THE SYMPLECTIC GEOMETRY OF POLYGONS IN THE 3-SPHERE

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ABSTRACT. We study the symplectic geometry of the moduli spaces $M_r = M_r(\mathbb{S}^3)$ of closed n-gons with fixed side-lengths in the 3-sphere. We prove that these moduli spaces have symplectic structures obtained by reduction of the fusion product of n conjugacy classes in $SU(2)$, denoted C_r^n, by the diagonal conjugation action of $SU(2)$. Here C_r^n is a quasi-Hamiltonian $SU(2)$-space. An integrable Hamiltonian system is constructed on M_r in which the Hamiltonian flows are given by bending polygons along a maximal collection of nonintersecting diagonals. Finally, we show the symplectic structure on M_r relates to the symplectic structure obtained from gauge-theoretic description of M_r. The results of this paper are analogues for the 3-sphere of results obtained for $M_r(\mathbb{H}^3)$, the moduli space of n-gons with fixed side-lengths in hyperbolic 3-space [KMT], and for $M_r(\mathbb{E}^3)$, the moduli space of n-gons with fixed side-lengths in $\mathbb{E}^3$ [KM1].

1. INTRODUCTION

In this paper we study the symplectic geometry of the space of polygons in $\mathbb{S}^3$ with fixed side-lengths modulo the group of isometries. We denote this moduli space by $M_r = M_r(\mathbb{S}^3)$. This paper is continuation of [KM1] and [KMT], which studied the polygonal linkages in Euclidean 3-space and hyperbolic 3-space, respectively.

An (open) n-gon P in $\mathbb{S}^3$ is an ordered $(n+1)$-tuple $(x_1, ..., x_{n+1})$ of points in $\mathbb{S}^3 \subset \mathbb{C}^2$ called the vertices. We join the vertex x_i to the vertex x_{i+1} by the unique geodesic segment e_i, called the i-th edge (here we must make the restriction x_i and x_{i+1} are not antipodal points). We let Pol_n denote the space of n-gons in $\mathbb{S}^3$. An n-gon is said to be closed if $x_{n+1} = x_1$. We let $CPol_n$ denote the space of closed n-gons. The group $G = SU(2) \times SU(2)$ acting on $\mathbb{S}^3$ by $g \cdot x = g_1 x g_2^{-1}$, $x \in \mathbb{S}^3$, $g = (g_1, g_2) \in G$, is the group of isometries of $\mathbb{S}^3$. Two n-gons $P = (x_1, ..., x_{n+1})$ and $P' = (x'_1, ..., x'_{n+1})$ are equivalent if there exists $g \in G$ such that $g \cdot P = P'$, that is $g \cdot x_i = x'_i$, for all $1 \le i \le n+1$.

Let $r = (r_1, ..., r_n) \in \mathbb{R}_+^n$ be an n-tuple of positive numbers with $r_i < \pi$ for $1 \le i \le n$. We denote by $\widetilde{N}_r$ the space of open n-gons in which the side e_i a has fixed length $d(x_i, x_{i+1}) = r_i$. We then let $\widetilde{M}_r = \widetilde{N}_r \cap CPol_n$, $N_r = \widetilde{N}_r/G$, and $M_r = \widetilde{M}_r/G$. This paper examines the symplectic geometry of the space M_r.

We have $G = SU(2) \times SU(2)$, K is the diagonal subgroup in G, and $P = G/K$ which we identify with $SU(2)$. We equip G, K, P with the quasi-Poisson structures associated to the standard Manin pair $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{g} = \{(x, y) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)\}$ and $\mathfrak{k} = \{(x, x) \in \mathfrak{g} : x \in \mathfrak{su}(2)\}$.

The main theorem of this paper is:

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Theorem 1.1. *The space M_r is a symplectic manifold with the symplectic structure obtained from reduction of the fusion product of n conjugacy classes in $SU(2)$, $C_{r_1} \circledast \cdots \circledast C_{r_n}$, by the diagonal dressing action (conjugation) of the quasi-Poisson Lie group K.*

We are also interested in finding an integrable system on M_r. We denote by d_{ij} a geodesic connecting the vertices x_i and x_j (we always assume $i < j$), which we call a diagonal. Let ℓ_{ij} be the length of the diagonal d_{ij}. Then ℓ_{ij} is a continuous function on M_r, but it is not smooth when either $\ell_{ij} = 0$ or $\ell_{ij} = \pi$. If d_{ij} and d_{km} are nonintersecting diagonals, then

$$\{\ell_{ij}, \ell_{km}\} = 0.$$

By considering a maximal collection of nonintersecting diagonals, we obtain $\frac{1}{2} dim(M_r)$ Poisson commuting Hamiltonians.

The Hamiltonian flow Ψ_{ij}^t associated to a ℓ_{ij} has the following nice description. Separate the polygon into two pieces via the diagonal d_{ij}, the Hamiltonian flow is given by leaving one piece fixed while rotating the other piece about the diagonal at constant angular velocity 1. The flow Ψ_{ij}^t is called the "bending flow" along the diagonal d_{ij}.

