spherical harmonics of degree 2; that is, the "traceless" symmetric 2-tensors. Here the "trace" is the inner product with the $SO(n,1)$ invariant bilinear form $(\,,)$ using the form on the symmetric 2-tensors induced by $(\,,)$.

H	G	g	infinitesimal deformations
SO(n,1)	SO(n+1,1)	$h \oplus V$	$H^1(\Gamma,V)$
SO(n,1)	$PGL_{n+1}(\mathbb{R})$	$h \oplus V$	$H^1(\Gamma,S_0^2V)$
SU(n,1)	SU(n+1,1)	$h \oplus V$	$H^1(\Gamma,V)$

We will discuss the first and second examples in detail in this paper. We note in the first example $\mathfrak{g}$ may be identified with $\Lambda^2(V \oplus L)$ where $V \oplus L$ is the standard representation of $SO(n+1,1)$ and L is a line invariant under $SO(n,1)$. As representation spaces for $SO(n,1)$ we have:

$$\Lambda^2(V \oplus L) = \Lambda^2 V \oplus (L \otimes V) = \Lambda^2 V \oplus V.$$

We will use this identification extensively in Sections 6 and 7. In the second case we have $s\ell_{n+1} \approx \Lambda^2 V \oplus S_0^2 V$ as $SO(n,1)$ modules. In the first case ρ_0 is good if and only if n is even, in the second case ρ_0 is good for all n.

In the third case a more subtle rigidity theorem holds and there are no interesting deformations - see Goldman-Millson [28].

2. Infinitesimal Deformations and Obstructions.

In this section, we review standard material concerning infinitesimal deformations. We begin by recalling the definitions of Eilenberg-MacLane 1-cocycles and coboundaries.

Let V be a vector space and $\rho:\Gamma \to \mathrm{Aut}\ V$ a representation. Then a 1-cocycle on Γ with coefficients in ρ is a map $c:\Gamma \to V$ such that for $\gamma,\delta \in \Gamma$ we have:

$$c(\gamma\delta) = c(\gamma) + \rho(\gamma)\cdot c(\delta).$$

We let $Z^1(\Gamma,V)$ denote the space of 1-cocycles on Γ with values in V. Elements of $Z^1(\Gamma,V)$ are often called crossed-homomorphisms (with values in V).

A 1-cocycle c is said to be a 1-coboundary if there exists $v \in V$ such that:

$$c(\gamma) = \rho(\gamma)v - v \quad \text{for all} \quad \gamma \in \Gamma$$

We denote the subspace of 1-coboundaries by $B^1(\Gamma,V)$ and define the first cohomology group of Γ with values in V by:

$$H^1(\Gamma,V) = \frac{Z^1(\Gamma,V)}{B^1(\Gamma,V)} .$$

There are similar but more complicated definitions for $Z^p(\Gamma,V)$, $B^p(\Gamma,V)$ and $H^p(\Gamma,V)$ for all $p \geq 1$, see Eilenberg-MacLane [9].

Let X be a real algebraic set in $\mathbb{R}^n$ and $x \in X$. Let $\alpha:(-\varepsilon,\varepsilon) \to X$ be a real analytic curve such that $\alpha(0) = x$. Let $\alpha(t) = \Sigma_{k=0}^{\infty}\alpha_k t^k$ be the Taylor series for α about $t = 0$. We define the <u>leading coefficient</u> of α at $t = 0$ to be α_n if $n > 0$, $\alpha_n \neq 0$ and $\alpha_m = 0$ for $0 < m < n$. We then define the <u>tangent cone</u> TC_x of X at x to be the set of all leading coefficients of curves α as above. If X is smooth at x then TC_x coincides with the tangent space to X at x.

Now let $\rho:\Gamma \to G$ be a representation of Γ into the real points G of an algebraic group $\underline{G}$ defined over $\mathbb{R}$. Let ρ_t be a curve in $\mathrm{Hom}(\Gamma,G)$ with $\rho_0 = \rho$. Let $\dot{\rho}(\gamma) \in T_{\rho(\gamma)}(G)$ be the leading coefficient at $t = 0$ to the curve $\rho_t(\gamma)$ in G. Define a function c from Γ to the Lie algebra $\mathfrak{g}$ of G by:

$$c(\gamma) = \dot{\rho}(\gamma)\rho(\gamma)^{-1}.$$

The following lemma is immediate, observe that Γ acts on $\mathfrak{g}$ by the composition of ρ with the adjoint action of G on $\mathfrak{g}$.

<u>Lemma 2.1</u>. c <u>is a cocycle</u>.

One obtains in this way an embedding of the tangent cone at ρ to $\mathrm{Hom}(\Gamma,G)$ into $Z^1(\Gamma,\mathfrak{g})$. For this reason, the space $Z^1(\Gamma,\mathfrak{g})$ will be called the space of infinitesimal deformations of ρ. We let TC_ρ denote the tangent cone to $\mathrm{Hom}(\Gamma,G)$ at ρ.

Suppose now that ρ_t is a trivial deformation of ρ ; that is, suppose there exists a curve g_t in G with $g_0 = 1$, the identity in $\underline{G}$, such that $\rho_t = \mathrm{Ad}\, g_t \cdot \rho$. Let $\dot{g}$ be the tangent vector to g_t at $t = 0$; hence $\dot{g} \in \mathfrak{g}$. Upon differentiating we obtain:

$$c(\gamma) = \dot{g} - \mathrm{Ad}\,\rho(\gamma)\dot{g}$$

and we have proved the following lemma.

<u>Lemma 2.2</u>. <u>If</u> c <u>is tangent to a trivial deformation then</u> c <u>is a 1-coboundary. Conversely, every 1-coboundary is a tangent to a trivial deformation</u>.

<u>Corollary</u>. <u>If</u> $c \in TC_\rho$ <u>then</u> $c + b \in TC_\rho$ <u>for all</u> $b \in B^1(\Gamma,\mathfrak{g})$.

<u>Proof</u>. $c + b$ is tangent to the deformation $\mathrm{Ad}\, g_t \cdot \rho_t$.

<u>Remark</u>. By the previous lemma, the map $d\pi$ annihilates $B^1(\Gamma,\mathfrak{g})$ and induces a map $\overline{d\pi}$ from the image of TC_ρ in $H^1(\Gamma,\mathfrak{g})$ to the tangent cone to $X(\Gamma,G)$ at $\pi(\rho)$. We can obtain more information in case ρ is a good representation. In this case $\pi|S^*(\Gamma_N)$ is a principal bundle so $\overline{d\pi}|H^1(\Gamma,\mathfrak{g})$ is injective. In this case we may identify $d\pi$ with the projection $Z^1(\Gamma,\mathfrak{g})$ to $H^1(\Gamma,\mathfrak{g})$ restricted to TC_ρ.

<u>Lemma 2.3</u>. <u>If</u> ρ <u>is a good representation then</u> $d\pi$ <u>maps</u> TC_ρ <u>onto the tangent cone of</u> $X(\Gamma,G)$ <u>at</u> $\pi(\rho)$.

<u>Proof</u>. If z is an element in the tangent cone to $X(\Gamma,G)$ at $\pi(\rho)$ then there exists a curve x_t in $X(\Gamma,G)$ with tangent vector z at $t = 0$. But by the corollary to Theorem 1.1 we can lift x_t near $t = 0$ to a curve in $\mathrm{Hom}(\Gamma,G)$. The surjectivity of $d\pi$ follows.

<u>Remark</u>. We call an element of $H^1(\Gamma,\mathfrak{g})$ an infinitesimal deformation of $\pi(\rho)$.

We now derive a necessary condition for an element $c \in Z^1(\Gamma,\mathfrak{g})$ to be the leading coefficient at $t = 0$ to a curve ρ_t in $\mathrm{Hom}(\Gamma,G)$. Recall that the cup-square of $c \in Z^1(\Gamma,\mathfrak{g})$ is the element $[c,c] \in Z^2(\Gamma,\mathfrak{g})$ defined by:

$$[c,c](\gamma,\delta) = [c(\gamma),\mathrm{Ad}\,\rho(\gamma)c(\delta)].$$

Here $[,]$ denotes the bracket operation in $\mathfrak{g}$. The following proposition follows from Lemma 2.4 of Goldman-Millson [28].

<u>Proposition 2.1</u>. (i) <u>If</u> c <u>is an element in</u> $Z^1(\Gamma,\mathfrak{g})$ <u>which is the leading coefficient at</u> $t = 0$ <u>to a curve</u> ρ_t <u>in</u> $R(\Gamma,G)$ <u>then</u> $[c,c]$ <u>is the zero element in</u> $H^2(\Gamma,\mathfrak{g})$.

(ii) <u>If</u> $z \in H^1(\Gamma,\mathfrak{g})$ <u>is such that</u> $d\pi(z)$ <u>is the leading coefficient at</u> $t = 0$ <u>to a curve in</u> $X(\Gamma,G)$ <u>and</u> ρ_0 <u>is good, then</u> $[z,z] = 0$ <u>in</u> $H^2(\Gamma,\mathfrak{g})$.

The second part of the proposition requires some comment. First, it is standard that the cup-product is a well-defined map from $H^1(\Gamma,\mathfrak{g}) \otimes H^1(\Gamma,\mathfrak{g})$ to $H^2(\Gamma,\mathfrak{g})$. Second, by the corollary to Theorem 1.2, a germ in $X(\Gamma,G)$ with leading coefficient z can be lifted to a germ in $\mathrm{Hom}(\Gamma,G)$ with leading coefficient c, where c is a representative cocycle for z. Then $[c,c]$ represents $[z,z]$ and is a coboundary by (i).

Definition. Given an infinitesimal deformation $z \in H^1(\Gamma,\mathfrak{g})$ the class $[z,z] \in H^2(\Gamma,\mathfrak{g})$ is called the first obstruction to the existence of a deformation tangent to z.

Remark. There is an infinite sequence of obstructions to the existence of a deformation tangent to z. Their construction follows the general scheme of Kodaira-Spencer deformation theory. The second obstruction is an analogue of the Massey product and may be interesting for three manifolds.

3. Quasi-Fuchsian Groups in Hyperbolic n-space.

In this section, we will specialize the considerations of Section 1 to the case $\underline{H} = \underline{SO}(n,1)$ and $\underline{G} = \underline{SO}(n+1,1)$. By $\underline{SO}(n,1)$ we will mean the complex points of the algebraic group of orientation preserving isometries of the quadratic form for $\mathbb{C}^{n+1}$ given by $f(x_1,x_2,\ldots,x_{x+1}) = -x_1^2 + x_2^2 + \cdots + x_{n+1}^2$. The symbol $SO(n,1)$ will denote the real points of $\underline{SO}(n,1)$. We will embed $\underline{SO}(n,1)$ into $\underline{SO}(n+1,1)$ as the isotropy subgroup of the last standard basis vector.

We let Γ be a torsion free group embedded as a uniform discrete subgroup by $\rho_0:\Gamma \to SO(n,1)$. We assume for convenience that $\rho_0(\Gamma) \subset SO_0(n,1)$, the connected component of the identity. We wish to study the space $\mathrm{Hom}(\Gamma,SO(n+1,1))$, its orbit space $\mathrm{Hom}(\Gamma,SO(n+1,1))/SO(n+1,1)$ and its algebraic geometrical quotient $X(\Gamma,SO(n+1,1))$. For many reasons (among them to describe a nice neighborhood of ρ_0 in the quotient) it is useful to impose a technical condition on the representations considered. We observe any representation ρ of Γ in $SO(n+1,1)$ defines an action of Γ on S^n, the boundary of hyperbolic space $\mathbb{H}^{n+1}$.

Definition. A representation ρ in $\mathrm{Hom}(\Gamma,SO(n+1,1))$ is said to be quasi-Fuchsian if the action of Γ via ρ on S^n is quasi-conformally conjugate to the action via ρ_0.

We will call ρ_0 or any representation conjugate by an element of $SO(n+1,1)$ to ρ_0 a Fuchsian representation. Our terminology is classical in the case $n = 2$. We let $R_n(\Gamma)$ denote the space of all quasi-Fuchsian representations of Γ and $T_n(\Gamma)$ the space of conjugacy classes of quasi-Fuchsian representations.

We now prove that the representations in $R_n(\Gamma)$ have several good properties. We note first that for any $\rho \in R_n(\Gamma)$ the subgroup $\rho(\Gamma) \subset SO(n+1,1)$ is discrete since it has a non-empty domain of discontinuity on S^n. Also since the action of any such ρ is topologically conjugate to ρ_0, the group $\rho(\Gamma)$ does not fix any point of S^n and consequently is not contained in any parabolic subgroup of $SO(n+1,1)$. By Morgan [16], Lemma 1.1, we have the following lemma.

Lemma 3.1. If $\rho \in R_n(\Gamma)$ then the $SO(n+1,1)$ orbit of ρ in $\mathrm{Hom}(\Gamma,SO(n+1,1))$ is closed.

Corollary. If $\rho \in R_n(\Gamma)$ then the $\underline{SO}(n+1,1)$ orbit of ρ in $\mathrm{Hom}(\Gamma,\underline{SO}(n+1,1))$ is closed.

Proof. The corollary follows from Birkes [4].

We now show that if $\rho \in R_n(\Gamma)$ and ρ is not Fuchsian then the image of ρ is Zariski dense in $\underline{SO}(n+1,1)$.

By [7], Theorem 4.4.2, we see that it is sufficient to prove that Γ does not leave invariant a totally geodesic subspace of $\mathbb{H}^{n+1}$. But if $\rho(\Gamma)$ leaves invariant a totally geodesic subspace of dimension k with $k < n$, then, since $\rho(\Gamma)$ is discrete, it would operate properly discontinuously on some $\mathbb{H}^k$ and consequently have homological dimension less than or equal to k. But $H_n(\Gamma,\mathbb{R}) = \mathbb{R}$. Finally, if $\rho(\Gamma)$ leaves an $\mathbb{H}^n$ invariant then we transform this $\mathbb{H}^n$ to the standard ^{n} by an element of $SO(n+1,1)$. But $M = \rho(\Gamma)\backslash\mathbb{H}^n$ must be compact since $H_n(M,\mathbb{R}) = H_n(\Gamma,\mathbb{R}) = \mathbb{R}$. We can apply Mostow rigidity to conclude ρ is Fuchsian. We obtain:

Lemma 3.2. A quasi-Fuchsian representation which is not Fuchsian is Zariski dense.

With these two theorems we have established that $R_n(\Gamma)$ is contained in the subset S of stable representations (Section 1); moreover, if $\rho \in R_n(\Gamma)$ is not Fuchsian then it is good.

The image of S in the variety $X(\Gamma,\underline{SO}(n+1,1))$ is its orbit space – Newstead [19], Proposition 3.8; that is, $\underline{\pi}(\rho_1) = \underline{\pi}(\rho_2)$ if and only if ρ_1 and ρ_2 are conjugate in $\underline{SO}(n+1,1)$.

Lemma 3.3. If $\rho_1,\rho_2 \in \mathrm{Hom}(\Gamma,SO(n+1,1))$ and ρ_1 is Zariski dense then $\underline{\pi}(\rho_1) = \underline{\pi}(\rho_2)$ if and only if ρ_1 and ρ_2 are conjugate by an element of $SO(n+1,1)$.

Proof. Since $\underline{\pi}(\rho_1) = \underline{\pi}(\rho_2)$ there exists $g \in \underline{SO}(n+1,1)$ so that $g\rho_1 g^{-1} = \rho_2$. Applying complex conjugation σ we find $\sigma(g)\rho_1\sigma(g)^{-1} = \rho_2$. Hence $\sigma(g)^{-1}g$ centralizes ρ_1. Since ρ_1 is Zariski dense, its centralizer $Z(\rho_1)$ is Z_G, the center of G. Thus either $\sigma(g) = g$ and we are done or $\sigma(g) = -g$. In this second case $g = ih$ with h in $GL_{n+2}(\mathbb{R})$. But we claim that $\underline{SO}(n+1,1)$ contains no pure imaginary matrices (for $n \geq 2$). Indeed h would transform the matrix A of the form f relative the standard basis into its negative. Note ${}^t(ih)A(ih) = A$. But f and $-f$ have different signatures. With this the lemma is proved.

Corollary. $T_n(\Gamma)$ embeds in $X(\Gamma, SO(n+1,1))$.

We are now ready to prove that $R_n(\Gamma)$ is open. First we need a lemma, the main idea of which we owe to John Morgan. We refer the reader to Thurston [25], 8.1 and 8.2, for the definitions and properties of the limit set $\Lambda(\rho(\Gamma))$ (denoted L_Γ in Thurston) and the regular set $\Omega(\rho(\Gamma)) = S^n - \Lambda(\rho(\Gamma))$ for the action of $\rho(\Gamma)$ on S^n.

Let ρ be quasi-Fuchsian. Then we know $\Lambda(\rho(\Gamma))$ is homeomorphic to a sphere and $\Omega(\rho(\Gamma))$ is homeomorphic to two disjoint open hemi-spheres Ω_+ and Ω_-.

Lemma 3.4. $M(\Gamma) = (\mathbb{H}^{n+1} \cup \Omega(\rho(\Gamma)))/\rho(\Gamma)$ is compact.