The paper is organized ad follows:

In section 2, we give background material for Manin pairs and quasi-Poisson Lie groups.

In section 3, we define a symplectic structure on M_r by quasi-Hamiltonian reduction on the fusion product of conjugacy classes.

In section 4, we study the Hamiltonians ℓ_{ij} and their associated Hamiltonian flows.

In section 5, we study the an action of the pure braid group on M_r given by the time 1 Hamiltonian flows of a certain family of functions.

In section 6, we relate the symplectic form on M_r to symplectic form given on the relative character varieties on n-punctured 2-spheres.

We note that the moduli spaces of polygons in the spaces of constant curvature give examples of completely integrable systems obtained from the theory of Manin pairs associated to a compact simple Lie group [AMM2]. The Manin pairs corresponding to the various moduli spaces are:

- $\big(\mathfrak{su}(2) \ltimes \mathfrak{su}(2)^*, \mathfrak{su}(2)\big)$ for polygons in the zero curvature space (Lie-Poisson theory);
- $\big(\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}(2)^{\mathbb{C}}, \mathfrak{su}(2)\big)$ for polygons in negative curvature space (Poisson-Lie theory);
- $\big(\mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathfrak{su}(2)\big)$ for polygons in positive curvature space (quasi-Poisson Lie theory).

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2. Manin Pairs and quasi-Poisson Lie groups

2.1. **quasi-Poisson Structures.** In this section, we let K be any compact simple Lie group with Lie algebra denoted by $\mathfrak{k}$. Let $G = K \times K$ be the double of K with Lie algebra

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}$. The Killing form on $\mathfrak{k}$, which we denote by $(,)$, defines a nondegenerate bilinear form $B(,)$ on $\mathfrak{g}$ given by

$$B((X_1, X_2), (Y_1, Y_2)) = (X_1, Y_1) - (X_2, Y_2), \text{ for } (X_1, X_2), (Y_1, Y_2) \in \mathfrak{g}.$$

If we now let K denote the diagonal subgroup of G then its Lie algebra $\mathfrak{k}$ is a maximal isotropic subalgebra of $\mathfrak{g}$. The pair $(\mathfrak{g}, \mathfrak{k})$ is a Manin pair. We will construct a quasi-Poisson Lie group structure on G associated to the Manin pair $(\mathfrak{g}, \mathfrak{k})$ which restricts to a (trivial) quasi-Poisson Lie group structure on K. For background on quasi-Poisson Lie groups, quasi-Poisson structures, Manin pairs, etc. we refer the reader to [AKS], [Le], [KS1], [KS2].

Let $\mathfrak{p} = \{(\frac{1}{2}X, -\frac{1}{2}X) \in \mathfrak{g}\}$ be the anti-diagonal in $\mathfrak{g}$. Then $\mathfrak{p}$ is an isotropic complement of $\mathfrak{k}$. Note that $\mathfrak{p}$ is not a Lie subalgebra of $\mathfrak{g}$ ($[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$), so the triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ is a Manin quasi-triple, rather than a Manin triple which arises in the theory of Poisson Lie groups. We call this triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ the standard Manin quasi-triple.

A Manin quasi-triple gives rise to a Lie quasi-bialgebra $(\mathfrak{k}, F, \varphi)$. We can identify $\mathfrak{p}$ with $\mathfrak{k}^*$ via the bilinear form of $\mathfrak{g}$. The cobracket on $\mathfrak{k}$ is a map $F : \mathfrak{k} \to \mathfrak{k} \wedge \mathfrak{k}$ which is the transpose of the map from $\mathfrak{p} \wedge \mathfrak{p} \to \mathfrak{p}$, also denoted by F, defined by

$$F(\xi, \eta) = \rho_{\mathfrak{p}}[\xi, \eta], \ \xi, \eta \in \mathfrak{p}.$$

We can also define the element $\varphi \in \wedge^3 \mathfrak{k}$ by the map $\mathfrak{p} \wedge \mathfrak{p} \to \mathfrak{k}$ given by

$$\varphi(\xi, \eta) = \rho_{\mathfrak{k}}[\xi, \eta], \ \xi, \eta \in \mathfrak{p}.$$

For the Manin quasi triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ given above, we have $F = 0$ and $\varphi = \frac{1}{24} \sum_{ijk} f^i_{jk} e_i \wedge e_j \wedge e_k$, where $[e_j, e_k] = \sum_i f^i_{jk} e_i$.