Proof. $M(\Gamma)$ is a (possibly non-compact) manifold with boundary components the quotients of the two hemi-spheres Ω_+ and Ω_-. Since the action of ρ is topologically conjugate to that of ρ_0, we know $N_+ = \Omega_+/\rho(\Gamma), N_- = \Omega_-/\rho(\Gamma)$ and $\mathbb{H}^n/\rho_0(\Gamma)$ are homeomorphic. Hence N_+ and N_- are compact orientable n-manifolds. Since the universal cover Ω_+ of N_+ embeds into $\mathbb{H}^{n+1} \cup \Omega_+ \cup \Omega_-$ we know $\pi_1(N_+)$ injects into $\pi_1(M(\Gamma))$. But this map is clearly surjective since $\rho_0(\Gamma)$ stabilizes Ω_+. Hence the inclusions $N_+ \subset M(\Gamma)$ and $N_- \subset M(\Gamma)$ are homotopy equivalences. Hence $H_n(M(\Gamma), \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and the homology classes represented by N_+ and N_- are each generators. Hence N_+ and N_- (with the opposite orientation) are homologous. But then there is a finite chain on $M(\Gamma)$ with boundary $N_+ \cup N_-$ and the lemma is proved.

Corollary. $N(\Gamma) = H(\Lambda(\rho(\Gamma)))/\rho(\Gamma)$ is compact (here $H(X)$ denotes the convex hull of X).

Proof. $N(\Gamma)$ and $M(\Gamma)$ are homotopy equivalent manifolds with boundary, Thurston [25], 8.3.5, via a boundary preserving homotopy equivalence. Hence $N(\Gamma)$ is compact.

We are now ready to prove the main theorem of this section. We will make frequent use of Lok [13], Theorem 2.123 and also use an unpublished argument of Lok.

Theorem 3.1. $R_n(\Gamma)$ is open in $\mathrm{Hom}(\Gamma, SO(n+1,1))$.

Proof. Let ρ be a quasi-Fuchsian representation and α be a small positive number.

Consider the manifold M obtained as the quotient of the open ε-neighborhood of $H(\Lambda(\rho(\Gamma)))$ in $\mathbb{H}^{n+1}$ by $\rho(\Gamma)$. Then ρ is the holonomy of the resulting (incomplete) hyperbolic structure on M. Then by Lok [13], Theorem 2.123, for any $\rho' \in R(\Gamma, SO(n+1,1))$ sufficiently close to ρ there exists an open hyperbolic manifold M' and ψ, a diffeomorphism from M to M' which has the property that ψ maps any geodesic arc in M to an arc in M' of curvature less than α.

We first claim that the developing map $D: \tilde{M}' \to \mathbb{H}^{n+1}$ is injective provided $\alpha < 1$ (here $\tilde{M}'$ denotes the universal cover of M'). Suppose that there exist x', y' in $\tilde{M}'$ so that $D(x') = D(y')$. We can join the preimages of x' and y' under $\tilde{\psi}$ (the lift of ψ) by a geodesic arc since $\tilde{M}$ is convex. Hence x' and y' can be joined by an arc γ of curvature less than 1. But if $D(x') = D(y')$ then $D(\gamma)$ is a closed curve in $\mathbb{H}^{n+1}$ with a single corner and curvature less than 1. No such curve exists in $\mathbb{H}^{n+1}$, see Lok [13] Proposition 2.112.

As a consequence of the result of the previous paragraph we may identify the universal cover of M' with a subset of $\mathbb{H}^{n+1}$ (via the developing map). Of course we may do the same for M. The convex hull of a connected subset X of M' is then defined as the quotient by $\rho'(\Gamma)$ of the convex hull in $\mathbb{H}^{n+1}$ of a connected component of the inverse image of X in M'. Let $\bar{M}' = H^{n+1}/\rho'(\Gamma)$ so $M' \subset \bar{M}'$.

Let $C \subset M$ be the Nielsen convex core of M; that is, $C = H(\Lambda(\rho(\Gamma)))/\rho(\Gamma)$ and let N be the closed ε'-neighborhood of C in M (we assume $\varepsilon' < \varepsilon$). Then N is a strictly convex hyperbolic manifold with C^1 boundary and with holonomy ρ. We claim we can construct a hyperbolic manifold $N' \subset \tilde{M}'$ which is diffeomorphic to N, has holonomy ρ' and is strictly convex. We first consider

$\psi(N) \subset M'$. Unfortunately $\psi(N)$ is not necessarily convex but we claim its convex hull is within the $(n+1)\delta(\alpha)$ neighborhood of $\psi(N)$ where $\delta(\alpha) = \cosh^{-1}(1/\sqrt{1-\alpha^2})$. In particular, $\lim_{\alpha\to 0}\delta(\alpha) = 0$. We owe the proof to Larry Lok.

Let p and q be points in $\psi(\tilde{N})$. Then by the argument of the previous paragraph we may join p and q by a curve σ in $\psi(\tilde{N})$ with small curvature α. By Lok [13], Corollary 2.113, the segment σ is homotopic (but not necessarily with endpoints fixed) to a geodesic γ so that σ is within the standard equidistant neighborhood of γ of radius $\delta(\alpha)$. Since this neighborhood is convex, we may find a geodesic γ' joining p and q within this neighborhood. Hence, all geodesic segments in $\mathbb{H}^{n+1}$ joining points of $\psi(\tilde{N})$ lie within the $\delta(\alpha)$ neighborhood of $\psi(\tilde{N})$. We define the k-hull of $\psi(\tilde{N})$ to be the set of convex combinations of k-tuples of points of $\psi(\tilde{N})$. We now show by induction that the k-hull of $\psi(\tilde{N})$ lies within $k\delta(\alpha)$ of $\psi(\tilde{N})$. The previous argument proves the assertion for $k = 1$. Assume that the assertion is proved for the (k-1)-hull. We observe that the k-hull is the 1-hull of the (k-1)-hull. Let γ be a geodesic segment connecting two points x and y of the (k-1) hull. By the induction hypothesis, there exist points x' and y' in $\psi(\tilde{N})$ so that $d(x',x) < (k-1)\delta(\alpha)$ and $d(y',y) < (k-1)\delta(\alpha)$ – here d denotes the hyperbolic distance. Let γ' be the geodesic segment in $\mathbb{H}^{n+1}$ joining x' and y'. Then, by the case $k = 1$, for any z' on γ' there exists z'' in $\psi(\tilde{N})$ so that $d(z',z'') < \delta(\alpha)$. But the function $d(z,\gamma')$ is a convex function on $\mathbb{H}^{n+1}$ and hence its restriction to γ takes its maximum value at either x or y. Hence, for any z on γ, there exists z' on γ' so that $d(z,z') < (k-1)\delta(\alpha)$. But choosing a z'' as above we find z'' in $\psi(\tilde{N})$ so that $d(z,z'') < k\delta(\alpha)$. We conclude that the k-hull is within $k\delta(\alpha)$ of $\psi(\tilde{N})$. Taking $k = n+1$ we find that the convex hull of $\psi(\tilde{N})$ is within $(n+1)\delta(\alpha)$ of $\psi(\tilde{N})$.

If we choose α small enough, the boundary of $H(\psi(N))$ will be within a tubular neighborhood of the boundary of $\psi(N)$ and will be transverse to the fibers of that tubular neighborhood. Hence, we may construct a self-diffeomorphism f of $\bar{M}'$ which carries the boundary of $\psi(N)$ to the boundary of $H(\psi(N))$ and consequently carries $\psi(N)$ to $H(\psi(N))$. Now let N' be an ε''-neighborhood of $H(\psi(N))$. Clearly $H(\psi(N))$ and N' are diffeomorphic. Thus, we have found a strictly convex hyperbolic manifold N' with holonomy ρ' and a diffeomorphism

φ from N to N' as required.

We lift φ to a diffeomorphism $\tilde{\varphi}$ from $\tilde{N}$ to $\tilde{N}'$. The sets $\tilde{N}$ and $\tilde{N}'$ are strictly convex submanifolds of $\mathbb{H}^{n+1}$. We may then extend $\tilde{\varphi}$ to $\mathbb{H}^{n+1}$ by mapping normal rays to normal rays as in Thurston [26], 8.3.4, to obtain a quasi-isometry conjugating the action of ρ on $\mathbb{H}^{n+1}$ to the action of ρ' on $\mathbb{H}^{n+1}$. The boundary value of this quasi-isometry gives the required quasi-conformal conjugacy. With this Theorem 3.1 is proved.

Corollary. $T_n(\Gamma)$ is an open subset of the real quasi-algebraic set determined by the image of $\mathrm{Hom}(\Gamma, SO(n+1,1))$ in $X(\Gamma, SO(n+1,1))$. We recall that a real quasi-algebraic set is a subset of $\mathbb{R}^m$ determined by polynomial equations and inequalities.

Proof. The image of $\mathrm{Hom}(\Gamma, SO(n+1,1))$ is the image of a real algebraic set by a polynomial mapping and consequently it is quasi-algebraic by the Tarski- Seidenberg Theorem [8].

In order to improve on this result, we must make a more careful study of the real algebraic set $X(\Gamma, SO(n+1,1))$ embedded into $\mathbb{C}^m$ as described in Section 1. We know $T_n(\Gamma)$ is an open subset of the image of $\mathrm{Hom}(\Gamma, SO(n+1,1)))$ under π. We now determine when the image of $\mathrm{Hom}(\Gamma, SO(n+1,1))$ under π is open in $X(\Gamma, SO(n+1,1))$.

Recall that $S^* \subset S$ is the set of representations ρ with the properties:

(i) $\underline{G}\cdot\rho$ is closed in $R(\Gamma, \underline{G})$

(ii) $Z(\rho) = Z_{\underline{G}}$.

We have seen in Section 1 that S^* is Zariski open in $R(\Gamma, \underline{G})$.

We now compute the real points of $S^*/\underline{G}$. Let σ be the conjugation of $\underline{G}$ relative the real form G. We note that the action of σ descends to $X(\Gamma, \underline{SO}(n+1,1))$. A superscript σ on a set will denote the subset of fixed-points for σ. We define an action of τ, the non-trivial element of the Galois group of $\mathbb{C}$ over $\mathbb{R}$, on the regular functions on $R(\Gamma, \underline{SO}(n+1,1))$ by $\tau\cdot f(\rho) = \overline{f(\sigma(\rho))}$ where $\bar{\ }$ denotes complex conjugation.

Lemma 3.6. $(S/\underline{G})^\sigma$ is the set of real points $\underline{\pi}(S)_{\mathbb{R}}$ of $\underline{\pi}(S)$.

Proof. By definition $\underline{\pi}(S)_{\mathbb{R}}$ is the set of points in $\underline{\pi}(S)$ where the real invariants take real values. Since the orbits of $\underline{G}$ on S are closed we may identify $S/\underline{G}$ and $\underline{\pi}(S)$. Let f be a real invariant; hence, $\overline{f(\rho)} = f(\sigma(\rho))$. Now $f(\rho)$ is real if and only if $\overline{f(\rho)} = f(\rho)$

or $f(\sigma(\rho)) = f(\rho)$. Since the real invariants separate $\underline{G}$ orbits in S we conclude that ρ and $\sigma(\rho)$ are conjugate under $\underline{G}$ or $\sigma(\underline{\pi}(\rho)) = \underline{\pi}(\rho)$. With this the lemma is proved.

By the previous lemma we know that to compute $\underline{\pi}(S)_{\mathbb{R}}$ we have only to compute the fixed-point set $(S/\underline{G})^{\sigma}$. In fact, it is considerably easier to compute $\underline{\pi}(S^*)_{\mathbb{R}} = (S^*/\underline{G})^{\sigma}$. This we now do.

Let $x \in S^*/\underline{G}$ be a fixed-point of σ. Choose $\rho \in \underline{\pi}^{-1}(x)$. Then $\sigma(\rho) = \mathrm{Ad}\, h\cdot\rho$ for some $h \in \underline{G}$. Applying σ again we find $\sigma(h)h = \pm 1$ since $Z(\rho) = \pm 1$ (in fact if $n+2$ is odd we have $\sigma(h)h = +1$). Hence h is a cocycle, $h \in Z^1(\sigma, \mathrm{Ad}\, \underline{G})$. We recall that h_1 and h_2 in $Z^1(\sigma, \mathrm{Ad}\, G)$ are cohomologous if there exists $g \in \underline{G}$ so that $h_1 = \sigma(g)h_2g^{-1}$. The set of cohomology classes of cocycles is the cohomology set $H^1(\sigma, \mathrm{Ad}\, \underline{G})$. We enumerate this set as $\{h_i\}$ with $i \in I$; in fact, this set is known to be finite. However, this will follow from Lemma 3.7 combined with the fact that a real algebraic set has a finite number of connected components.

For each $h_i \in H^1(\sigma, \mathrm{Ad}\, \underline{G})$ let $(S^*)^{h_i} = \{\rho \in S^* : \sigma(\rho) = h_i\rho h_i^{-1}\}$. Then $(S^*)^{h_i} = (S^*)^{-h_i}$ and $\underline{\pi}((S^*)^{h_i}) \subset (S^*/\underline{G})^{\sigma}$.

<u>Lemma 3.7</u>. <u>The map</u> $h \to \underline{\pi}((S^*)^h)$ <u>induces a one-to-one correspondence between</u> $H^1(\sigma, \mathrm{Ad}\, \underline{G})$ <u>and the connected components of</u> $(S^*/\underline{G})^{\sigma}$.

<u>Proof</u>. The lemma follows easily since $\mathrm{Ad}\, \underline{G}$ acts freely on S^*.

<u>Corollary</u>. $\pi((S^*)^{\sigma})$ <u>is a connected component of</u> $(S^*/G)^{\sigma}$.

The main result of this section now follows.

<u>Theorem 3.2</u>. (i) $T_n(\Gamma) - \pi(\rho_0)$ is an open subset of the real algebraic set $X(\Gamma, SO(n+1,1))$.

(ii) If n is even, $T_n(\Gamma)$ is an open subset of the real algebraic set $X(\Gamma, SO(n+1,1))$.

<u>Proof</u>. $R_n(\Gamma)$ is open in $\mathrm{Hom}(\Gamma, SO(n+1,1))$ by Theorem 3.1. But by Lemma 3.1 we know $G\cdot\rho_0$ is closed in S^{σ} hence $R_n(\Gamma) - G\rho_0$ is open in S^{σ}. But, by Lemma 3.2, we know $R_n(\Gamma) - G\rho_0 \subset (S^*)^{\sigma}$ and $R_n(\Gamma) - G\rho_0$ is an open G-invariant subset; consequently, its image $T_n(\Gamma) - \pi(\rho_0)$ is open in $\pi((S^*)^{\sigma})$. The statement (i) follows.

In case n is even, an easy calculation shows that for ρ Fuchsian we have $Z(\rho) = Z_{\underline{G}}$. Hence, in this case $R_n(\Gamma) \subset (S^*)^{\sigma}$. With this the theorem is proved.

<u>Remark</u>. In case n is odd, $Z(\rho_0)$ is larger than $Z_{\underline{G}}$ and there is

another component of $(S/\underline{G})^{\sigma}$ containing $\pi(\rho_0)$ namely $\pi(\mathrm{Hom}(\Gamma, SO(n,2))$ where $\underline{SO}(n,2)$ is identified with $\underline{SO}(n+1,1)$ by conjugating by the diagonal matrix with diagonal entries $(1,1,\ldots,1,i)$. Thus, part (ii) of the theorem is false for n odd if $\pi(\mathrm{Hom}(\Gamma, SO(n,2)))$ contains non-trivial curves through ρ_0. The methods of Section 5 show that this is often the case for the standard arithmetic examples.

4. <u>Homology and Cohomology with Local Coefficients and the Crossed Homomorphism Associated to a Hypersurface with Coefficient</u>.

Let X be the underlying space of a simplicial complex and E a flat bundle over X. We wish to define homology and cohomology groups with values in E. We define a p-chain with values in E to be a formal sum $\Sigma_{i=1}^{m} c_i\sigma_i$ where σ_i is an ordered p-simplex and c_i is an element of the fiber of E over the first vertex of σ_i. We denote the group of such chains by $C_p(X,E)$ and define the boundary operator:

$$\partial_p : C_p(X,E) \to C_{p-1}(X,E)$$

by

$$\partial_p(c\sigma) = \langle v_0,v_1\rangle^*(c)\sigma_0 + \sum_{j=1}^{p} (-1)^j c\sigma_j.$$

Here σ_j is the jth face of σ and $\langle v_0,v_1\rangle^*$ is parallel translation from v_0 to v_1 along the edge $\langle v_0,v_1\rangle$. Then $\partial_p^2 = 0$ and we may define homology groups with coefficients in E.

In a similar way cohomology with coefficients in a flat bundle is defined. An E-valued p-cochain on X is a function which assigns to each ordered p-simplex σ an element of the fiber of E over the first vertex of σ. The coboundary $\delta\alpha$ of a p-cochain α is defined on a (p+1)-simplex σ by:

$$\delta\alpha(\sigma) = \langle v_1,v_0\rangle^*\alpha(\sigma_0) + \sum_{j=1}^{p} (-1)^j \alpha(\sigma_j).$$

Here σ_j is the jth face of σ. Then $(\delta)^2 = 0$ and we may define cohomology groups with values in E.

Choose a base-point $x_0 \in X$ and a base-point $\tilde{x}_0$ for $\tilde{X}$, the universal cover of X, such that $\pi(\tilde{x}_0) = x_0$ where $\pi : \tilde{X} \to X$ is the covering. Let E_0 be the fiber of E over x_0, so E_0 is also the fiber of π^*E over $\tilde{x}_0$. Note there is a map $\varepsilon : C^0(\tilde{X}, \pi^*E) \to E_0$ using parallel translation of fibers of π^*E to E_0 (this is independent of path on $\tilde{X}$). We use ε to construct a map from $Z^1(X,E)$, the

1-cocycles on X with values in E, to $Z^1(\pi_1(X,x_0),E_0)$, the crossed-homomorphisms on $\pi_1(X,x_0)$ with values in E_0. Let $\alpha \in Z^1(X,E)$ be given. Define $\varphi_\alpha:\pi_1(X,x_0) \to E_0$ on $\gamma \in \pi_1(X,x_0)$ as follows. Let $\tilde{\gamma}$ be a simplicial lift of γ to $\tilde{X}$ starting at x_0. Then:

$$\varphi_\alpha(\gamma) = \varepsilon(\pi^*\alpha(\tilde{\gamma})).$$

It is easily seen that φ_α is a crossed-homomorphism and that the map $\alpha \to \varphi_\alpha$ induces an isomorphism from $H^1(X,E)$ to $H^1(\pi_1(X,x_0),E_0)$.

If α is a p-cochain with coefficients in E and b is a p-chain with coefficients in M and $v:E \otimes M \to N$ is a parallel section of $\mathrm{Hom}(E \otimes M,N)$, then the Kronecker index $\langle\alpha,b\rangle$ is defined by:

$$\langle\alpha,b\rangle = \sum_{\text{simplices } \sigma \text{ in } b} v(\alpha(\sigma) \otimes b_\sigma).$$

Here b_σ is the coefficient of σ in b.

The Kronecker index is well-behaved under ∂ and δ and we get a map:

$$\langle,\rangle: H^p(X,E) \otimes H_p(X,M) \to N.$$

The Kronecker index allows us to identify $H^p(X,E^*)$ and $H_p(X,E)^*$. Indeed, if $\alpha \in H^p(X,E^*)$ and $\langle\alpha,b\rangle = 0$ for all $b \in H_p(X,E)$, then α annihilates the kernel of ∂_p, so α is in the image of δ. This is true because the chain groups are vector spaces so the image of ∂_p is a direct summand.

Now assume X is an oriented n-manifold and E, M, N are flat bundles over X and $v:E \otimes M \to N$ is a parallel section of the flat bundle $\mathrm{Hom}(E \otimes M,N)$. Let $a \in H_{n-p}(X,E)$ and $b \in H_{n-q}(X,M)$ and assume that the simplices of a are in general position with respect to the simplices of b. Then the intersection product $a \cdot b$ of a and b is defined by:

(1) Intersect each simplex σ of a with τ of b to get an $n - (p+q)$ simplex as usual.

(2) Give the resulting simplex the coefficient $v(a_\sigma \otimes b_\tau)$; this has to be given at the initial vertex of the intersection; however, since σ is contractible there is a unique way to move a_σ to any other point of σ and the same for b_τ. In this way we obtain the intersection pairing:

$$H_{n-p}(X,E) \otimes H_{n-q}(X,M) \to H_{n-(p+q)}(X,N)$$

The geometric version of Poincare duality for coefficients

states that the intersection pairing:

$$H_{n-p}(X,E) \otimes H_p(X,E^*) \to \mathbb{R}$$

is a perfect pairing. We then obtain an isomorphism from $H_{n-p}(X,E)$ to $H_p(X,E^*)^*$. Composing this isomorphism with that obtained from the Kronecker index we obtain an isomorphism:

$$PD: H_{n-p}(X,E) \to H^p(X,E).$$

Using v and the usual formula for cup-product of simplicial cochains we obtain a cup-product to be denoted $\cup$:

$$H^p(X,E) \otimes H^q(X,M) \to H^{p+q}(X,N).$$

The cohomological version of Poincare duality tells us that the following pairing is perfect:

$$H^p(X,E) \otimes H^{n-p}(X,E^*) \to \mathbb{R}.$$

Remark. If Z is an $(n-p)$-cycle with coefficients in E, then $PD(Z)$ is characterized by the equation:

$$\langle\eta \cup PD(Z),X\rangle = \langle\eta,Z\rangle$$

for all $\eta \in H^{n-p}(X,E^*)$.

There is a formula relating PD, $\cdot$ and $\cup$ which will be critical to us (in a special case). Let $v:E \otimes M \to N$ be as before. We will not prove the following lemma but we will prove the special case we need, Lemma 4.3.

Lemma 4.1. The following diagram is commutative:

$$\begin{array}{ccc} H^p(X,E) \otimes H^q(X,M) & \xrightarrow{\cup} & H^{p+q}(X,N) \\ PD \otimes PD \uparrow & & PD \uparrow \\ H_{n-p}(X,E) \otimes H_{n-q}(X,M) & \to & H_{n-(p+q)}(X,N) \end{array}$$

There is a particularly simple construction of cycles with coefficients in E. Let Y be a closed, oriented submanifold of X of codimension p and let s be a parallel section of the restriction of E to Y. Let $[Y]$ denote the fundamental cycle of Y so $[Y] = \Sigma_i \sigma_i$, a sum of ordered $n-p$ simplices.

Definition (Notation). $Y \otimes s$ denotes the $(n-p)$ chain with values in E given by $Y \otimes s = \Sigma\sigma_i \otimes s_i$ where s_i is the value of s on the first vertex of σ_i.

Clearly $Y \otimes s$ is a cycle. Suppose now that Y_1 and Y_2 are closed, oriented submanifolds of codimension p and q respectively,

s_1 and s_2 are parallel sections of $E|Y_1$ and $E|Y_2$ respectively, and Y_1 and Y_2 intersect transversely. Then we may simplify the previous general formula defining $(Y_1 \otimes s_1)\cdot(Y_2 \otimes s_2)$ as follows:

(1) Intersect Y_1 and Y_2 in the usual way to obtain a (possibly disconnected) codimension $p+q$ submanifold Z with an intersection multiplicity ± 1 (see Section 7).

(2) Assign to Z the parallel section of $N|Z$ given by $v(s_1,s_2)$.

We wish to relate the intersection product of cycles with coefficients of the previous special type and the cup product of their Poincare duals. We first give a formula for the Poincare dual of a cycle of the type $Y \otimes s$.

Let $U(Y)$ be a tubular neighborhood of Y and φ be a p-cochain with compact support in $U(Y)$ representing the Thom class of $U(Y)$. Let $[U(Y)]$ denote the relative fundamental cycle of $U(Y)$. We extend s to a parallel section of $E|U(Y)$. Then extending $\varphi \otimes s$ by zero we obtain an E-valued p-cocycle on all of Y which we continue to denote $\varphi \otimes s$. We then have the following lemma.

Lemma 4.2. $PD(Y \otimes s) = \varphi \otimes s$.

Proof. Let η be any E^*-valued cocycle. Then, letting $(,)$ denote the pairing between E^* and E we obtain:

$$\begin{aligned}\langle \eta \cup (\varphi \otimes s), X\rangle &= \langle \eta \cup (\varphi \otimes s), [U(Y)]\rangle = \langle (s,\eta) \cup \varphi, [U(Y)]\rangle \\ &= \langle (s,\eta), Y\rangle = \langle \eta, Y \otimes s\rangle .\end{aligned}$$

The next to last inequality follows because (s,η) is a scalar-valued cocycle on $U(Y)$ and φ is the Thom class. The claim now follows from the preceding remark characterizing $PD(Y \otimes s)$.

We can now prove the relation between the intersection product of cycles and the cup product of their duals.

Lemma 4.3. $PD(Z \otimes v(s_1,s_2)) = PD(Y_1 \otimes s_1) \cup PD(Y_2 \otimes s_2)$.

Proof. We choose tubular neighborhoods $U_1(Y_1)$ and $U_2(Y_2)$ and Thom classes φ_1 and φ_2 respectively. But then $\varphi_1 \cup \varphi_2$ represents the Thom class of a suitable tubular neighborhood of $Y_1 \cap Y_2$. We find then

$$\begin{aligned}PD(Y_1 \otimes s_1) \cup PD(Y_2 \otimes s_2) &= (\varphi_1 \otimes s_1) \cup (\varphi_2 \otimes s_2) \\ &= (\varphi_1 \cup \varphi_2) \otimes v(s_1,s_2) \\ &= PD(Z \otimes v(s_1,s_2)).\end{aligned}$$

The last equality follows from Lemma 4.2. The lemma is now proved.

Now suppose Y is an oriented hypersurface in X and $Y \otimes s$ is a cycle with values in E as before. We now give a formula for the crossed homomorphism $\varphi \in H^1(\pi_1(X,x_0),E_0)$ which corresponds to $PD(Y \otimes s)$. Let $\gamma \in \pi_1(X,x_0)$. We suppose γ is transverse to Y. Suppose γ intersects Y at points that are $y_1,y_2,\ldots,y_r$ in γ, that the signs of the intersections at these points $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_r$ and that $\alpha_i = s(y_i)$ for $i = 1,2,\ldots,r$. Parallel translate α back along γ to x_0 to get α_i'.

Lemma 4.4. $\varphi(\gamma) = \sum_{i=1}^{r} \varepsilon_i \alpha_i'$.

The proof follows immediately from the definition of $\varphi = \varphi_{PD(Y \otimes s)}$ given in the beginning of this section.

There is a decomposition of the fundamental group of X associated to the hypersurface Y. We suppose first that Y separates X into S_1 and S_2. We assume S_2 contains the positive side of Y. Choose a base-point x_1 for X which does not lie on Y and with $x_1 \in S_1$. Let x_2 be a base-point for S_2 and c be a directed arc joining x_1 to x_2 which intersects Y at one point y with multiplicity $+1$. Let a be the segment from x_1 to y and b that from y to x_2, so $c = ab$ in the path groupoid. Let $\{\mu_2,\mu_2,\ldots,\mu_k\}$ be a set of generators for $\pi_1(S_1,x_1)$ and $\{\eta_1,\eta_2,\ldots,\eta_e\}$ be a set of generators for $\pi_1(S_2,x_2)$. Then $\{\mu_1,\ldots,\mu_j,\nu_1,\ldots,\nu_e\}$ is a set of generators for $\pi_1(X,x_1)$ with $\nu_j = c\eta_j c^{-1}$ for $j = 1,2,\ldots,\ell$ by van Kampen's Theorem. Now let $s(y) = \beta$ and $a^*\beta = \alpha$ where a^* denotes parallel translation along a from y to x_0 (a similar notation is used below for other paths). Then by Lemma 4.4 we find for $j = 1,2,\ldots,\ell$:

$$\begin{aligned}\varphi(\nu_j) &= a^*\beta - (c\eta_j b^{-1})^*\beta = \alpha - (c\eta_j b^{-1} a^{-1})^*\alpha \\ &= \alpha - \nu_j^*\alpha = \alpha - \rho(\nu_j)\alpha .\end{aligned}$$

Clearly $\varphi(\mu_j) = 0$ for $j = 1,2,\ldots,k$.

In case $X-Y$ remains connected the $\pi_1(X,x_1)$ has an $H\cdot N\cdot N$ presentation with generators $\{\mu_1,\mu_2,\ldots,\mu_p,\nu\}$ with μ_j in the image of $\pi_1(X - Y,x_1)$ in $\pi_1(X,x_1)$. The extra generator ν meets Y at a single point. An argument similar to the previous one gives:

$$\begin{aligned}\varphi(\mu_j) &= 0 \quad \text{for } j = 1,2,\ldots,p \\ \varphi(\nu) &= \alpha .\end{aligned}$$

We summarize these formulas in a lemma. The notation is as above.

Lemma 4.5. (i) If Y separates then $PD(Y \otimes s)$ corresponds to the unique crossed homomorphism φ given by:

$$\varphi(\mu_j) = 0 \quad \text{for } j = 1,2,\ldots,k$$
$$\varphi(\nu_j) = \alpha - \rho(\nu_j)\alpha.$$

(ii) If Y does not separate then $PD(Y \otimes s)$ corresponds to the unique crossed homomorphism φ given by:

$$\varphi(\mu_j) = 0 \quad \text{for } j = 1,2,\ldots,p$$
$$\varphi(\nu) = \alpha.$$

We will often identify a parallel section s along a submanifold Y containing the base-point x_0 for X with its value α at x_0. Then α is an invariant for $\pi_1(Y,x_0)$. Given such an invariant α, we will often denote the corresponding cycle by $Y \otimes \alpha$.

5. Algebraic Bending and a Lower Bound on the Dimensions of the Deformation Spaces.

In this section, we construct deformations of Γ in the conformal and projective groups corresponding to disjoint, non-singular, two-sided, totally geodesic hypersurfaces. By a two-sided hypersurface we mean one with a trivial normal bundle - it does not necessarily separate M. We begin with the case of a single hypersurface.

Suppose M is a compact manifold and M_1 is an embedded two-sided connected hypersurface in M. We suppose moreover, that we are given a representation $\rho: \pi_1(M) \to G$ where G is a Lie group with Lie algebra $\mathfrak{g}$. We abbreviate $\pi_1(M)$ to Γ and $\pi_1(M_1)$ to A.

Lemma 5.1. Suppose $\rho(\pi_1(M_1))$ has an invariant x_1 in $\mathfrak{g}$ such that x_1 is not invariant under $\rho(\Gamma)$. Then $R(\Gamma,G)$ contains a non-constant curve through ρ.

Proof. We first assume M_1 separates M into 2 parts S_1 and S_2. Then Γ is an amalgam $\pi_1(S_1) *_A \pi_1(S_2)$. The vector x_1 cannot be invariant under both $\rho(\pi_1(S_1))$ and $\rho(\pi_1(S_2))$ since they generate $\rho(\Gamma)$. We assume x_1 is not invariant under $\rho(\pi_1(S_2))$. Then we define a curve ρ_t in $R(\Gamma,G)$ by:

$$\rho_t | \pi_1(S_1) = \rho$$
$$\rho_t | \pi_1(S_2) = \mathrm{Ad}\, R(t) \cdot \rho \quad (\text{where } R(t) = \exp t\, x_1).$$

The representations $\rho_t|\pi_1(S_1)$ and $\rho_t|\pi_1(S_2)$ agree on A; hence, by the universal property of amalgams we obtain a representation of Γ.

We next consider the case in which $S = M - M_1$ remains connected. Then Γ is an H·N·N group, $\Gamma = \pi_1(S)_{*A}$. Hence we have a generator ν of Γ such that the only relations involving ν are of the form $\nu^{-1} j_1(a) \nu = j_2(a)$ where j_1 and j_2 are the inclusions of the fundamental groups of the two sides of M_1 into $\pi_1(S)$. We let $R(t) = \exp t x_1$ where x_1 is invariant under $\rho(j_1(A))$ and we define:

$$\rho_t | \pi_1(S) = \rho$$
$$\rho_t(\nu) = R(t)\rho(\nu).$$

Note $\rho_t(\nu)^{-1}\rho(j_1(a))\rho_t(\nu)$ is constant in t and so we obtain representation $\rho_t : \Gamma \to G$ for all t.

Definition. A trivial deformation of ρ parametrized by a set T is one obtained by conjugating ρ by a family of elements of G parametrized by T.

We may prove the non-triviality of the above deformation by computing the class of the cocycle $c \in Z^1(\Gamma,\mathfrak{g})$ tangent to ρ_t. Since c is a crossed homomorphism, it is determined by its values on a set of generators. We choose generators for $\pi_1(M)$ as described at the end of Section 4.

A straightforward calculation then yields the following lemma.

Lemma 5.2. In case M_1 separates we have:

$$c(\mu) = 0 \qquad \text{for } \mu \in \pi_1(S_1)$$
$$c(\nu) = x - \rho(\nu)x\rho(\nu)^{-1} \qquad \text{for } \nu \in \pi_1(S_2)$$

In case M_1 does not separate we have:

$$c(\mu) = 0 \quad \text{for } \mu \in \pi_1(S)$$
$$c(\nu) = x.$$

Lemma 5.2 together with Lemma 4.5 gives the following theorem of fundamental importance in what follows.

Theorem 5.1. The derivative of the bending deformation ρ_t and the Poincare dual of the cycle with coefficients $M_1 \otimes x$ coincide as elements in $H^1(\Gamma,\mathfrak{g})$.

To show that the deformation ρ_t is non-trivial it is sufficient to prove that c is a non-zero element of $H^1(\Gamma,\mathfrak{g})$. To do this and to treat the general case of r embedded hypersurfaces we introduce the graph associated to a collection of two-sided hypersurfaces of a manifold M.

Let M be a connected n-dimensional manifold and $\widetilde{M}$ the universal cover of M. In the next lemma Γ denotes the group of covering transformations of $\pi:\widetilde{M} \to M$.

Lemma 5.3. Suppose $M_1, M_2, \ldots, M_r$ are disjoint non-singular two-sided connected hypersurfaces in M. Then there exists an oriented tree X such that Γ acts on X without inversions and such that $Y = X/\Gamma$ has r edges.

Proof. We define the oriented graph Z associated to any collection of disjoint two-sided connected hypersurfaces $\{M_i\}_{i \in I}$ in a manifold M. The vertices of Z are the components $\{S_j\}_{j \in J}$ of $M - \bigcup_{i \in I} M_i$ and the edges are the M_i's. We choose a tubular neighborhood around M_i with boundary $M_i^+ \amalg M_i^-$. Then, the origin $o(M_i)$ of the edge M_i is the component of $M - \bigcup_{i \in I} M_i$ containing M_i^- and the terminus $t(M_i)$ of M_i is the component of $M - \bigcup_{i \in I} M$ containing M_i^+. We observe that if M is the disjoint union of two submanifolds M' and M'' then the graph Z is the disjoint union of two subgraphs (possibly empty) Z' and Z''.