We can also identify $\mathfrak{g}$ with $\mathfrak{k} \oplus \mathfrak{k}^*$ via the bilinear form B(,). The canonical r-matrix on $\mathfrak{g}$ associated to the Manin quasi-triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ is an element $r_{\mathfrak{g}} \in \mathfrak{g} \otimes \mathfrak{g}$ defined by the map $r_{\mathfrak{g}} : \mathfrak{g}^* \to \mathfrak{g}$ given by $r_{\mathfrak{g}}(\xi, X) = (0, \xi)$ where $X \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. Let $\{e_i\}$ be an orthonormal basis of $\mathfrak{k}$ and $\{\varepsilon^i\}$ be the dual basis in $\mathfrak{k}^*$, then

$$r_{\mathfrak{g}} = \sum_i e_i \otimes \varepsilon^i.$$

The multiplicative 2-tensor $w_G = dL_g r_{\mathfrak{g}} - dR_g r_{\mathfrak{g}}$ actually defines a bivector on G, since the symmetric part of $r_{\mathfrak{g}}$ is a multiple of the bilinear form $B(,)$ on $\mathfrak{g}$. $w_{\mathfrak{g}}$ gives us a quasi-Poisson Lie group structure on G. $w_{\mathfrak{g}}$ naturally restricts to the trivial bivector on the subgroup $K \subset G$. There is also a natural projection of $w_{\mathfrak{g}}$ to $G/K = P$, which can identified with K, via the map $p : G \to P$ defined by $p(g_1, g_2) = g_1 g_2^{-1}$. The bivector w_P is given by

$$w_P = \frac{1}{2} \sum_i e_i^{\lambda} \wedge e_i^{\rho}.$$

Here e_i^{λ} (e_i^{ρ}) denotes the left-invariant (resp. right-invariant) vector field on P with value e_i at the identity. We will use this notation for vector fields on P throughout the rest of the paper. Note that w_P is not multiplicative, so P is not a quasi-Poisson Lie group. We will see that in the next section that P is the target space of a generalized moment map.

2.2. **Moment map and reduction.** The action of G on itself is by left multiplication induces an action of K on P, the dressing action, which is given by conjugation.

We denote by x_M the vector field, more generally the multivector field, on M induced by the action of K on M and $x \in \mathfrak{k}$ satisfying

$$(x_M f)(m) = \frac{d}{dt}|_{t=0} f(\exp(-tx) \cdot m)$$

where $f \in C^\infty(M)$ and $m \in M$. This is a Lie algebra homomorphism, i.e. $[x_M, y_M] = [x, y]_M$ for $x, y \in \mathfrak{k}$.

We have the following definition of a quasi-Poisson action.

Definition 2.1. *Let (K, w_K, φ) be a connected quasi-Poisson Lie group acting on a manifold M with bivector w_M. The action of K on M is said to be a quasi-Poisson action if and only if*

(i) $\frac{1}{2}[w_M, w_M] = \varphi_M$
(ii) $\mathcal{L}_{x_M} w_M = -(F(x)_M)$

for all $x \in \mathfrak{k}$.

The dressing action of K on P is a quasi-Poisson action. There is also a notion of a generalized moment map associated to a quasi-Poisson action.

Definition 2.2. *A map $\mu : M \to P$, equivariant with respect to the action of K on M and the dressing action of K on P, is called a moment map for the action of K on (M, w_M) if, on any open subset of M,*

$$w^\sharp(\mu^* \alpha_x) = x_M.$$

Here $\alpha_x \in \Omega^1(P)$ is defined by $< \alpha_x, \xi_P >= -(x, \xi)$ for $x \in \mathfrak{k}$ and $\xi \in \mathfrak{p}$.

Definition 2.3. *The action of K on M is called quasi-Hamiltonian if it admits a moment map. A quasi-Hamiltonian space is a manifold with bivector on which a quasi-Poisson Lie group acts by a quasi-Hamiltonian action.*

The following lemma will be useful in this paper for the proofs of Proposition 2.8 and Theorem 2.7.

Lemma 2.4. *Let (M, w_M) be a manifold with bivector on which the compact simple Lie group K act in a quasi-Poisson manner. Then (M, w_M) is a quasi-Hamiltonian space if and only if there exists a map $\mu : M \to P$ which is equivariant with respect to action of K on M and the action of K on P by conjugation which satisfies*

$$w^\sharp(\mu^*(x, \theta)) = \frac{1}{2}((1_\mathfrak{k} + Ad_\mu)x)_M$$

for all $x \in \mathfrak{k}$. Here $w^\sharp : T^*M \to T_*M$ is given by $w^\sharp(\alpha) = w(\alpha, \cdot)$ for $\alpha \in T^*M$, and $\theta : T_*K \to \mathfrak{k}$ is the left-invariant Maurer-Cartan on K. For K a matrix group $\theta = k^{-1}dk$.

Proof: See [AKS, Proposition 5.33]. □

Example 2.5. *The basic example of a quasi-Hamiltonian space is the space P. The action of K on P is the dressing action and the associated moment map is the identity map. The bivector on P is given by $w_P = \frac{1}{2}\sum_i e_i^\lambda \wedge e_i^\rho$.*

In general, any K-invariant embedded submanifold of P is also a quasi-Hamiltonian space with moment map given inclusion.

Example 2.6. *Let $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ be the standard Manin quasi-triple. Let $C \subset P$ be a conjugacy class in P. The action of K on C given by conjugation is a quasi-Poisson action. The momentum map associated to this action of is the inclusion map (i.e. $\mu : C \to P$ given by $\mu(g) = g$). Since the bivector w_P is K-invariant, the bivector on C is given by the restriction $w_P|_C$*

Even though a quasi-Hamiltonian space (M, μ, w_M) is not in general a Poisson manifold, $\frac{1}{2}[w_M, w_M] = \varphi_M$, there is still a notion of reduction to a symplectic manifold.

Lemma 2.7. *Let (M, w_M, μ) be a quasi-Hamiltonian space such that the bivector w_M is everywhere nondegenerate. Assume M/G is a smooth manifold in a neighborhood U of $p(x_0)$, where $p : M \to M/G$ and $x_0 \in M$. Let $x \in M$ be such that $p(x) \in U$ and $s = \mu(x) \in D/G$ is a regular value of the moment map μ. Then the symplectic leaf through $p(x)$ in the Poisson manifold U is the connected component of the intersection with U on the projection of the manifold $\mu^{-1}(s)$.*

Proof: See [AKS, Theorem 5.5.5]

2.3. **Fusion product of quasi-Poisson manifolds.** Given quasi-Hamiltonian spaces M_1 and M_2 each acted on by K with associated moment maps $\mu_1 : M_1 \to P$ and $\mu_2 : M_2 \to P$, it is not true that $M_1 \times M_2$ with the product bivector structure is a quasi-Hamiltonian K-space with the action being the diagonal action of K on $M_1 \times M_2$. We can define a new bivector on $M_1 \times M_2$ such that diagonal action is a quasi-Poisson action with respect to this new bivector. $M_1 \times M_2$ with this bivector is called the fusion product and is due to [AKSM].

As defined in the previous section, the subscript M denotes the vector field, or multivector field, induced by the action of K on M.

Proposition 2.8. *Let (M_1, w_1, μ_1) and (M_2, w_2, μ_2) be quasi-Hamiltonian K-spaces in the sense of* [AKS]. *Then $M = M_1 \times M_2$ with the action of K on M given by the diagonal action, bivector on M given by*

$$w_M = w_1 + w_2 + \frac{1}{2}\sum_j (e_j)_{M_1} \wedge (e_j)_{M_2}$$

and moment map $\mu = \mu_1\mu_2$ is a quasi-Hamiltonian K-space. Recall $\{e_i\}$ is an orthonormal basis of $\mathfrak{k}$. M with this structure is called the fusion product of M_1 and M_2 and is denoted by $M = M_1 \circledast M_2$.

Proof: We begin by showing the diagonal action of K on (M, w_M) is a quasi-Poisson action. For this we need to show,

(i) $\frac{1}{2}[w_M, w_M] = \varphi_M$
(ii) $\mathcal{L}_{x_M} w_M = 0$.

We will then show that $\mu : M_1 \times M_2 \to P$ given above is the moment map associated to the diagonal action.

It is a straightforward calculation to show (i):

$$
\begin{aligned}
\frac{1}{2}\Big[w_M, w_M\Big] &= \frac{1}{2}\Big[w_1 + w_2 + \frac{1}{2}\sum_j (e_j)_{M_1} \wedge (e_j)_{M_2}, w_1 + w_2 + \frac{1}{2}\sum_k (e_k)_{M_1} \wedge (e_k)_{M_2}\Big] \\
&= \frac{1}{2}\Big[w_1, w_1\Big] + \frac{1}{2}\Big[w_2, w_2\Big] + \Big[w_1 + w_2, \frac{1}{2}\sum_{j=1}^{n} (e_j)_{M_1} \wedge (e_j)_{M_2}\Big] \\
&\quad + \frac{1}{2}\Big[\frac{1}{2}\sum_j (e_j)_{M_1} \wedge (e_j)_{M_2}, \frac{1}{2}\sum_k (e_k)_{M_1} \wedge (e_k)_{M_2}\Big] \\
&= \frac{1}{2}\Big[w_1, w_1\Big] + \frac{1}{2}\Big[w_2, w_2\Big] + \Big[w_1 + w_2, \sum_j (e_j)_{M_1} \wedge (e_j)_{M_2}\Big] \\
&\quad + \frac{1}{8}\sum_{j,k}\Big(\Big[(e_j)_{M_1}, (e_k)_{M_1}\Big] \wedge (e_j)_{M_2} \wedge (e_k)_{M_2} + \Big[(e_j)_{M_2}, (e_k)_{M_2}\Big] \wedge (e_j)_{M_1} \wedge (e_k)_{M_1}\Big)
\end{aligned}
$$