Now, given M as in the statement of the lemma, we have the graph Z attached to the collection $\{M_i\}_{i=1}^r$. We also have the collection of hypersurfaces in $\widetilde{M}$ formed from the connected components M'_α of inverse images of the M_i under the covering $\pi:\widetilde{M} \to M$. The set $\{M'_\alpha\}$ separates $\widetilde{M}$ into regions S'_β. We let $\widetilde{Z}$ denote the corresponding graph.

We claim $\widetilde{Z}$ is a tree. $\widetilde{Z}$ is connected for if S'_β and S'_γ are components we may connect them by a path in $\widetilde{M}$ crossing a finite number of hypersurfaces. This gives an edge path between the vertices S'_β and S'_γ in $\widetilde{Z}$. To show $\widetilde{Z}$ is a tree it is now sufficient to show that the removal of any non-extreme edge from $\widetilde{Z}$ disconnects $\widetilde{Z}$ (an extreme edge is an edge containing an extreme vertex). But the graph obtained upon removing a corresponding M'_α is the graph associated to the collection of hypersurfaces $\{M'_\beta : \beta \neq \alpha\}$ in the manifold $\widetilde{M} - M'_\alpha$. But $\widetilde{M} - M'_\alpha$ has two components (since any closed hypersurface in $\widetilde{M}$ must separate $\widetilde{M}$), each of which contains hypersurfaces from the collection $\{M'_\beta : \beta \neq \alpha\}$. Hence the new graph is the union of two disjoint proper subgraphs by the observation from the first paragraph. With this the claim is established.

Since the collection $\{M'_\alpha\}$ is Γ-invariant, we see that Γ acts on the tree $\widetilde{Z}$. Since the M_i's are two-sided, Γ maps the positive

and negative sides of M' to the corresponding sides of $\gamma M'_\alpha$. Hence Γ acts on $\widetilde{Z}$ without inversions. We take $X = \widetilde{Z}$.

We claim that Z is the quotient of $\widetilde{Z}$ under the action of Γ. There is a map $p:\widetilde{Z} \to Z$ given by sending the vertex on $\widetilde{Z}$ corresponding to S'_β to the vertex of Z corresponding to $\pi(S'_\beta)$ and the edge in $\widetilde{Z}$ corresponding to M'_α to the edge of Z corresponding to $\pi(M'_\alpha)$. Clearly p is incidence preserving, bijective on Γ-orbits and factors through $\widetilde{Z}/\Gamma$. If we take $Y = Z$, the lemma is proved.

Corollary. Choose a maximal tree T in Y. Then Γ is isomorphic to the fundamental group $\pi_1(\Gamma, Y, T)$ (here the notation is as in Serre [23], page 42).

Proof. The corollary is the Bass-Serre Theorem, Serre [23], Theorem 13.

We can now give a sufficient condition for the deformations of Lemma 5.1 to be non-trivial. We first treat the case in which M_1 separates M into two parts S_1 and S_2 with S_2 the positive side. ρ_t will denote the deformation of Lemma 5.1. We abbreviate $\pi_1(M_1)$ to A and $\pi_1(S_1)$ and $\pi_1(S_2)$ to B_1 and B_2 respectively.

Lemma 5.4. Suppose neither $\rho(B_1)$ nor $\rho(B_2)$ has a non-zero invariant in $\mathfrak{g}$. Then ρ_t is a non-trivial deformation.

Proof. Since Γ acts on the tree X we have a cohomology exact sequence relating the cohomology of Γ with coefficients in $\mathfrak{g}$ to that of the stabilizers of the vertices and edges of X (Serre [23], Proposition 13). In particular we have:

$$H^0(B_1,\mathfrak{g}) \oplus H^0(B_2,\mathfrak{g}) \to H^0(A,\mathfrak{g}) \to H^1(\Gamma,\mathfrak{g})$$

By hypothesis $H^0(B\ ,\mathfrak{g})$ and $H^0(B_2,\mathfrak{g})$ are both zero; hence $\delta_*:H^0(A\ \ \mathfrak{g}) \to H^1(\Gamma,\mathfrak{g})$ is injective. But $\delta_* x_1 = -c$ (lift x_1 back to $(0,x_1)$), and $\delta(0,x_1)(\mu) = 0$ and $\delta(0,x_1)(\nu) = (0, x_1 - \rho(\nu)x_1\rho(\nu)^{-1})$. With this the lemma is proved.

We next treat the case in which $M - M_1$ remains connected. We abbreviate $\pi_1(S)$ to B. Again ρ_t denotes the deformation of Lemma 5.1.

Lemma 5.5. Suppose $\rho(B)$ has no non-zero invariant in $\mathfrak{g}$. Then ρ_t is a non-trivial deformation.

Proof. The cohomology exact sequence associated to the action of Γ on X now becomes in part:

$$H^0(B,g) \to H^0(A,g) \to H^1(\Gamma,g).$$

Since $H^0(B,g)$ is zero by hypothesis, we find again that $\delta_*:H^0(A,g) \to H^1(\Gamma,g)$ is injective.

We claim that again we have $\delta_* x_1 = -c$. To prove this claim we consider the diagram of short exact sequences of inhomogeneous cochains: (Serre [23], page 126):

$$\begin{array}{ccccccccc} 0 \to & C^1(\Gamma,g) & \to & C^1(\Gamma,\mathrm{ind}_B^\Gamma 1 \otimes g) & \to & C^1(\Gamma,\mathrm{ind}_A^\Gamma 1 \otimes g) & \to 0 \\ & & & \uparrow \delta_1 & & \uparrow & \\ 0 \longrightarrow & g & \longrightarrow & \mathrm{ind}_B^\Gamma 1 \otimes g & \xrightarrow{\delta_2} & \mathrm{ind}_A^\Gamma 1 \otimes g & \longrightarrow 0 \end{array}$$

Here $\mathrm{ind}_B^\Gamma 1 \otimes g$ means the representation with underlying vector space $f:\Gamma \to g : f(\gamma\mu) = f(\gamma), \mu \in B\}$ and Γ action given by:

$$\gamma \cdot f(\eta) = \rho(\gamma) f(\gamma^{-1}\eta)\rho(\gamma)^{-1}.$$

The previous diagram arises from the double complex of Eilenberg-MacLane cochains with values in the g-valued cellular cochains of X together with Γ-equivariant identifications of $\mathrm{ind}_B^\Gamma 1 \otimes g$ with $C^0(X) \otimes g$ and $\mathrm{ind}_A^\Gamma 1 \otimes g$ with $C^1(X) \otimes g$. Under these identifications the cellular coboundary $d:C^0(X) \otimes g \to C^1(X) \otimes g$ is identified with $\delta_2:\mathrm{ind}_B^\Gamma 1 \otimes g \to \mathrm{ind}_A^\Gamma 1 \otimes g$ given by:

$$\delta_2 f(\gamma) = f(\gamma\nu) - f(\gamma).$$

We now compute $\delta_* x_1$. Our strategy is to identify $x_1 \in g^A$ with an element h in $\mathrm{ind}_A^\Gamma 1 \otimes g$ (Frobenius reciprocity), compute a suitable preimage f of h under δ_2, apply δ_1 to f and find a preimage b to $\delta_1 f$ in $C^1(\Gamma,g)$. The class of b will then by definition be $\delta_* x_1$.

Clearly h is given by the formula $h(\gamma) = \rho(\gamma)x_1\rho(\gamma)^{-1}$. Let $f \in \mathrm{ind}_B^\Gamma 1 \otimes g$ satisfy:

(i) $f(1) = 0$

(ii) $\delta_2 f(\gamma) = f(\gamma\nu) - f(\gamma) = h(\gamma)$.

We note that f exists because X is a tree and $\nu^{-1}A\nu \subset B$. Now $\delta_1 f \in C^1(\Gamma,\mathrm{ind}_B^\Gamma 1 \otimes g)$. We may identify the space of such cochains F with functions $\hat{F}$ from $\Gamma \times \Gamma$ into g satisfying:

$$\hat{F}(\gamma_1,\gamma_2\mu) = \hat{F}(\gamma_1,\gamma_2) \quad \text{for } \mu \in B.$$

Then we have $\delta_1^\wedge f(\gamma_1,\gamma_2)$ is independent of γ_2 and the cochain $b \in C^1(\Gamma,g)$ – see above, is defined by $b(\gamma) = \delta_1^\wedge f(\gamma,1)$. Now according to the formula for the Eilenberg-MacLane coboundary we have:

$$\delta_1^\wedge f(\gamma_1,\gamma_2) = \rho(\gamma_1) f(\gamma_1^{-1}\gamma_2)\rho(\gamma_1)^{-1} - f(\gamma_2)$$

and

$$b(\gamma) = \rho(\gamma)f(\gamma^{-1})\rho(\gamma)^{-1} - f(1) = \rho(\gamma)f(\gamma^{-1})\rho(\gamma)^{-1}.$$

We claim $b = -c$. We have only to check that they coincide on B and on ν since they are both crossed homomorphisms.

If $\mu \in B$ then:

$$b(\mu) = \rho(\mu)f(\mu^{-1})\rho(\mu)^{-1} = \rho(\mu)f(1)\rho(\mu)^{-1} = 0 = -c(\mu).$$

Before evaluating b on ν we note that as a consequence of (ii) with $\gamma = \nu^{-1}$ we have:

$$f(1) - f(\nu^{-1}) = \rho(\nu^{-1})x_1\rho(\nu) \quad \text{or} \quad f(\nu^{-1}) = -\rho(\nu^{-1})x_1\rho(\nu)$$

and

$$\rho(\nu)f(\nu^{-1})\rho(\nu)^{-1} = -x_1.$$

Finally then we obtain:

$$b(\nu) = \rho(\nu)f(\nu^{-1})\rho(\nu)^{-1} = -x_1 = -c(\nu).$$

With this the lemma is proved.

We now consider the general case in which M is a compact manifold containing r disjoint two-sided embedded connected hypersurfaces $M_1,M_2,\ldots,M_r$. We suppose we have a representation $\rho:\Gamma \to G$ where G is the set of real points of a reductive algebraic group $\underline{G}$ defined over $\mathbb{R}$. In what follows we will consider a graph Y with r edges containing a maximal tree T with b edges. Our strategy is to construct a deformation of the group associated to T.

<u>Lemma 5.6</u>. <u>Suppose</u> ρ <u>is a representation into</u> G <u>of the fundamental group of a tree</u> T <u>of groups such that every edge group has a non-zero invariant in</u> g. <u>Suppose</u> T <u>has</u> b <u>edges and</u> P <u>is a vertex of</u> T. <u>Then there exists a</u> b-<u>parameter family of deformations of</u> ρ <u>which is constant on</u> Γ_P <u>and is trivial when restricted to any vertex group</u>.

<u>Proof</u>. For each integer n we let T_n be the set of vertices at distance n from P. If $Q \in T_n$, with $n \geq 1$, there is a single vertex Q' at distance strictly less than n from Q to which Q is adjacent. The correspondence $Q \to Q'$ defines a map of T_n into T_{n-1}. If $Q \in T_n$ then we call the vertices in $\cup_{m>n} f_m^{-1}(Q)$ the predecessors of Q and the vertices of $f_{n+1}^{-1}(Q)$ the immediate predecessors of Q. The vertex Q together with its predecessors form the vertex set of a subtree of T.

Let Q_1 be an immediate predecessor of P and e_1 be the edge joining P and Q_1. Let A_1 be the edge group associated to A_1 and x_1 an invariant of $\rho(A_1)$ in g. Let $R(t) = \exp tx_1$. Then we define a 1-parameter family ρ_t of representations of Γ by defining:

$$\rho_t|\Gamma_Q = \text{Ad } R(t)\rho|\Gamma_Q \quad \text{if } Q = Q_1 \text{ or } Q \text{ is a predecessor of } Q_1$$
$$\rho_t|\Gamma_Q = \rho|\Gamma_Q \quad \text{otherwise.}$$

As Q_1 varies through T_1, we obtain a b_1-parameter family ρ_{t_1}, for $t_1 \in \mathbb{R}^{b_1}$, of deformations of ρ. Here b_1 is the number of immediate predecessors of P. Clearly the family ρ_{t_1} satisfies all the hypotheses of the lemma.

Now we choose a vertex $Q_1 \in T_1$ and let Q_2 be an immediate predecessor of Q_1. Let e_2 be the edge joining Q_1 and Q_2. Let A_2 be the edge group associated to e_2. Choose $t_1 \in \mathbb{R}^{b_1}$. Then the representation $\rho_{t_1}|A_2$ has a non-zero invariant $x_2 = x_2(t_1)$ in g because it is conjugate to $\rho|A_2$. Moreover we may choose x_2 to be an analytic function of t_1. Let $R(t_1,t) = \exp tx_2(t_1)$. We define a (b_1+1)-parameter family $\rho_{t_1,t}$ of deformations of ρ by defining:

$$\rho_{t_1,t}|\Gamma_Q = \text{Ad } R(t_1,t)\cdot\rho_{t_1}|\Gamma_Q \quad \text{if } Q = Q_2 \text{ or } Q \text{ is a predecessor of } Q_2$$

$$\rho_{t_1,t}|\Gamma_Q = \rho_{t_1}|\Gamma_Q \quad \text{otherwise.}$$

Continuing in this way we obtain the lemma.

Lemma 5.7. Suppose ρ is a representation of the fundamental group of a graph of groups $\pi_1(\Gamma,Y,T)$ into G so that every edge group has a non-zero invariant in g. Suppose that Y has r edges and P is a base vertex. Then there exists an r-parameter family of deformations of ρ which is constant on Γ_P and is a trivial deformation of ρ restricted to any vertex group.

Proof. Let Ω be the subgroup of Γ generated by the vertex groups. Then Ω is the fundamental group of the subgraph of groups corresponding to T and we may apply Lemma 5.6 to obtain a b-parameter family ρ_t of deformations of $\rho|\Omega$ satisfying the hypotheses of Lemma 5.6.

We claim we may extend ρ_t to a b-parameter family of representations $\tilde{\rho}_t$ of Γ. It remains to extend ρ_t to the generators ν corresponding to the edges e of Y which are not in T. There are two cases. We assume first that e is a loop and let Q be the origin (and terminus) of e. Then the relations involving ν in $\pi_1(\Gamma,Y,T)$ are of the form:

$$\nu j_2(a)\nu^{-1} = j_1(a) \quad \text{for } a \in A$$

where A is the edge group associated to e and j_1 and j_2 are two embeddings of A into Γ_Q. Since $\rho_t|\Gamma_Q$ is trivial there is a b-parameter family R_t of elements of G so that $\rho_t|\Gamma_Q = \text{Ad } R_t\cdot\rho|\Gamma_Q$. We define $\tilde{\rho}_t(\nu) = \text{Ad } R_t\cdot\rho(\nu)$. Then the relations involving ν are satisfied.

We now suppose that e is not a loop. Let P and Q be the origin and terminus of e. Then the relations involving ν are of the form $\nu j_2(a)\nu^{-1} = j_1(a)$ where j_1 is the inclusion of A into Γ_P and j_2 is the inclusion of A into Γ_Q. Now there exist b-parameter families R'_t and R_t of elements of G so that $\rho_t|\Gamma_P = \text{Ad } R'_t\cdot\rho|\Gamma_P$ and $\rho_t|\Gamma_Q = \text{Ad } R_t\cdot\rho|\Gamma_Q$. Define $\tilde{\rho}_t(\nu) = R'_t\rho(\nu)R_t^{-1}$. Then the relations involving ν are satisfied and we have proved the claim.

We now extend $\tilde{\rho}_t$ to an r-parameter family $\rho_{t,u}$ of deformations of ρ for $u = (u_1,u_2,\ldots,u_\ell)$ with $\ell = r - b$.

We define $\rho_{t,u}$ on the generators of $\pi_1(\Gamma,Y,T)$. If $\gamma \in \Omega$, then $\rho_{t,u}(\gamma) = \tilde{\rho}_t(\gamma)$. Consider the generator ν_j associated to an edge e_j of $Y - T$. Either e_j is a loop with vertex Q or it is an edge with origin Q and terminus Q'. If A_j is the edge group associated to e_j then we have the embedding $j_1:A_j \to \Gamma_Q$. Now $\tilde{\rho}_t|j_1(A_j)$ is trivial; hence $\tilde{\rho}_t|j_1(A_j)$ admits a non-zero invariant x'_j in g. Put $R(u_j) = \exp u_j x'_j$ and define $\rho_{t,u}(\nu_j) = R(u_j)\tilde{\rho}_j(\nu_j)$. With this the lemma is proved.

We now compute the derivative of $\rho_{t,u}$. We use the exact cohomology sequence obtained from the action of Γ on X, Serre [23], Proposition 13. We require some more notation. For $j = 1,2,\ldots,r$, let A_j be the edge group corresponding to the edge e_j of Y. We assume the edges are ordered so that $e_1,e_2,\ldots,e_b$ are in T. We enumerate the vertices of Y as $P = P_1,P_2,\ldots,P_m$ so that the vertices of T_n (see Lemma 5.6) come before those of Y_m for $m > n$. We orient the edges of Y so that $o(e)$ comes before $t(e)$ in the enumeration. Let B_k for $k = 1,2,\ldots,m$ be the corresponding vertex groups. Then we have the sequence (exact at the middle):

$$\bigoplus_{k=1}^{m} H^0(B_k,g) \to \bigoplus_{j=1}^{r} H^0(A_j,g) \to H^1(\Gamma,g) \qquad (*)$$

In Lemma 5.7 we constructed an analytic map $\Phi(t,u):\mathbb{R}^r \to R(\Gamma,G) \subset G^N$ given by $\Phi(t,y) = \rho_{t,u}$.