But $\frac{1}{2}\Big[w_i, w_i\Big] = \varphi_{M_i}$ for $i = 1, 2$ since the K- actions on M_1 and M_2 are quasi-Poisson actions. Also, we have $[(e_k)_{M_i}, w_i] = \mathcal{L}_{(e_k)_{M_i}} w_1 = -\Big(F(e_k)\Big)_{M_i}$ where $F : \mathfrak{k} \to \wedge^2\mathfrak{k}$ is the cobracket. But $F \equiv 0$ for the standard quasi-Poisson Lie group K we have, thus $[(e_k)_{M_i}, w_i] = 0$. Let f^i_{jk} denote the structure constants on $\mathfrak{k}$. The above equations then become

$$
\begin{aligned}
&= \varphi_{M_1} + \varphi_{M_2} + 0 + \frac{1}{8}\sum_{j,k}\Big[e_j, e_k\Big]_{M_1} \wedge (e_j)_{M_2} \wedge (e_k)_{M_2} \\
&\quad + \frac{1}{8}\sum_{j,k}\Big[e_j, e_k\Big]_{M_2} \wedge (e_j)_{M_1} \wedge (e_k)_{M_1} \\
&= \frac{1}{24}\sum_{ijk} f^i_{jk}(e_i)_{M_1} \wedge (e_j)_{M_1} \wedge (e_k)_{M_1} + \frac{1}{24}\sum_{ijk} f^i_{jk}(e_i)_{M_2} \wedge (e_j)_{M_2} \wedge (e_k)_{M_2} \\
&\quad + \frac{1}{8}\sum_{ijk} f^i_{jk}(e_i)_{M_1} \wedge (e_j)_{M_2} \wedge (e_k)_{M_2} + \frac{1}{8}\sum_{ijk} f^i_{jk}(e_i)_{M_2} \wedge (e_j)_{M_1} \wedge (e_k)_{M_1} \\
&= \frac{1}{24}\sum_{ijk} f^i_{jk}\Big((e_i)_{M_1} + (e_i)_{M_2}\Big) \wedge \Big((e_j)_{M_1} + (e_j)_{M_2}\Big) \wedge \Big((e_k)_{M_1} + (e_k)_{M_2}\Big) \\
&= \frac{1}{24}\sum_{ijk} f^i_{jk}(e_i)_M \wedge (e_j)_M \wedge (e_k)_M \\
&= \varphi_M
\end{aligned}
$$

To show (ii), we again use $\mathcal{L}_{(e_k)_{M_i}} w_{M_i} = 0$.

$$
\begin{aligned}
\mathcal{L}_{(e_k)_M} w_M &= \mathcal{L}_{(e_k)_{M_1}+(e_k)_{M_2}}\Big(w_1 + w_2 + \sum (e_j)_{M_1} \wedge (e_j)_{M_2}\Big) \\
&= \mathcal{L}_{(e_k)_{M_1}+(e_k)_{M_2}}\Big(\sum (e_j)_{M_2} \wedge (e_j)_{M_2}\Big) \\
&= \sum \Big[(e_k)_{M_1}, (e_j)_{M_1}\Big] \wedge (e_j)_{M_2} - \sum \Big[(e_k)_{M_2}, (e_j)_{M_2}\Big] \wedge (e_j)_{M_1} \\
&= \sum_{i,j} C^i_{kj} (e_i)_{M_1} \wedge (e_j)_{M_2} - \sum_{i,j} C^i_{kj} (e_i)_{M_2} \wedge (e_j)_{M_1} \\
&= 0
\end{aligned}
$$

We next use Lemma 2.4 to show that $\mu = \mu_1\mu_2 : M_1 \times M_2 \to P$ is indeed the moment map associated to the diagonal action.

$$
\begin{aligned}
w^\sharp(\mu^*(x,\theta)) &= w^\sharp((\mu_1\mu_2)^*(x,\theta)) \\
&= w^\sharp((x, \mu_2^*\theta + Ad_{\mu_2^{-1}}\mu_1^*\theta)) \\
&= w^\sharp(\mu_2^*(x,\theta) + \mu_1^*(Ad_{\mu_2}x,\theta)) \\
&= w_1^\sharp\Big(\mu_1^*(Ad_{\mu_2}x,\theta)\Big) + w_2^\sharp\Big(\mu_2^*(x,\theta)\Big) + \frac{1}{2}\sum_j \Big((\mu_1^*(Ad_{\mu_2}x,\theta))(e_j)_{M_1}\Big)(e_j)_{M_2} \\
&\quad -\frac{1}{2}\sum_j \Big((\mu_2^*(x,\theta))(e_j)_{M_2}\Big)(e_j)_{M_1}
\end{aligned}
$$

(M_i, w_i) is a quasi-Hamiltonian space with moment map $\mu_i : M_i \to P_i$, so we have by Lemma 2.4

$$
w_i^\sharp\Big(\mu_i^*(x,\theta)\Big) = \frac{1}{2}((1 + Ad_{\mu_i})x)_{M_i}.
$$

We can also see that

$$
\begin{aligned}
\sum_i \Big((\mu_j^*(x,\theta))(e_i)_{M_j}\Big)(e_i)_{M_k} &= \sum_i (x, Ad_{\mu_j^{-1}}e_i - e_i)(e_i)_{M_k} \\
&= \sum_i (Ad_{\mu_j}x - x, e_i)(e_i)_{M_k} \\
&= (Ad_{\mu_j}x - x)_{M_k}
\end{aligned}
$$