Lemma 5.8. Assume that no vertex group has a non-zero invariant in g (under ρ). Then the differential of Φ has rank r at the origin of $\mathbb{R}^r$.

Proof. We remind the reader of the enumeration of the vertices of Y in the previous paragraph. Let $\{\gamma_{ki}: i \in I_k\}$ be a set of generators for Γ_{P_k}. We let $\nu_1, \nu_2, \ldots, \nu_\ell$ be the generators for $\pi_1(\Gamma,Y,T)$ corresponding to the positively oriented edges of Y that are not in T. We have assumed that there are N generators for Γ in all. This choice of generators gives an embedding of $R(\Gamma,G)$ into G^N by:

$$\rho \to (\rho(\gamma_1),\ldots,\rho(\gamma_m),\rho(\nu_1),\ldots,\rho(\nu_\ell)).$$

Here we have abbreviated the coordinates corresponding to the generators $\{\gamma_{ki}: i \in I_k\}$ by a single symbol $\rho(\gamma_k)$.

We may then consider Φ as a map from $\mathbb{R}^r$ into G^N. It is convenient to define $\Phi_1:\mathbb{R}^r \to G^m$ and $\Phi_2:\mathbb{R}^r \to G$ by:

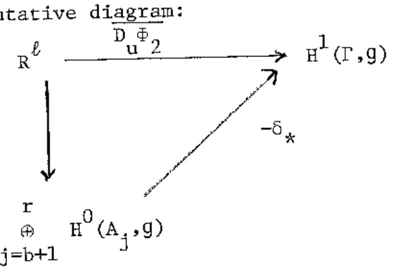
$$\Phi_1(t,u) = (\rho_{t,u}(\gamma_1),\ldots,\rho_{t,u}(\gamma_m))$$
$$\Phi_2(t,u) = (\rho_{t,u}(\nu_1),\ldots,\rho_{t,u}(\nu_\ell))$$

Then $\Phi(t,u) = (\Phi_1(t,u),\Phi_2(t,u))$ and $\Phi_1(t,u)$ does not depend on u. Thus to prove the lemma it is sufficient to prove that $D_t\Phi_1(0,0)$ has rank b and $D_u\Phi_2(0,0)$ has rank ℓ. Now $D_t\Phi_1(0,0)$ and $D_u\Phi_2(0,0)$ take values in $Z^1(\Gamma,g)$. We denote their compositions with the projection to $H^1(\Gamma,g)$ by $\overline{D_t\Phi_1}$ and $\overline{D_u\Phi_2}$. It is sufficient to prove these latter two maps have ranks b and ℓ.

We first compute $\overline{D_u\Phi_2}$. From Lemma 5.7 we have:

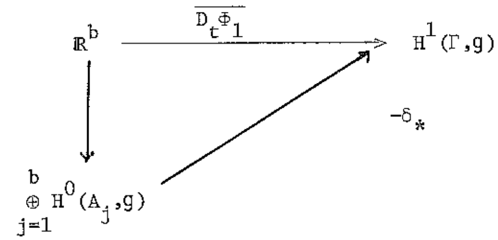
$$\Phi_2(0,u) = (R(u_1)\rho(s_1),\ldots,R(u_\ell)\rho(s_\ell)).$$

But the calculation of $\partial\Phi_2/\partial u_i(0,0)$ is identical to that of Lemma 5.5. We find a commutative diagram:



Here the vertical arrow is an isomorphism mapping $(u_1,u_2,\ldots,u_\ell)$ to $(u_1x_1,\ldots,u_\ell x_\ell)$. Since $H^0(B_j,g)$ is zero for all j it follows from the exactness of (*) that δ_* is injective and consequently $\overline{D_u\Phi_2}$ has rank ℓ.

We now compute $\overline{D_t\Phi_1}$. We claim that again we have a commutative diagram:



Here the vertical map sends $(t_1,t_2,\ldots,t_b)$ to $(t_1x_1,\ldots,t_bx_b)$. The claim is equivalent to the formula:

$$\frac{\partial\Phi}{\partial t_j}(0,0) = -\delta_*(0,\ldots,x_j,0,\ldots,0) \text{ where } x_j \text{ is in the jth component.}$$

By the construction of Lemma 5.6 there exists an edge e with origin Q and terminus Q' and edge group A so that x_j is invariant under $\rho(A)$ and Φ satisfies:

$$\Phi((0,\ldots,t_j,\ldots,0),(0,\ldots,0)) = (\rho_{t_j}(\gamma_1),\ldots,\rho_{t_j}(\gamma_m))$$

where:

$$\rho_{t_j}(\gamma_k) = R(t_j)\rho(\gamma_k)R(t_j)^{-1} \text{ if } P_k \text{ is } Q' \text{ or a predecessor of } Q'$$
$$\rho_{t_j}(\gamma_k) = \rho(\gamma_k) \quad \text{otherwise.}$$

Hence $\partial\Phi/\partial t_j(0,0)$ is identified with the cocycle c_j given by:

$$c_j(\gamma_k) = x_j - \rho(\gamma_k)x_j\rho(\gamma_k)^{-1} \text{ if } P_k \text{ is } Q' \text{ or a predecessor of } Q'$$
$$c_j(\gamma_k) = 0 \quad \text{otherwise.}$$

But to compute $\delta_*(0,\ldots,x_j,\ldots,0)$ we observe that an inverse image of $(0,\ldots,x_j,\ldots,0)$ in $\oplus_{k=1}^m H^0(B_k,g)$ is given by $\alpha = (\alpha_k)$, where α is given by:

$$\alpha_k = x_j \quad \text{if } P_k \text{ is } Q' \text{ or a predecessor of } Q'$$
$$\alpha_k = 0 \quad \text{otherwise.}$$

Clearly $\delta\alpha = -c_j$ and the lemma is proved (here δ is the Eilenberg-MacLane coboundary).

Remark. In the course of the proof we have proved the following diagram commutative (here $\overline{d\Phi}$ is $d\Phi(0,0)$ followed by the projection to

$H^1(\Gamma,\mathfrak{g}))$.

$$\begin{array}{ccc} \mathbb{R}^r & \xrightarrow{\overline{d\Phi}} & H^1(\Gamma,\mathfrak{g}) \\ \downarrow & \nearrow_{-\delta_*} & \\ \bigoplus_{j=1}^{r} H^0(A_j,\mathfrak{g}) & & \end{array}$$

Proposition 5.1. *Suppose that* ρ *is a stable representation (Section 1) of the fundamental group of a graph of groups into the real points of an algebraic group* $\underline{G}$ *defined over* R *(with Lie algebra* $\mathfrak{g}$*) such that every edge group has an invariant in* $\mathfrak{g}$ *and no vertex group has a non-zero invariant in* $\mathfrak{g}$. *Then:*

$$\dim X(\Gamma,G) \geq r$$

where r *is the number of edges of the graph*.

Proof. We use the previous lemmas to construct an embedded r-ball B around ρ in $R(\Gamma,G)$. We may assume that B is contained in the set of stable representations. We claim that the image of B in $X(\Gamma,G)$ is the quotient of B by a finite group. To check this, it is sufficient to compute the image of B under the orbit map $\pi: R(\Gamma,G) \to R(\Gamma,G)/G$ since B consists entirely of stable representations. Suppose $\pi(\rho_1) = \pi(\rho_2)$. Then there exists $g \in \underline{G}$ with $\mathrm{Ad}\, g \cdot \rho_1 = \rho_2$. Hence $\mathrm{Ad}\, g \cdot \rho_1|\Gamma_P = \rho_2|\Gamma_P$. Here P is the base vertex (see Lemma 5.7). But by construction $\rho_1|\Gamma_P = \rho_2|\Gamma_P = \rho|\Gamma_P$. Hence $g \in Z(\rho(\Gamma_P))$, the centralizer of $\rho(\Gamma_P)$ in $\underline{G}$. Hence $g \in Z(H)$ where H is the Zariski closure of $\rho(\Gamma_P)$ in $\underline{G}$. But H has no non-zero invariant in $\mathfrak{g}$, hence $Z(H)$ is discrete, hence finite and the proposition is proved since the quotient of an r-ball by a finite group contains a small r-ball.

Remark. If ρ_0 is good then $X(\Gamma,G)$ contains an r-ball around ρ_0.

We are now ready to prove the required lower bounds for the dimension of the spaces of conformal and projective structures on a compact hyperbolic n-manifold M. Let $\Gamma = \pi_1(M)$ and $\rho: \Gamma \to SO(n,1)$ be the standard uniformization. We first treat the case of the space of conformal structures.

Theorem 5.2. *Suppose* M *contains* r *disjoint, embedded, totally geodesic, two-sided connected hypersurfaces* $M_1, M_2, \ldots, M_r$. *Then the*

dimension of $X(\Gamma, SO(n+1,1))$ *is greater than* r.

Proof. We first check that $\rho(\pi_1(M_j))$ has a non-zero invariant in $so(n+1,1)$ for $j = 1,2,\ldots,r$. We may identify $so(n+1,1)$ with $\Lambda^2\mathbb{R}^{n+2}$ by using $(,)$. Now $\rho(\pi_1(M_j))$ leaves invariant a vector v_j in $\mathbb{R}^{n+1}$ with $(v_j,v_j) > 0$. Also $\rho(\pi_1(M))$ leaves invariant e_{n+1} so a fortiori $\rho(\pi_1(M_j))$ leaves invariant e_{n+1}. Thus $\rho(\pi_1(M_j))$ leaves invariant $v_j \wedge e_{n+1}$. To prove the theorem it suffices to check that $\rho(\pi_1(S_k))$ has no invariant in $so(n+1,1)$ for $k = 1,2,\ldots,b+1$. This follows from the next lemma.

Lemma 5.9. *Suppose* $n \geq 2$ *and let* S *be a compact hyperbolic manifold with totally geodesic boundary. Let* $\rho: \pi_1(S) \to SO(n,1)$ *be the uniformization representation. Then* $\rho(\pi_1(S))$ *is Zariski dense in* $SO(n,1)$.

Proof. We first prove that $\rho(\pi_1(S))$ has no invariant line in $\mathbb{R}^{n+1}$. Let M be a boundary component of S. Then $\rho(\pi_1(M))$ has a unique invariant line L in $\mathbb{R}^{n+1}$. Since $\pi_1(M) \subset \pi_1(S)$ we see that if $\rho(\pi_1(S))$ has an invariant line then it must be L. Suppose this to be the case. Then $\rho(\pi_1(S))$ is contained in the subgroup H of $SO(n,1)$ which leaves L invariant. Since $\rho(\pi_1(S))$ is discrete in $SO(n,1)$, it is discrete in H. Since $\rho(\pi_1(M))$ is uniform in H so is $\rho(\pi_1(S))$. Hence $M' = \rho(\pi_1(S))\backslash H/K \cap H$ is a compact hyperbolic $(n-1)$-manifold and M is a compact manifold covering M' with $[\pi_1(S):\pi_1(M)]$ sheets. Hence, if we can prove $[\pi_1(S):\pi_1(M)] = \infty$ we are done.

To establish this, assume that $\pi_1(M)$ has finite index in $\pi_1(S)$. The universal cover $\tilde{M}$ of M embeds into the universal cover $\tilde{S}$ of S. Now divide out $\tilde{S}$ by $\rho(\pi_1(M))$. We obtain a cover $S' \to S$ so that the image of $\pi_1(S')$ in $\pi_1(S)$ is precisely $\pi_1(M)$; hence, a finite cover. By construction $M \subset S'$ and the inclusion $\pi_1(M) \to \pi_1(s')$ is an isomorphism. We rename S' by S. We now claim that M is the only boundary component of S. Indeed suppose M' were another. Choose a closed geodesic α' in M'. Because $\pi_1(M)$ maps onto $\pi_1(S)$, α' is freely homotopic to a closed geodesic β in M. Since $M \cap M' = \varphi$, the closed geodesics α' and β are different. But this leads to a contradiction because two different closed geodesics in a hyperbolic manifold are never freely homotopic.

Now we have $M = \partial S$ and the inclusion of $\pi_1(M)$ into $\pi_1(S)$ is an isomorphism. Double S along M to obtain a compact hyperbolic manifold N. By van Kampen's Theorem we have $\pi_1(N) = \pi_1(M)$ but this is impossible because $H_n(\pi_1(M),\mathbb{Z}/2) = 0$ whereas $H_n(\pi_1(N),\mathbb{Z}/2) = \mathbb{Z}/2$.

Now let R be the Zariski closure of $\rho(\pi_1(S))$ in $SO(n,1)$. Then R is not discrete. Also R properly contains H; hence R leaves no totally geodesic subspace of $\mathbb{H}^n$ invariant nor does it fix any point of the closed ball $\mathbb{H}^n \cup S^{n-1}$. Hence, by Theorem 4.4.2 of [7] we have $R \supset SO_0(n,1)$. But R is a real algebraic subgroup of $SO(n,1)$ so $R = SO(n,1)$.

Corollary. $\rho(\pi_1(S))$ has no non-zero invariant in $so(n+1,1)$.

Proof. Any invariant of $\rho(\pi_1(S))$ would be an invariant of $R = SO(n,1)$. But $SO(n,1)$ has no invariants in $so(n+1,1)$.

As a consequence of Theorem 5.2 and the Holonomy Theorem, we obtain the following theorems.

Theorem 5.2 (bis). $\dim(\mathcal{C}(M)) \geq r$.

Proof. We have seen that $\mathrm{Hom}(\Gamma,G)/G$ contains embedded r-balls around points arbitrarily close to ρ_0. The theorem now follows from the holonomy theorem.

Theorem 5.2 (tertio). $\dim \mathcal{H}(M \times \mathbb{R}) \geq r$.

Proof. The proof is the same as above.

We now treat the projective case.

Theorem 5.3. Suppose M contains r disjoint embedded two-sided connected totally geodesic hypersurfaces $M_1, M_2, \ldots, M_r$. Then we have:

$$\dim X(\Gamma, PGL_{n+1}(\mathbb{R})) \geq r.$$

Proof. We may identify the Lie algebra $\mathfrak{g}$ of $PGL_{n+1}(\mathbb{R})$ with $s\ell_{n+1}(\mathbb{R})$, the Lie algebra of n+1 by n+1 matrices of trace zero. As a module for $SO(n,1)$, we may identify the n+1 by n+1 real matrices with $\otimes^2(\mathbb{R}^{n+1})^*$ where the identity matrix is identified with the form $(,)$. Then $so(n,1)$ is identified with $\Lambda^2(\mathbb{R}^{n+1})^*$ and the orthogonal complement M of $so(n,1)$ in $s\ell_{n+1}(\mathbb{R})$ is identified with $S_0^2(\mathbb{R}^{n+1})^*$, the traceless symmetric 2-tensors. Let $\rho: \Gamma \to SO(n,1) \to PGL_{n+1}(\mathbb{R})$ be the uniformization representation followed by the natural map.

We now observe that $\rho(\pi_1(M_j))$ has a non-zero invariant in $S_0^2(\mathbb{R}^{n+1})^*$ for $j = 1,2,\ldots,r$. We know that in the uniformization representation on $\mathbb{R}^{n+1}$ (or $(\mathbb{R}^{n+1})^*$) the group $\pi_1(M_j)$ has a non-

zero invariant v_j. Let h_{v_j} be the traceless projection of the symmetric 2-tensor $v_j \otimes v_j$. Then $\rho(\pi_1(M_j))$ leaves h_{v_j} invariant for $k = 1,2,\ldots,b+1$. But we know $\rho(\pi_1(S_k))$ is Zariski dense in $SO(n,1)$ and $SO(n,1)$ has no non-zero invariant in $S_0^2(\mathbb{R}^{n+1})^*$ – in fact this latter module is irreducible, nor does $SO(n,1)$ have a non-zero invariant in $\Lambda^2(\mathbb{R}^{n+1})^*$. With this the theorem is proved.

As a consequence of Theorem 5.3 and the Holonomy Theorem, we obtain the following theorem.

Theorem 5.3 (bis). $\mathcal{P}(M)$ has dimension greater than or equal to r.

Proof. In the course of the proof of Theorem 5.3 we saw that $\mathrm{Hom}(\Gamma,G)/G$ contained an r-ball around ρ_0. The theorem now follows from the Holonomy Theorem.

6. Singularities in the Deformation Spaces.

In this section, we give a criterion in terms of the topology of M for the spaces $\mathrm{Hom}(\Gamma,G)$ and $\mathrm{Hom}(\Gamma,\underline{G})$ to be singular at a representation ρ and $X(\Gamma,G)$ and $X(\Gamma,\underline{G})$ to be singular at the class of a good representation ρ. In Section 7 we show that this criterion is satisfied for the standard arithmetic examples. In what follows we let $\mathcal{D}$ be a symbol denoting any of the four above spaces.

Lemma 6.1. Suppose M_1 and M_2 are embedded hypersurfaces of M and ρ is any representation of Γ. Suppose the following hold.

(i) $\rho(\pi_1(M_1))$ leaves invariant a non-zero $x \in \mathfrak{g}$.

(ii) $\rho(\pi_1(M_2))$ leaves invariant a non-zero element $y \in \mathfrak{g}$.

(iii) $(M_1 \otimes x)\cdot(M_2 \otimes y) \neq 0$.

Then $\mathrm{Hom}(\Gamma,G)$ and $\mathrm{Hom}(\Gamma,\underline{G})$ are singular at ρ; moreover, if ρ is good then $X(\Gamma,G)$ and $X(\Gamma,\underline{G})$ are singular at the class of ρ.

Proof. For simplicity we assume M_1 is not a boundary and M_2 is not a boundary. Then Γ has an H·N·N decomposition corresponding to M_1 given by $\Gamma = B_1 *_{A_1}$ where $B_1 = \pi_1(M - M_1)$ and $A_1 = \pi_1(M_1)$. Let R_α be the one parameter group in G (or $\underline{G}$) tangent to x. As in Lemma 5.1, we obtain a curve ρ_α in $\mathcal{D}$ constant on B_1 and changing $\rho(\nu_1)$ to its product by R_α. The tangent vector $\dot{\rho}_\alpha$ to ρ_α at $\alpha = 0$ is, by Theorem 5.1, dual to $M_1 \otimes x$. Let R_β be the one parameter group in G (or $\underline{G}$) tangent to y. Then, as above, we obtain a curve ρ_β in $\mathcal{D}$ leaving B_2 fixed and changing $\rho(v_2)$ to its product by R_β. The tangent vector $\dot\rho_\beta$ to ρ_β at $\beta = 0$ is dual to $M_2 \otimes y$ by Theorem 5.1. Now consider a linear combination $c\dot\rho_\alpha + d\dot\rho_\beta$ with $c \neq 0$ and $d \neq 0$. We compute the first obstruction μ (see the end of Section 2) to finding a curve in $\mathcal{D}$ tangent to $c\dot\rho_\alpha + d\dot\rho_\beta$. We have:

$$\mu = [c\dot\rho_\alpha + d\dot\rho_\beta, c\dot\rho_\alpha + d\dot\rho_\beta] = c^2[\dot\rho_\alpha,\dot\rho_\alpha] + 2cd[\dot\rho_\alpha,\dot\rho_\beta] + d^2[\dot\rho_\beta,\dot\rho_\beta].$$

Now $[\dot\rho_\alpha,\dot\rho_\alpha]$ and $[\dot\rho_\beta,\dot\rho_\beta]$ are zero because $\dot\rho_\alpha$ and $\dot\rho_\beta$ are tangent to curves in $\mathcal{D}$. Hence:

$$\mu = 2cd[\dot\rho_\alpha,\dot\rho_\beta].$$

But by Lemma 4.3, the class $[\dot\rho_\alpha,\dot\rho_\beta]$ is dual to $(M_1 \otimes x)\cdot(M_2 \otimes y)$. Thus the tangent cone to $\mathcal{D}$ is not a vector space and the lemma is proved.

<u>Remark</u>. In the cases $\mathcal{D} = X(\Gamma,G)$ and $\mathcal{D} = X(\Gamma,\underline{G})$ we must check that the tangent vectors $\dot\rho_\alpha$ and $\dot\rho_\beta$ are non-trivial and distinct in $H^1(\Gamma,\mathfrak{g})$ (or $H^1(\Gamma,\underline{\mathfrak{g}})$). But this follows because $[\dot\rho_\alpha,\dot\rho_\beta] \neq 0$ and $[\dot\rho_\alpha,\dot\rho_\alpha] = 0$.

In this case what is actually proved here is that the slice through ρ in $\mathrm{Hom}(\Gamma,G)$ is not a <u>smooth</u> analytic subvariety of $\mathrm{Hom}(\Gamma_N,G)$ because the tangent cone to the intersection is not a linear subspace of $Z^1(\Gamma,\mathfrak{g})$. This implies that $X(\Gamma,G)$ and $X(\Gamma,\underline{G})$ are singular at $\pi(\rho)$ by the remark following Theorem 1.2.

Before proving the two main theorems of this section we need the following observation. Suppose M_1 and M_3 are disjoint totally geodesic hypersurfaces of M. Let ρ_θ be the deformation of the Fuchsian representation ρ corresponding to the hypersurface M_3. Let v_1 be a non-zero invariant of $\rho(\pi_1(M_1))$. Then v_1 is an invariant of $\rho_\theta(\pi_1(M_1))$ - the curve ρ_θ is constant on $\pi_1(M_1)$ since $M_1 \cap M_3 = \emptyset$. Here we have chosen the base-point of M to lie on M_1. Hence if V_θ denotes the vector space V with Γ acting by ρ_θ then we can form a curve of classes $M_1 \otimes v_1 \in H_{n-1}(M,V_\theta)$. We can now state our main theorems of this section - in what follows we assume the base-point of M is chosen to lie on $M_1 \cap M_2$.

<u>Theorem 6.1</u>. <u>Suppose</u> M_1,M_2,M_3 <u>are embedded totally geodesic hypersurfaces in</u> M <u>such that</u> $M_1 \cap M_3 = \emptyset$ <u>and</u> $M_2 \cap M_3 = \emptyset$. <u>Let</u> ρ_θ <u>be the deformation of</u> ρ <u>as above corresponding to</u> M_3 <u>and</u> v_1 <u>and</u> v_2 <u>be non-zero invariants of</u> $\rho(\pi_1(M_1))$ <u>and</u> $\rho(\pi_1(M_2))$ <u>respectively</u>. <u>Assume that for all</u> θ <u>the cycle</u> $M_1 \otimes v_1 \cdot M_2 \otimes v_2$ <u>is non-zero in</u> $H_{n-2}(M,\Lambda^2 V_\theta)$.

<u>Then there exists</u> $\varepsilon > 0$ <u>such that for every</u> θ <u>in</u> $(-\varepsilon,\varepsilon)$ <u>the point</u> ρ_θ <u>(or its class) is a singular point of</u> $\mathcal{D}$.

<u>Proof</u>. We have only to check that the hypotheses of the previous lemma are satisfied. We take $x = v_1 \wedge e_{n+1}$ and $y = v_2 \wedge e_{n+1}$, then $[x,y] = v_1 \wedge v_2$. The theorem follows since ρ_θ is quasi-Fuchsian but not Fuchsian (hence good) for θ in $(-\varepsilon,\varepsilon) - \{0\}$ for some positive ε.

<u>Corollary</u>. <u>If</u> M_1, M_2 <u>and</u> M_3 <u>exist as above then</u> $\mathcal{D}$ <u>has non isolated singularities</u>.

The projective version of Theorem 5.1 goes as follows. We apply Lemma 6.1 with $\underline{G} = PGL_{n+1}$.

<u>Theorem 6.2</u>. <u>Suppose that for all</u> θ <u>the cycle</u> $M_1 \otimes h_{v_1} \cdot M_2 \otimes h_{v_2}$ <u>is non-zero in</u> $H_{n-2}(M,\Lambda^2 V_\theta)$ <u>(here</u> h_v <u>is as in Theorem 5.3)</u>.

<u>Then there exists</u> $\varepsilon > 0$ <u>such that for every</u> θ <u>in</u> $(-\varepsilon,\varepsilon)$ <u>the point</u> ρ_θ <u>is a singular point of</u> $\mathcal{D}$.

<u>Proof</u>. The proof is identical to that of Theorem 6.1.

We conclude this section with a determination of when $[h_v,h_w] = 0$. Recall we are identifying the traceless symemtric 2-tensor h_v with an element of $s\ell(n+1,\mathbb{R})$ using the form $(,)$. This element is easily seen to be the endomorphism of $\mathbb{R}^{n+1}$ given by:

$$h_v(u) = (u,v)v - \frac{(v,v)}{n+1}u.$$

We find the following formula for the bracket:

$$[h_v,h_w] = (v,w)w \wedge v$$

where by $v \wedge w$ we mean the transformation given by:

$$(w \wedge v)\cdot u = (w,u)v - (v,u)w.$$

Hence $[h_v,h_w] = 0$ if and only if v and w are either proportional or orthogonal. Note that the bracket carries $S_0^2 V$ into $\Lambda^2 V$.

7. Configurations of Totally Geodesic Submanifolds in the Standard Arithmetic Examples.

In this section we verify that the hypotheses of Theorems 6.1 and 6.2 are satisfied for the compact hyperbolic n-manifolds obtained from the standard arithmetic subgroups of $SO(n,1)$. These groups are obtained as follows.

Let p be a positive, square - free integer and $Q:\mathbb{R}^{n+1} \to \mathbb{R}$ be the quadratic form given by:

$$Q(x_1,x_2,\ldots,x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 - \sqrt{p}\, x_{n+1}^2$$

We let $(,)$ denote the symmetric bilinear form associated to Q. Let $\mathcal{O}$ be the ring of algebraic integers in the quadratic field $k = \mathbb{Q}(\sqrt{p})$. Then the group Φ of matrices with entries in $\mathcal{O}$ which are isometries of Q is a uniform (cocompact) discrete subgroup of the group of matrices with entries in $\mathbb{R}$ which are isometries of Q - see for example Borel [5]. Since this latter group can be identified with $O(n,1)$ in an obvious way, we obtain a uniform, discrete subgroup of $O(n,1)$. The group Φ is often called the group of units of Q, a terminology motivated by the case $n = 1$. By Millson-Raghunathan [15], we can pass to a suitable congruence subgroup $\Gamma = \Gamma(\mathfrak{a})$ of Φ, for $\mathfrak{a}$ an ideal in $\mathcal{O}$, and obtain a uniform, discrete, torsion - free subgroup of $SO_0(n,1)$ and consequently a compact hyperbolic n-manifold $M = \Gamma\backslash\mathbb{H}^n$. We let $\pi:\mathbb{H}^n \to M$ denote the quotient map.

We will use the (upper sheet of the) hyperboloid model for $\mathbb{H}^n$; that is:

$$\mathbb{H}^n = \{z \in \mathbb{R}^{n+1}:(z,z) = -\sqrt{p} \text{ and } (z,e_{n+1}) < 0\}.$$

Here $\{e_1,e_2,\ldots,e_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$. We will often write V for $\mathbb{R}^{n+1}$ and L for $\mathcal{O}^{n+1}$, the set of vectors with coordinates in $\mathcal{O}$.

We now construct compact orientable non-singular totally geodesic submanifolds in suitable (congruence subgroup) covers of M. Let $X = \{x_1,x_2,\ldots,x_k\}$ be a k-tuple of vectors in L chosen so that:

(i) $\dim \operatorname{span} X = k$

(ii) $(,)|\operatorname{span} X$ is positive definite.

We let $\mathbb{H}^n_X$ denote the totally geodesic, codimension k submanifolds of $\mathbb{H}^n$ given by:

$$\mathbb{H}^n_X = \{z \in \mathbb{H}^n:(z,x) = 0 \text{ for span } X\}$$

We put $M_X = \pi(\mathbb{H}^n_X)$. Usually M_X will have self-intersections; however the following lemma shows that the self-intersections may be removed upon passing to a suitable cover. In what follows we let r_X denote the involution of V given by:

$$r_X(x) = -x \text{ for } x \in \operatorname{span} X$$
$$r_X(x) = x \text{ for } x \in (\operatorname{span} X)^{\perp}.$$

Then $\mathbb{H}^n_X$ is the fixed-point set of r_X acting on $\mathbb{H}^n$. A subscript X on a subgroup of $SO(n,1)$ will denote the subgroup of elements fixing $\{x_1,x_2,\ldots,x_k\}$. In particular $\Gamma_X = \{\gamma \in \Gamma:\gamma x_j = x_j$ for $j = 1,2,\ldots,k\}$. For a subgroup $\Gamma' \subset \Gamma$ we let $\pi':\mathbb{H}^n \to M' = \Gamma'\backslash\mathbb{H}^n$ denote the quotient map. A prime superscript on an object in M which is the image under π of an object on $\mathbb{H}^n$ will denote the corresponding image under π'; for example, $M'_X = \pi'(\mathbb{H}^n_X)$ and $M'_Y = \pi'(\mathbb{H}^n_Y)$.

Lemma 7.1. There exists a congruence subgroup $\Gamma' \subset \Gamma$ so that $\pi'(\mathbb{H}^n_X) = \Gamma'_X\backslash\mathbb{H}^n$. In this case $\pi'(\mathbb{H}^n_X)$ is an orientable submanifold. Moreover if $\Gamma'' \subset \Gamma'$ and $\gamma'' \in \Gamma''$ satisfies $\gamma''\mathbb{H}^n_X \cap \mathbb{H}^n_X \neq \emptyset$ then $\gamma'' \in \Gamma''_X$.

Proof. Choose Γ' so that $r_X\Gamma' r_X = \Gamma'$. By the Jaffee Lemma, Millson [14], Lemma 2.1, we find that $\pi'(\mathbb{H}^n_X) = \Lambda\backslash\mathbb{H}^n_X$ where $\Lambda = \{\gamma \in \Gamma':r_X\gamma r_X = \gamma\}$; that is, γ preserves the splitting $V = \operatorname{span} X + (\operatorname{span} X)^{\perp}$. But consider the action on $\operatorname{span} X$ induced by Λ. The projection of Λ is a discrete subgroup of the direct product of the orthogonal group of $\operatorname{span} X$ with itself (because the projection of Λ leaves invariant a lattice in $\operatorname{span} X \oplus \operatorname{span} X$). But the restriction of Q to $\operatorname{span} X$ is positive definite and consequently the projection of Λ is finite. Hence if Γ is neat (so no element of Γ has an eigenvalue equal to a non-trivial root of unity) we find that $\Lambda = \Gamma'_X$. With this the first statement is proved. The second statement follows because Γ'_X preserves the orientation of $\mathbb{H}^n_X$ (see remarks below) and Γ'_X is torsion free. The third statement follows because $\Gamma'' \cap \Gamma'_X = \Gamma''_X$.

Remark. In the course of the proof, we showed that if γ preserves $\operatorname{span} X$ and Γ is neat then γ fixes the elements of X.

To orient M_X it is sufficient to orient $\mathbb{H}^n_X$. The normal bundle of $\mathbb{H}^n_X$ may be canonically identified with $\operatorname{span} X$; thus, it is sufficient to orient $\operatorname{span} X$. We orient $\mathbb{H}^n_X$ so that the orientation of $\mathbb{H}^n_X$ at z followed by the orientation of $\operatorname{span} X$ followed by z is the orientation of the standard basis of V.

We now rename Γ' by Γ and suppress all primes. By Millson [14], Section 4, for any positive integer m we can find a cover of M containing at least m disjoint non-singular orientable totally geodesic hypersurfaces (which in addition are homologically independent). By the results of Section 5, we deduce the following theorem.

Theorem 7.1. For any $m > 0$ and any $n \geq 2$, there exists a compact hyperbolic n-manifold M with fundamental group Γ, a standard arithmetic subgroup of $SO(n,1)$, such that the dimensions of $\mathcal{C}(M)$, $\mathcal{P}(M)$, $\mathcal{H}(M \times R)$, $X(\Gamma, SO(n+1,1))$ and $X(\Gamma, PGL_{n+1}(R))$ are all greater than or equal to m.

We now assume $X = \{x_1,\ldots,x_p\}$ and $Y = \{y_1,\ldots,y_q\}$ are chosen so that $X \cup Y$ spans a subspace U of dimension $p+q$ so that $(,)|U$ is positive definite. This assumption on U implies that $\mathbb{H}^n_X$ and $\mathbb{H}^n_Y$ intersect transversely in a codimension $p+q$ totally geodesic subspace. We do not assume that the vectors in X are orthogonal to those in Y. We let E and F be flat bundles over M and $v: E \otimes E \to F$ be a parallel bundle map as in Section 4. We choose a point w_0 on $\mathbb{H}^n_X \cap \mathbb{H}^n_Y$ as a base-point for $\mathbb{H}^n$ and let $z_0 = \pi(w_0)$ be a base-point for M. We let E_0 and F_0 denote the fibers of E and F over z_0. We assume Γ_X has an invariant α_X in E_0 and Γ_Y has an invariant β_Y in F_0. The invariant α_X corresponds to a parallel section s_X of $E|M_X$ satisfying $s_X(z_0) = \alpha_X$. The invariant β_Y corresponds to a parallel section s_Y of $E|M_Y$ satisfying $s_Y(z_0) = \beta_Y$. Then $v(s_X,s_Y)$ is a parallel section of $F|M_X \cap M_Y$. We now give a formula for the intersection cycle $(M_X \otimes s_X)\cdot(M_Y \otimes s_Y)$. We assume M_X and M_Y intersect transversely in disjoint codimension $p+q$ submanifolds $P_1,P_2,\ldots,P_\ell$. We first show how to orient each P_j.

Choose an orientation ω for P_j. The orientation ω induces an orientation ω_1 of the normal bundle of P_j in M_X and an orientation ω_2 of the normal bundle of P_j in M_Y by requiring that ω followed by ω_1 be the orientation of M_X and ω followed by ω_2 be the orientation of M_Y. Then $\omega_1 \wedge \omega_2$ is independent of the choice of ω. We define $\varepsilon(\omega)$ to be $+1$ if the orientation of $\omega \wedge \omega_1 \wedge \omega_2$ is the orientation of M and $\varepsilon(\omega)$ to be -1 otherwise. We will call the orientation of P_j such that $\varepsilon(\omega) = +1$ the intersection orientation.

Remark. The intersection orientation may also be described as the orientation ω for P_j so that the induced orientation of the normal bundle of P_j in M_X coincides with that of the restriction to P_j of the normal bundle of M_Y in M - note that this second bundle already has an orientation.