So the above becomes

$$\begin{aligned}
w^\sharp(\mu^*(X,\theta)) &= \frac{1}{2}(Ad_{\mu_2}+Ad_{\mu_1\mu_2}X)_{M_1}+\frac{1}{2}(1+Ad_{\mu_2}X)_{M_2}+\frac{1}{2}(Ad_{\mu_1\mu_2}X-Ad_{\mu_2}X)_{M_2}\\
&\quad -\frac{1}{2}(Ad_{\mu_2}X-X)_{M_1}\\
&= \frac{1}{2}((1+Ad_{\mu_1\mu_2})X)_{M_1}+\frac{1}{2}((1+Ad_{\mu_1\mu_2})X)_{M_2}\\
&= \frac{1}{2}((1+Ad_{\mu_1\mu_2})X)_{M}
\end{aligned}$$

□

Remark 2.9. *It is a quick calculation to show the fusion product is associative, that is* $M_1 \circledast (M_2 \circledast M_3) \simeq (M_1 \circledast M_2) \circledast M_3$. *The bivector is given by*

$$w = w_1 + w_2 + w_3 + \frac{1}{2}\sum_i (e_i)_{M_1}\wedge(e_i)_{M_2} + \frac{1}{2}\sum_i (e_i)_{M_1}\wedge(e_i)_{M_3} + \frac{1}{2}\sum_i (e_i)_{M_2}\wedge(e_i)_{M_3}.$$

The quasi-Hamiltonian space we are most interested in for this paper is the fusion product of n conjugacy classes in P. Recall from Example 2.6 that $C_{r_i} \subset P$ is a quasi-Hamiltonian space with action given by conjugation and the associated moment map given by inclusion. The fusion product of n conjugacy classes $C_r^n = C_{r_1} \circledast \cdots \circledast C_{r_n}$, $r = (r_1, ..., r_n) \in \mathbb{R}_+$ is also a quasi-Hamiltonian space with action given by the diagonal conjugation and moment map $\widetilde{\mu} : M \to P$ given by multiplication, $\widetilde{\mu}(g_1, g_2, ..., g_n) = g_1 g_2 \cdots g_n$. The bivector on this space is given by

$$\widetilde{w} = \frac{1}{2}\sum_{i=1}^{n}\sum_k \left(e_k^\lambda \wedge e_k^\rho\right)_i + \frac{1}{2}\sum_{i<j}^{n}\sum_k \left(e_k^\lambda - e_k^\rho\right)_i \wedge \left(e_k^\lambda - e_k^\rho\right)_j$$

where the subscripts i, j denote the vector field on $C_{r_i}, C_{r_j} \subset C_r^n$.

2.4. **Poisson bracket on $C^\infty(P^n)^K$.** For a general quasi-Hamiltonian space (M, w_M), the bracket on $C^\infty(M)$ defined by the bivector w_M is not a Poisson bracket. This is easy to see since the Shouten bracket $[w_M, w_M] = \varphi_M$ is an invariant trivector field. The bracket does however define a Poisson bracket when we restrict to the space $C^\infty(M)^K$ of smooth K-invariant functions on M.

Lemma 2.10. *Let K be a connected quasi-Poisson Lie group acting on a manifold (M, w_M) in a quasi-Poisson manner. Then the bivector w_M defines a Poisson bracket on the space $C^\infty(M)^K$ of the smooth K-invariant functions in M.*

Proof: See [AKS, Theorem 4.2.2] □

For $\psi \in C^\infty(P^n)$ we define

$$D_i\psi : P^n \to \mathfrak{k}_i, \quad D_i'\psi : P^n \to \mathfrak{k}_i$$

as follows. Let $g = (g_1, ..., g_n) \in P^n$ and $x = (x_1, ..., x_n) \in \mathfrak{k}^n$, then

$$d_i\psi_g(x^\rho) = (D_i\psi, x) = \frac{d}{dt}|_{t=0}\psi(g_1, ..., e^{tx_i}g_i, ..., g_n)$$

$$d_i\psi_g(x^\lambda) = (D_i'\psi, x) = \frac{d}{dt}|_{t=0}\psi(g_1, ..., g_i e^{tx_i}, ..., g_n).$$

Here $(,)$ is the Killing form extended to $\mathfrak{k}^n$ by $(x,y)=\sum_{i=1}^n (x_i,y_i)$ for $x,y\in\mathfrak{k}^n$.

Remark 2.11. *It is easy to see that*

$$Ad_{g_i} D'_i\psi(g) = D_i\psi$$

We also define

$$\Psi_j(g) = \sum_{i=1}^{j-1}\Big[D_i\psi(g) - D'_i\psi(g)\Big] + D_j\psi(g)$$

We now define the Poisson bracket on $C^\infty(P^n)^K$.