We give each component P_j for $j = 1,2,\ldots,\ell$, the intersection orientation. We then have an equality of oriented cycles:

$$M_X \cdot M_Y = \sum_{j=1}^{\ell} P_j .$$

By definition of the intersection of cycles with coefficients we also have:

$$(M_X \otimes s_X)\cdot(M_Y \otimes s_Y) = \sum_{j=1}^{\ell} P_j \otimes v(s_X,s_Y)|P_j$$

We wish to obtain a formula which will enable us to determine when $v(s_X,s_Y)|P_j$ is zero.

In order to simplify notation we suppress the subscript j, replacing P_j by P and γ_j by γ. We let t denote the section $v(s_X,s_Y)$ of $F|P$. We have chosen a component $\tilde{P} = \gamma(H^n_X) \cap H^n_Y$ of $\pi^{-1}(P)$. We lift t to a section $\tilde{t}$ of the pull-back of F to $\tilde{P}$. We then parallel translate $\tilde{t}$ to w_0 and evaluate, obtaining an element $\varphi(t|P)$ which is zero if and only if $v(s_X,s_Y)|P$ is zero. We wish to evaluate $\varphi(t|P)$ in terms of γ, α_X and β_Y. Choose $w_2 \in \tilde{P}$, let $z_2 = \pi(w_2)$ and let $w_1 = \gamma^{-1}(w_2)$ so $w_1 \in H^n_X$. We choose a path $\tilde{a}$ in H^n_X from w_0 to w_1 and a path $\tilde{b}$ in H^n_Y from w_0 to w_2. We let $a = \pi(\tilde{a})$ and $b = \pi(\tilde{b})$. Then ab^{-1} represents γ^{-1} in $\pi_1(M,z_0)$ since it lifts to $\tilde{a}\gamma^{-1}(\tilde{b}^{-1})$. By definition $s_X(z_2) = a_* X$ where a_* denotes parallel translation along a. Also $s_Y(z_2) = b_*\beta_Y$. Hence $t(z_2) = v(a_*\alpha_X, b_*\beta_Y)$ and hence $\tilde{t}(w_2) = v(a_*\alpha_X, b_*\beta_Y)$. We obtain $\varphi(t|P)$ by parallel translating $\tilde{t}(w_2)$ back along $\tilde{b}$; that is $\varphi(t|P) = (\tilde{b}^{-1})_* v(a_*\alpha_X, b_*\beta_Y)$. But $(\tilde{b}^{-1})_* = (b^{-1})_*$ and hence $\varphi(t|P) = (b)_*^{-1} v(a_*\alpha_X, b_*\beta_Y) = v(b_*^{-1}a_*\alpha_X, \beta_Y) = v((ab^{-1})_*\alpha_X, \beta_Y)$. Now $(ab^{-1})_*\alpha_X$ is the parallel translate of α_X around a loop representing γ^{-1}. This is the way γ acts on α_X via its action on the standard fiber. We obtain the following lemma.

Lemma 7.2. $v(s_X,s_Y)|P = 0$ if and only if $v(\gamma\alpha_X,\beta_Y) = 0$ in E_0.

Remark. If we choose a different γ, say $\gamma' = \eta\gamma\mu$ with $\eta \in \Gamma_Y$ and $\mu \in \Gamma_X$, then P would change to ηP and the coefficient would change to $\eta v(\gamma\alpha_X, \beta_Y)$.

We define a subset $\Delta \subset \Gamma$ by:

$$\Delta = \{\gamma \in \Gamma : \gamma(\mathbb{H}^n_X) \cap \mathbb{H}^n_Y \neq \emptyset\}.$$

Then $\Gamma_X \times \Gamma_Y$ acts on Δ by $(\gamma_1, \gamma_2)\cdot\gamma = \gamma_2\gamma\gamma_1^{-1}$. The map $\gamma \to \pi(\gamma(\mathbb{H}^n_X) \cap \mathbb{H}^n_X)$ induces a one-to-one correspondence between the orbits of $\Gamma_X \times \Gamma_Y$ in Δ and the components of $M_X \cap M_Y$. Hence Δ consists of a finite number of $\Gamma_X \times \Gamma_Y$ orbits (or Γ_Y, Γ_X double cosets). For any ideal $b \subset \mathcal{O}$ we define:

$$\Delta(b) = \Delta \cap \Gamma(b).$$

We observe that if $c \subset b$ then $\Delta(c) \subset \Delta(b)$.

We have the following theorem under the assumptions $p+q \neq n-1$ and $n \geq 4$. In the next theorem we consider congruence subgroups $\Gamma'' \subset \Gamma' \subset \Gamma$. We let $M' = \Gamma'\backslash\mathbb{H}^n$ and $M'' = \Gamma''\backslash\mathbb{H}^n$. We let π' and π'' be the covering projections and $M'_X = \pi'(\mathbb{H}^n_X)$ and $M''_X = \pi''(\mathbb{H}^n_X)$ and similarly for Y. We assume in what follows that Γ is the congruence subgroup of Φ, the group of units of $(\,,\,)$, of level a.

Theorem 7.2. There exists a congruence cover M' of M so that $M'_X \cap M'_Y$ consists of the single component $\pi'(\mathbb{H}^n_X \cap \mathbb{H}^n_Y)$. Moreover for any congruence cover M'' of M' the intersection $M''_X \cap M''_Y$ again consists of the single component $\pi''(\mathbb{H}^n_X \cap \mathbb{H}^n_Y)$.

Theorem 7.2 will be a consequence of the following proposition.

Proposition 7.1. If $p+q \neq n-1$, there exists an ideal b so that $\Delta(b) \subset \Gamma_Y\Gamma_X$.

In what follows $R_b(\cdot)$ will denote reduction modulo the ideal b. We define $\Delta' \subset L^p$ by:

$$\Delta' = \{X' \in L^p : X' = \gamma' X \text{ for some } \gamma' \in \Delta\}.$$

The proof of Proposition 7.1 will follow two lemmas. The next lemma shows how to pass to a congruence cover and eliminate certain intersection components.

Lemma 7.3. Let $X' \in \Delta'$ and suppose $b \subset \mathcal{O}$ is an ideal such that:

$$R_b(\Gamma_Y X') \cap R_b(\Gamma_Y X) = \emptyset.$$

Then $\Delta(b) \cap \Gamma_Y\gamma'\Gamma_X = \emptyset$ where $\gamma' \in \Delta$ satisfies $\gamma' X = X'$.

Proof. If $\gamma \in \Delta(b) \cap \Gamma_Y\gamma'\Gamma_X$ so $\gamma = \gamma_2\gamma'\gamma_1$ with $\gamma_2 \in \Gamma_Y$ and $\gamma_1 \in \Gamma_X$ then:

$$R_b(X) = R_b(\gamma X) = R_b(\gamma_2\gamma'\gamma_1 X) = R_b(\gamma_2 X').$$

With this the lemma is proved.

We now use Lemma 7.3 to eliminate all double cosets so that the orbit of X under the double coset can be separated modulo some ideal c from the trivial double coset.

Lemma 7.4. There exists an ideal b so that $\gamma \in \Delta(b)$ implies $R_c(\Gamma_Y\gamma X) \cap R_c(\Gamma_Y X) \neq \emptyset$ for any c.

Proof. There are a finite number of Γ_Y, Γ_X double cosets in Δ. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ be a set of representatives. Either there exists an ideal b_1 such that $R_{b_1}(\Gamma_Y\gamma_1 X) \cap R_{b_1}(\Gamma_Y X) = \emptyset$ or no such ideal exists. If such an ideal exists then by Lemma 7.3 we know $\Delta(b_1) \cap \Gamma_Y\gamma_1\Gamma_X = \emptyset$ and we have eliminated the double coset containing γ_1. If no such ideal exists then for every element $\gamma \in \Gamma_Y\gamma_1\Gamma_X$ we have $R_c(\Gamma_Y\gamma X) \cap R_c(\Gamma_Y X) \neq \emptyset$ for every c. In this case we do not need to eliminate γ_1 so we take $b_1 = \mathcal{O}$. Continuing in this way we obtain ℓ (possibly non-proper) ideals $b_1, b_2, \ldots, b_\ell$. We put $b = b_1 b_2 \cdots b_\ell$ and the lemma is proved.

We now begin the proof of Proposition 7.1. For the course of this proof Γ_X and Γ_Y will be denoted Γ_1 and Γ_2 and G_X and G_Y by G_1 and G_2. For a ring R containing $\mathcal{O}$, the symbol $G_1(R)$ will denote the R-rational points of the algebraic subgroup of $SO(Q)$ fixing X and similarly for $G_2(R)$. The symbol $\mathcal{O}_P$ will denote the P-adic completion of $\mathcal{O}$ and the symbol $G_1(\mathcal{O}_P, a)$ will denote the subgroup of $G_1(\mathcal{O}_P)$ consisting of γ satisfying $\gamma \equiv 1 \bmod P^m$ where m is the largest power of P dividing a. We will assume Γ is chosen so that $\gamma \in \Gamma$ implies that the k spinor norm of γ is 1; this is possible by Millson-Raghunathan [16], Proposition 4.1.

The idea of the proof is to examine the elements $\gamma \in \Delta(b)$; that is, elements γ such that the corresponding vectors $X' = \gamma' X$ have an associated Γ_2 orbit, $\Gamma_2 X'$, which cannot be separated modulo any ideal c from $\Gamma_2 X$. We show all such γ' satisfy $\gamma' \in \Gamma_2\Gamma_1$. We let

$$\Delta'(b) = \{X' : X' = \gamma X \text{ for } \gamma \in \Delta(b)\}.$$

If $X \in V^p$, $Y \in V^q$, the symbol (X,Y) denotes the matrix

$((x_i,y_j))$. For $X' \in \Delta'$, let $B(X') = (X',Y)$, a p by q matrix with entries in k. Then for $\gamma_2 \in \Gamma_2$ we have $B(\gamma_2X') = B(X')$ and B is constant on Γ_2 orbits in Δ'. If $B(X') \neq B(X)$ then there exists some $y_j \in Y$ and some x_i' in X with $(x_i',y_j) \neq (x_i,y_j)$. Hence, for almost every prime P in $\mathcal{O}$ we have $(x_i',y_j) \neq (x_i,y_j) \bmod P$ and consequently $R_P(\gamma_2X') \neq R_P(X)$ for $\gamma_2 \in \Gamma_2$. Hence $R_P(\Gamma_2X')$ $R(\Gamma_2X) = \emptyset$. Thus, if $X' \in \Delta'(b)$, we have $B(X') = B(X)$.

But if $X' \in \Delta'$ then $X' = \gamma X$ with $\gamma \in \Delta$ so $(X',X') = (X,X)$. Consequently, the matrix of inner products of $(,)$ relative $X' \cup Y$ is the same as the matrix of $(,)$ relative $X \cup Y$. Consequently, if $X' \in \Delta'(b)$, there exists $g \in G(k)$ such that $gX = X'$ and $gY = Y$.

We claim, that in case $p + q \leq n - 2$, we may assume that g has spinor norm 1. For, in this case, the orthogonal complement W of span$(X \cup Y)$ is an indefinite space of dimension greater than or equal to 3. Hence, by O'Meara [18], 101·8, we may find an element $\eta \in SO(W)$ with entries in k and having the same spinor norm as g. Then, replacing g by $g\eta$, we prove the claim (we will need this later in the case $p + q \leq n - 2$).

In any case, since $gY = Y$, we have $g = g_2 \in G_2(k)$. But then $g_2^{-1}\gamma'X = X$ so $g_2^{-1}\gamma' = g_1 \in G_1(k)$ and we obtain:

$$\gamma' = g_2g_1 \in G_2(k)G_1(k).$$

By definition, if $X' \in \Delta'(b)$, we may suppose that for every prime ideal P in $\mathcal{O}$ and every integer $m > 0$ there exists an element $\gamma_2 = \gamma_2(P,m)$ with:

$$R_{P^m}(\gamma_2X') = R_{P^m}(X).$$

The infinite set $\{\gamma_2(P,m)\} \subset \Gamma_2 \subset G_2(\mathcal{O}_P,a)$ has a limit point ν_P^{-1} in $G_2(\mathcal{O}_P,a)$ satisfying $\nu_PX = X'$. We may assume that the spinor norm of ν_P is 1 since the kernel of the spinor norm is closed in $G_2(\mathcal{O}_P)$ - it is the intersection of $G_2(\mathcal{O}_P)$ with the image of the spin group in $G_2(\mathcal{O}_P)$. But then defining $\mu_P = \nu_P^{-1}\gamma$ we find that $\mu_P \in G_1(\mathcal{O}_P,a)$ and:

$$\gamma' = \nu_P\mu_P.$$

At this point we separate the proof of the theorem into two cases; the first in which $p + q = n$ and the second in which $p + q \leq n - 2$.

For the first case we note $G_1 \cap G_2 = \{1\}$ since any $g \in G_1 \cap G_2$ fixes a subspace of codimension 1 and has determinant 1. Thus,

we must have:

$$g_2 = \nu_P$$

and

$$g_1 = \mu_P$$

for all P. This concludes the proof of the theorem for the case $p + q = n$ since the above equality implies $g_1 \in \Gamma_1$ and $g_2 \in \Gamma_2$.

In case $p + q < n - 1$ we consider the adele $\{a_P\}$ with P th component a_P given by:

$$a_P = \nu_P^{-1}g_2.$$

Then a_P has spinor norm 1 by the previous claim and we may apply the Strong Approximation Theorem to the algebraic group $H = SO(W)$, see O'Meara [19], to conclude that there exists $\eta \in H(k)$ and an adele $\{b_P\} \in \Pi\, H(\mathcal{O}_P,a)$ such that:

$$\{b_P\eta^{-1}\} = \{\nu_P^{-1}g_2\}$$

From the previous equation we deduce $\nu_Pb_P = g_2\eta$. Consequently $g_2\eta$ is an element of Γ fixing Y and so $g_2\eta \in \Gamma_2$. Renaming $g_2\eta$ by ν and defining $\mu = \nu^{-1}\gamma$ we find $\mu \in \Gamma_1$ and $\gamma = \nu\mu \in \Gamma_2\Gamma_1$. With this Proposition 7.1 is proved.

We now show how Proposition 7.1 implies Theorem 7.2. Choose b so that $\Delta(b) \subset \Gamma_Y\Gamma_X$. Suppose first $p + q = n$. Then $\nu\mu X = \nu X$ and $\gamma X = \nu\mu X \equiv X \bmod b$. Hence $\nu X \equiv X \bmod b$. But also $\nu Y = Y$. Hence $\nu \equiv 1 \bmod b$ on span$(X \cup Y)$. But this span has codimension 1 and $\det \nu = 1$. Hence $\nu \equiv 1 \bmod b$ and consequently $\mu = \nu^{-1}\gamma$ also satisfies $\mu \equiv 1 \bmod b$.

Suppose now that $p + q \leq n - 2$. The previous argument shows that ν and μ are congruent to 1 modulo b on span$(X \cup Y)$. Also $\nu \equiv \mu^{-1} \bmod b$ on W. Let φ be the element of the finite group of isometries of W modulo b to which ν and μ^{-1} are congruent. Since φ has spinor norm 1 and the dimension of W is greater than or equal to 3, by the Strong Approximation Theorem we may find $\eta \in \Gamma \cap SO(W)$ so that $\eta^{-1} \equiv \varphi \bmod b$. We let $\nu' = \nu\eta$ and $\mu' = \eta^{-1}\mu$. Then $\nu' \equiv \mu' \equiv 1 \bmod b$ and $\gamma = \nu'\mu'$. This proves the first part of Theorem 7.2.

To prove the second part note that if $c \subset b$ then $\Delta(c) \subset \Delta(b) \subset \Gamma_Y\Gamma_X$ and we may repeat the previous argument.

We now apply Theorem 7.2 to the case $X = \{e_1,e_2\}$ and

$Y = \{y_1, y_2, \ldots, y_{n-2}\}$ an $\mathcal{O}$-integral $(n-2)$-frame chosen so that $X \cup Y$ spans a positive definite space of dimension n and so that $(e_1 \wedge e_2, y_1 \wedge y_2) > 0$. For example take $Y = \{e_1 + e_3, e_2 + e_4, e_5, \ldots, e_n\}$. We may assume, by the remark following Lemma 7.1, that Γ_Y acts trivially on the span of Y. Consequently we may form a cycle with coefficients in $\Lambda^2 V$ given by $M_Y \otimes y_1 \wedge y_2$. Similarly we have a cycle with coefficients in $\Lambda^2 V$ given by $M_Y \otimes e_1 \wedge e_2$. We use the form induced by $(,)$ on $\Lambda^2 V$ to define

$$(M_X \otimes e_2 \wedge e_2)\cdot(M_Y \otimes y_1 \wedge y_2).$$

Lemma 7.5. There exists a congruence subgroup $\Gamma(b) \subset \Gamma$ so that the corresponding cycles $M'_X \otimes e_1 \wedge e_2$ and $M'_Y \otimes y_1 \wedge y$ satisfy:

$$(M'_X \otimes e_1 \wedge e_2)\cdot(M'_Y \otimes y_1 \wedge y_2) \neq 0.$$

Proof. By Theorem 7.2 we may find b so that M'_X and M'_Y intersect at $\pi'(\mathbb{H}^n_X \cap \mathbb{H}^n_Y)$. We apply Lemma 7.2 with $\gamma = 1$ and find the coefficient contribution $(e_1 \wedge e_2, y_1 \wedge y_2) \neq 0$. With this Lemma 7.5 is proved.

Corollary. $M'_X \otimes e_1 \wedge e_1$ is a non-zero class in $H_{n-2}(\Gamma(b)\, \mathbb{H}^n, \Lambda^2 V)$.