Proposition 2.12. *Let $\phi,\psi\in C^\infty(P^n)^K$ then*

$$\{\phi,\psi\}(g) = \sum_{j=1}^{n}\Big(D'_j\phi(g) - D_j\phi(g), \Psi_j(g)\Big)$$

Proof:

Let us first note that for $x,y\in\mathfrak{k}$ $\sum_i (x,e_i)(y,e_i) = (x,y)$. Now,

$$\begin{aligned}
\{\varphi,\psi\}(g) &= w(d\varphi,d\psi)\\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_k \Big(e_k^\lambda\wedge e_k^\rho\Big)_i(d\phi,d\psi) + \frac{1}{2}\sum_{i<j}^{n}\sum_k\Big((e_k^\lambda - e_k^\rho)_i\wedge(e_k^\lambda-e_k^\rho)_j\Big)(d\phi,d\psi)\\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_k d_i\phi(e_k^\lambda)d_i\psi(e_k^\rho) - d_i\phi(e_k^\rho)d_i\psi(e_k^\lambda)\\
&\quad+\frac{1}{2}\sum_{i<j}^{n}\sum_k d_i\phi(e_k^\lambda-e_k^\rho)d_j\psi(e_k^\lambda-e_k^\rho) - d_j\phi(e_k^\lambda-e_k^\rho)d_i\psi(e_k^\lambda-e_k^\rho)\\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_k\Big(D'_i\phi,e_k\Big)\Big(D_i\psi,e_k\Big) - \Big(D_i\phi,e_k\Big)\Big(D'_i\psi,e_k\Big)\\
&\quad+\frac{1}{2}\sum_{i<j}\sum_k\Big(D'_i\phi - D_i\phi,e_k\Big)\Big(D'_j\psi-D_j\psi,e_k\Big) - \Big(D'_j\phi-D_j\phi,e_k\Big)\Big(D'_i\psi-D_i\psi,e_k\Big)\\
&= \frac{1}{2}\sum_{i=1}^{n}\Big(D'_i\phi,D_i\psi\Big) - \Big(D_i\phi,D'_i\psi\Big)\\
&\quad+\frac{1}{2}\sum_{i<j}\Big(D'_i\phi-D_i\phi,D'_j\psi-D_j\psi\Big) - \Big(D'_j\phi-D_j\phi,D'_i\psi-D_i\psi\Big)\\
&= \frac{1}{2}\sum_{i=1}^{n}\Big(D'_i\phi,D_i\psi\Big) - \Big(D_i\phi,D'_i\psi\Big)\\
&\quad+\frac{1}{2}\sum_{i<j}\Big(D'_i\phi-D_i\phi,D'_j\psi-D_j\psi\Big) - \sum_{i>j}\Big(D'_i\phi-D_i\phi,D'_j\psi-D_j\psi\Big)
\end{aligned}$$

But since $\psi\in C^\infty(P^n)^K$ is K-invariant, a quick calculation shows

$$\sum_{i=1}^{n}[D_i\psi - D_i'\psi] = 0$$

Using this fact and also that $(D_i'\phi, D_i'\psi) = (D_i\phi, D_i\psi)$ for all i, we can rewrite the above as,

$$\begin{aligned}
\{\phi,\psi\} &= \frac{1}{2}\sum_{i=1}^{n}\Big(D_i',\phi - D_i\phi, D_i\psi + D_i'\psi\Big) \\
&\quad -\frac{1}{2}\sum_{i\geq j}\Big(D_i'\phi - D_i\phi, D_j'\psi - D_j\psi\Big) - \frac{1}{2}\sum_{i>j}\Big(D_i'\phi - D_i\phi, D_j'\psi - D_j\psi\Big) \\
&= \sum_{i=1}^{n}\Big(D_i'\varphi - D_i\varphi, \Psi_i\Big)
\end{aligned}$$

□

From the above Proposition we can also define the Hamiltonian vector field X_ψ associated to $\psi \in C^\infty(P^n)^K$ by $X_\psi = w^\sharp(d\psi)$.

Corollary 2.13. *The Hamiltonian vector field $X_\psi(g) = ((X_1(g), ..., X_n(g))$ associated to the K-invariant function $\psi \in C^\infty(P^n)^K$ is given by*

$$X_j(g) = dL_{g_j}\Psi_j - dR_{g_j}\Psi_j,\ 1 \leq j \leq n.$$

and $g = (g_1, g_2, ..., g_n)$.

Proof: We use the convention $\{\phi,\psi\} = d\phi(X_\psi) = \sum_{j=1}^{n} d_j\varphi((X_j(g))$. Proposition 2.12 gives us

$$\begin{aligned}
d\phi(X_\psi(g)) &= \{\phi,\psi\} \\
&= \sum_{j=1}^{n}\Big(D_j'\phi - D_j\phi, \Psi_j\Big) \\
&= \sum_{j=1}^{n} d_j\phi(dL_{g_j}\Psi_j) - d_j\phi(dR_{g_j}\Psi_j) \\
&= \sum_{j=1}^{n} d_j\phi(dL_{g_j}\Psi_j - dR_{g_j}\Psi_j)
\end{aligned}$$

□

3. The symplectic structure on $M_r(\mathbb{S}^3)$

Throughout the rest of the paper, we let $G = SU(2) \times SU(2)$, $K = SU(2)$, and $P \simeq SU(2)$. In this section, we will define a symplectic structure on M_r obtained from the reduction of the fusion product of conjugacy classes to a symplectic manifold.

Recall, we defined $Pol_n(*)$ to be the open n-gons in $\mathbb{S}^3$ with side-length less than π, so that we can choose an unique geodesic between vertices. The map $\Phi : P^n \to Pol_n(*) \subset (S^3)^n$ defined by

$$\Phi(g) = (*, g_1*, g_1g_2*, ..., g_1g_2\cdots g_n*)$$

is a diffeomorphism.