We replace our original Γ by $\Gamma(b)$ and suppress all primes. We now apply Theorem 7.2 to the case $X = \{e_1\}$ and $Y = \{e_2\}$. We consider the cycles with coefficients in V given by $M_{e_1} \otimes e_1$ and $M_{e_2} \otimes e_2$. We use the exterior product from $V \otimes V$ to $\Lambda^2 V$ to define $(M_{e_1} \otimes e_1)\cdot(M_{e_2} \otimes e_2)$ as an element of $H_{n-2}(M, \Lambda^2 V)$. Let us denote $\pi(\mathbb{H}^n_{\{e_1,e_2\}}) \otimes e_1 \wedge e_2$ by $Z \otimes e_1 \wedge e_2$.

Remark. $Z \otimes e_1 \wedge e_2$ is not a boundary; hence if $\Gamma' \subset \Gamma$ is a subgroup of finite index, $\pi' : \mathbb{H}^n \to \Gamma' \backslash \mathbb{H}^n$ is the covering and $Z' = \pi'(\mathbb{H}^n_{e_1} \cap \mathbb{H}^n_{e_2})$ then $Z' \otimes e_1 \wedge e_2$ is not a boundary.

Lemma 7.6. There exists a congruence subgroup $\Gamma(c) \subset \Gamma$ so that:

$$(M'_{e_1} \otimes e_1)\cdot(M'_{e_2} \otimes e_2) \neq 0.$$

Proof. We apply Theorem 7.2 to deduce that there exists a congruence subgroup $\Gamma(c) \subset \Gamma$ so that $M'_{e_1} \cap M'_{e_2} = \pi'(\mathbb{H}^n_{e_1} \cap \mathbb{H}^n_{e_2})$. We denote this intersection by Z'. By the previous remark $Z' \otimes e_1 \wedge e_2 \neq 0$ and the lemma is proved since by Lemma 7.2 the coefficient contribution is non-zero – again applying Lemma 7.2 with $\gamma = 1$.

We have now proved the desired non-vanishing theorem for inter-

section products of hypersurfaces with coefficients in V. As a consequence of the results of Section 6 we have the following theorem, again assuming $n \geq 4$.

Theorem 7.3. $\mathrm{Hom}(\Gamma, SO(n+1,1))$ and $\mathrm{Hom}(\Gamma, \underline{SO}(n+1,1))$ each have a singularity at ρ_0.

Similar arguments based on Theorem 7.1 using coefficients in $S^2_0(V)$ yield the required theorem for projective structures. Note that $[h_{e_1}, h_{e_1+e_2}] = e_1 \wedge e_2$.

Lemma 7.7. For any subgroup Γ of finite index in the units of $(,)$ there exists a further congruence subgroup $\Gamma(c)$ so that:

$$(M'_{e_1} \otimes h_{e_1})\cdot(M'_{e_1+e_2} \otimes h_{e_1+e_2}) \neq 0.$$

Remark. In fact we obtain $(M'_{e_1} \otimes h_{e_1})\cdot(M'_{e_1+e_2} \otimes h_{e_1+e_2}) = M_{\{e_1,e_2\}} \otimes e_1 \wedge e_2$ which we proved to be non-zero in Lemma 7.5.

Since ρ_0 is good in the projective case, we obtain the following theorem, again assuming $n \geq 4$.

Theorem 7.4. $\mathrm{Hom}(\Gamma, PGL_{n+1}(\mathbb{R}))$, $\mathrm{Hom}(\Gamma, PGL_{n+1}(\mathbb{C}))$, $X(\Gamma, PGL_{n+1}(\mathbb{R}))$ and $X(\Gamma, PGL_{n+1}(\mathbb{C}))$ are singular at ρ_0.

We now wish to establish the existence of non-isolated singularities for the deformation spaces. By the results of Section 6, it is sufficient to find a two-sided, totally geodesic, non-singular hypersurface N disjoint from M_{e_1} and M_{e_2} (or M_{e_1} and $M_{e_1+e_2}$). We prove a more general theorem in the framework of Theorem 7.2 with X and Y as in that theorem. We suppose that $f : M' \to M$ is a cover and that Ψ is the group of covering transformations of f. We let Ψ_X denote the group of covering transformations of $M'_X \to M_X$ and Ψ_Y denote the group of covering transformations of $M'_Y \to M_Y$. Let $\eta \in \Psi$.

Lemma 7.8. $\eta(M'_X) \cap M'_X \neq \emptyset$ if and only if $\eta \in \Psi_X$.

Proof. Suppose $\eta(M'_X) \cap M'_X \neq \emptyset$. We choose $\gamma \in \Gamma$ representing η. Then there exist x_1 and x_2 in $\mathbb{H}^n_X$ such that $\pi'(\gamma x_1) = \pi'(x_2)$. Hence there exists $\gamma' \in \Gamma'$ so that $\gamma'\gamma x_1 = x_2$. But by Lemma 7.1, we have $\gamma'\gamma \in \Gamma_X$ and consequently $\eta \in \Psi_X$. The converse is clear and the lemma is proved.

We now examine when $\eta(M'_X)$ meets M'_Y. We assume M is chosen to satisfy the conclusions of Theorem 7.2; that is, we require that

$M_X \cap M_Y$ consist of a single component.

Lemma 7.9. $\eta(M'_X) \cap M'_Y \neq \emptyset$ if and only if $\eta \in \Psi_Y \Psi_X$.

Proof. Suppose $\eta(M'_X) \cap M'_Y \neq \emptyset$. Choose $\gamma \in \Gamma$ representing η. Then there exist $x_1 \in H^n_X$ and $x_2 \in H^n_Y$ such that $\pi'(x_1) = \pi'(x_2)$. Hence there exists $\gamma' \in \Gamma'$ such that $\gamma'\gamma x_1 = x_2$; that is, $\gamma'\gamma \in \Delta$. But by construction of Γ (Theorem 7.2) there exist $\nu \in \Gamma_Y$ and $\mu \in \Gamma_X$ such that $\gamma'\gamma = \nu\mu$. Reducing modulo Γ' we find $\eta \in \Psi_Y \Psi_X$. The converse is clear and the lemma is proved.

Theorem 7.5. Let $Y_1, Y_2, \ldots, Y_m$ be given such that for $j = 1,2,\ldots,m$ the set $X \cup Y_j$ spans a positive definite subspace of V of dimension $p+q$ with $p + q \neq n - 1$. Then there exists a covering $f: M' \to M$ and a covering transformation η of f such that $\eta(M'_X)$ does not intersect $M'_X, M'_{Y_1}, \ldots, M'_{Y_m}$.

Proof. We apply Theorem 7.2 successively to $Y_1, \ldots, Y_m$ to arrange that $M'_X \cap M'_{Y_j}$ consists of a single component for $j = 1,2,\ldots,m$. From Lemma 7.9 we find that it is sufficient to find a covering group Ψ such that $\Psi \neq \Psi_{Y_1}\Psi_X \cup \cdots \cup \Psi_{Y_n}\Psi_X$. Suppose no such cover exists. Choose $x \in X$ and $y_j \in Y_j$ for $j = 1,2,\ldots,m$. Then the equation $\Pi^m_{j=1}[(gx,y_j) - (x,y_j)] = 0$ is satisfied for all g in the congruence completion of Γ, hence for all $g \in \Gamma$ and hence by Zariski density for all $g \in \underline{G}$. Since $\underline{G}$ is irreducible one of the factors in the above equation must vanish identically on $\underline{G}$. But this is absurd.

We now prove the main theorem of this section assuming $n \geq 4$ and Γ as above.

Theorem 7.6. The spaces $R(\Gamma, SO(n+1,1))$, $R(\Gamma, \underline{SO}(n+1,1))$, $X(\Gamma, SO(n+1,1))$, $X(\Gamma, \underline{SO}(n+1,1))$, $R(\Gamma, PGL_{n+1}(\mathbb{R}))$, $R(\Gamma, PGL_{n+1}(\mathbb{C}))$, $X(\Gamma, PGL_{n+1}(\mathbb{R}))$ and $X(\Gamma, PGL_{n+1}(\mathbb{C}))$ all have non-isolated singularities.

Proof. We give the proof for the first case. We apply the previous theorem to the case $X = \{e_1\}$, $Y = \{e_2\}$ and $Y_2 = \{e_2 + e_4\}$. Then $\eta(M'_{e_1})$ is a totally geodesic hypersurface which does not intersect M'_{e_1}, M'_{e_2} or $M'_{e_2+e_4}$ and a fortiori does not intersect the surface $M'_{\{e_1+e_3, e_2+e_4, e_5, e_6, \ldots, e_m\}}$. Hence if ρ_t is the deformation of ρ corresponding to the totally geodesic hypersurface $\eta(M'_{e_1})$ then ρ_t is constant on the fundamental groups of the three above manifolds and

the intersection number calculations of Lemma 7.5 and Lemma 7.6 are independent of t.

Remark. Further work is required in order to make precise the statement that $C(M)$, $H(M \times \mathbb{R})$ and $P(M)$ are singular. First, we need a "completeness" theorem to the effect that each point in $S(M)$, the space of marked (G,X) structures for some G and X, has a neighborhood isomorphic to an analytic subvariety in $H^1(M,\Theta)$ where Θ is the sheaf of infinitesimal automorphisms of the G-structure. Second we need to know that the holonomy map preserves this structure. It appears that these results can be proved by imitating the proof of completeness for complex structures.

8. $C(M)$ and Riemannian Geometry.

In this section, we will regard $C(M)$ as the quotient space of Riemannian metrics with vanishing Weyl tensor by the group which is the semi-direct product of the group $C^\infty_+(M)$ of strictly positive smooth functions on M and the group of diffeomorphisms of M isotopic to the identity. Thus a point $c \in C(M)$ is an equivalence class of Riemannian metrics all of which have zero Weyl tensor. In what follows n will denote the dimension of the manifold M under consideration. The following theorem provides a canonical metric in an orbit under $C^\infty_+(M)$ of conformally flat metrics. We owe the theorem to S.Y. Cheng. Its proof will appear elsewhere. Of course M is always a compact hyperbolic manifold in what follows.

Theorem 8.1. Every orbit under $C^\infty_+(M)$ of conformally flat metrics contains a metric of constant scalar curvature. The metric is unique up to scalar multiples.

We will use two different normalizations of the scalar.

Corollary 1 (first normalization). Every orbit under $C^\infty_+(M)$ of conformally flat metrics contains a unique metric g of constant scalar curvature $-n(n-1)$.

Corollary 2 (second normalization). Every orbit under $C^\infty_+(M)$ of conformally flat metrics contains a unique metric g' of constant scalar curvature such that the volume of M (using the volume element associated to g') is 1.

Remark. The first corollary is the generalization of the theorem stating that every complex (conformal) structure on M^2 contains a unique hyperbolic metric.

Corollary 1 allows us to define an interesting function

$$\mathrm{vol}: \mathcal{C}(M) \to \mathbb{R}_+$$

as follows. Let $c \in \mathcal{C}(M)$ and g be the canonical metric with the first normalization. Then $\mathrm{vol}(c)$ is by definition the volume of M for the metric g. We can now relate the two normalizations g, g' in a conformal structure c, namely:

$$g' = \left(\frac{1}{\mathrm{vol}(c)}\right)^{2/n} g.$$

We now define a function $A: \mathcal{C}(M) \to \mathbb{R}$ closely related to vol but more convenient for computations by:

$$A(c) = \int_M \tau(g')\mathrm{vol}'$$

Here $\tau(g')$ is the scalar curvature of g'. Since $\tau(\lambda g) = 1/\lambda\tau(g)$ for λ a positive constant we find:

$$A(c) = -n(n-1)(\mathrm{vol}(c))^{2/n}.$$

Before studying the function vol further, we point out another consequence of Theorem 8.1, the existence of a Petersson-Weil metric on $\mathcal{C}(M)$. Now a Petersson-Weil metric on a space of structures is a consequence of a canonical metric in each structure and a Hodge theorem representing the infinitesimal deformations by "harmonic" tensor fields on M (as opposed to cohomology classes of tensor fields). The required Hodge Theorem has been proved by Gasqui and Goldschmidt [11].

We now prove some properties of the function vol. Of course, in the case $n = 2$, the function vol is constant by the Gauss-Bonnet Theorem. For all n, the unique hyperbolic structure, to be denoted c_0, is a critical point of A, Berger [3], page 29, hence, a critical point of vol. That the situation for $n > 2$ is altogether different from that of $n = 2$ is clear from the following theorem.

Theorem 8.2. If $n \geq 3$ the second derivative of $\mathrm{vol}: \mathcal{C}(M^n) \to \mathbb{R}_+$ at the hyperbolic structure c_0 is positive definite. In particular vol is not constant on $\mathcal{C}(M)$ provided $n \geq 3$.

Proof. The statement of the theorem is equivalent to the statement that the second derivative of A at the hyperbolic structure is negative definite. But the theorem now follows from [12], Theorem 2.5.

Indeed, we have only to check the eigenvalues of the operator L of [12], associated to the curvature transformation of the hyperbolic metric, on traceless symmetric 2-tensors. These eigenvalues are easily seen to be 0 on $e_i \otimes e_i - e_j \otimes e_j$ and -1 on $1/2(e_i \otimes e_j + e_j \otimes e_i)$. The minimum eigenvalue -1 is greater than $\min\{\frac{\tau}{n}, -\frac{\tau}{2n}\} = -(n-1)$ provided $n \geq 3$. With this the theorem is proved.

In the case in which $n = 4$ we find a remarkable and suggestive result using the Gauss-Bonnet Theorem.

Theorem 8.3. If $n = 4$; the function $\mathrm{vol}: \mathcal{C}(M^4) \to R_+$ has an absolute minimum at the hyperbolic structure c_0.

Proof. Let c be a conformal structure on M i.e. a canonical metric with the first normalization. In Berger [2], there is a formula for the Gauss-Bonnet integrand B as a universal linear combination of the norm $\|R\|^2$ of the curvature transformation R, the norm $\|\mathrm{Ric}\|^2$ of the Ricci transformation Ric and τ^2 the square of the scalar curvature. For a conformally flat manifold R is a linear function of Ric so B must be a universal combination of $\|\mathrm{Ric}\|^2$ and τ^2. By computing for S^4 and $S^1 \times S^3$ we find:

$$B = -2\|\mathrm{Ric}\|^2 + \frac{2}{3}\tau^2.$$

Hence

$$\frac{2}{3}\int_M \tau^2 = 32\pi^2 \chi(M) + 2\int_M \|\mathrm{Ric}\|^2$$

By Cauchy-Schwarz, we have for a symmetric transformation S:

$$(\mathrm{tr}\ S)^2 \leq \|S\|^2 n$$

if $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are the eigenvalues we have:

$$((\lambda_1, \lambda_2, \ldots, \lambda_n), (1, 1, \ldots, 1))^2 \leq (\lambda_1^2 + \cdots + \lambda_n^2)n.$$

We obtain then:

$$\tau^2 \leq \|\mathrm{Ric}\|^2 4$$

Substituting

$$\frac{2}{3}\int_M \tau^2 \geq 32\pi^2 \chi(M) + \frac{1}{2}\int_M \tau^2$$

and:

$$\int_M \tau^2 \geq 6(32\pi^2)\chi(M)$$

By the Hirzebruch Proportionality Principle we have:

$$\chi(M) = 2\frac{\mathrm{vol}(c_0)}{\frac{8\pi^2}{3}} = \frac{6\ \mathrm{vol}(c_0)}{8\pi^2}$$

where $vol(c_0)$ denotes the volume of M for the hyperbolic metric. Hence:

$$\int_M \tau^2 \geq 144 vol(c_0).$$

But the canonical metric is normalized so that $\tau = -12$. We obtain:

$$\int_M \tau^2 = 144 \text{ vol } M = 144 \text{ vol}(c)$$

and hence

$$vol(c) \geq vol(c_0).$$

With this the theorem is proved.

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ON DIVISION OF FUNCTIONS, SOLUTION OF MATRIX EQUATIONS, AND PROBLEMS IN DIFFERENTIAL GEOMETRY AND PHYSICS

by Mark Alan Mostow

Dedicated to my father on his sixtieth birthday

In this article we present some results on the continuity of division of smooth functions and discuss their applications to linear algebra, differential geometry, and physics. Much of the work was done jointly with Steven Shnider and will appear in greater detail elsewhere ([MS2], [MS3], [Mos2].

The basic division problem treated here, which we shall call the joint continuity of division of smooth functions, is the following:

> Consider the collection of triples (f,g,h) of smooth (C^∞) real-valued functions on $\mathbb{R}^n$, or more generally, on a manifold M, satisfying the relation $f = g \cdot h$ (product). Assuming that $g^{-1}(0)$ is nowhere dense (i.e. that its complement is dense), we can write $h = f/g$ without ambiguity. Is the quotient h a (jointly) continuous function of the pair $(f = gh, g)$, with respect to the Fréchet C^∞ topology of uniform convergence of a function and its derivatives on compact sets?

This question appears not to have been considered explicitly. What has been studied is the continuity in the numerator of division by a fixed smooth function g, that is, of the operator sending $f = gh$ to $h = f/g$. For example, Łojasiewicz [Loj] proved that division by a real analytic function is continuous. We refer to [MS2] for a discussion of the problem of continuity in the numerator and its relation to closedness of ideals in rings of smooth functions and to divisibility of distributions by smooth functions; see [Horm] for its relation to the existence of tempered solutions of partial differential equations. But continuity in the numerator does not imply joint continuity, as the