Proposition 3.1. *The map Φ is a K-equivariant diffeomorphism where K acts on P^n by the dressing action (diagonal conjugation) and on $Pol_n(*)$ by the diagonal action on $(\mathbb{S}^3)^n$.*

Proof: $* \in P$ is an element in P which is fixed by the K-action, that is $Ad_k(*) = *$ for all $k \in K$. For $k \in K$ and $g \in P^n$, $k \cdot p = (Ad_kg_1, ..., Ad_kg_n)$, so

$$\begin{aligned}\Phi(k\cdot g) &= (*, Ad_k(g_1)*, ..., Ad_k(g_1\cdots g_n)*)\\ &= (Ad_k*, Ad_k(g_1*, \cdots, Ad_k(g_1\cdots g_n*))\\ &= k\cdot(*, g_1*, ..., g_1\cdots g_n*).\end{aligned}$$

□

Remark 3.2. *The map Φ induces a diffeomorphism from $\{g \in P^n : g_1\cdots g_n = 1\}$ to $CPol(*)$.*

We have seen that the K-orbits in a quasi-Hamiltonian space are quasi-Hamiltonian spaces. In particular, a conjugacy class $C \subset P$ is a quasi-Hamiltonian space. Let $r \in \mathbb{R}^n$, with $r = (r_1, ..., r_n)$. Let $C_{r_i} \subset P$ denote the conjugacy class in P such that $r_i = d(*, g_i*) = \cos^{-1}\Big(-\frac{1}{2}trace(g_i)\Big) \in \mathbb{R}$ for all $g_i \in C_{r_i}$.

Lemma 3.3. *The map Φ induces a K-equivariant diffeomorphism from $C_{r_1}\times\cdots\times C_{r_n}$ to $\widetilde{N}_r$, the space of open n-gons with fixed side-lengths based at $*$, where $r_i = d(g_1{\cdot}g_i*, g_1{\cdot}g_{i-1}*)$, for all $1 \le i \le n$.*

Proof: Follows from the fact that k fixes side-lengths. □

Corollary 3.4. *Φ induces a diffeomorphism from the space $\{g \in C_r^n : g_1\cdots g_n = 1\}/K$ to M_r the moduli space of closed n-gons in $\mathbb{S}^3$.*

In §2.3 we saw that the fusion product of n conjugacy classes in P, $(C_r^n, \widetilde{\mu}, \widetilde{w})$, is a quasi-Hamiltonian space with the moment map $\widetilde{\mu}$ given by multiplication. So, $\widetilde{\mu}^{-1}(1)/K = \{g \in C_r^n : g_1\cdots g_n = 1\}/K$. We must determine when this restriction and quotient gives rise to symplectic manifold. Lemma 2.7 tells us that $\widetilde{\mu}^{-1}(1)/K$ is a symplectic manifold when

- $\widetilde{w}$ is everywhere nondegenerate on C_r^n
- 1 is a regular value of $\widetilde{\mu}$.

We use the following remark from [AKS, Example 5.5.4] to give the nondegeneracy condition.

Remark 3.5. *Let K be a quasi-Poisson Lie group arising from the standard quasi-triple and (M, μ, w) is a quasi-Hamiltonian space. Then (M, μ, w) is nondegenerate if and only if, for each $m \in M$,*

$$ker(w_m^\sharp) = \{\mu^*(x, \theta) : x \in ker(1 + Ad_{\mu(m)})\}.$$

Here $x \in \mathfrak{k}$.

It follows that the fusion product of conjugacy classes is nondegenerate.

Lemma 3.6. *1 is a regular value of* $\widetilde{\mu}$ *if and only if* $\mathfrak{k}_g = \{x \in \mathfrak{k} : x_{C_r^n} = 0\} = 0$ *for all* $g \in \widetilde{\mu}^{-1}(1)$.

Proof: We refer to Lemma 2.4. Let $x \in \mathfrak{k}$. Then $x \in (Im(d\widetilde{\mu}|_g))^{\perp} \Leftrightarrow (x, \widetilde{\mu}^*\theta) = 0 \Leftrightarrow 0 = \widetilde{w}^{\sharp}((x, \widetilde{\mu}^*\theta)) = ((1 + Ad_{\widetilde{\mu}(g)})x)_{C_r^n} = (2x)_{C_r^n}$. ☐

A polygon is said to be degenerate if it can be contained in a geodesic in $\mathbb{S}^3$. It follows from the above lemma that if there does not exist $g \in \widetilde{\mu}^{-1}(1) \subset C_r^n$ such that $\Phi(g)$ is a degenerate polygon, then 1 is a regular value of $\widetilde{\mu}$.

Theorem 3.7. *The moduli space* M_r *containing no degenerate polygons has a symplectic structure which is the transport structure from the moduli space* $\mu^{-1}(1)/K$.

In §6, we need a formula for the symplectic form on M_r i in §6.

Remark 3.8. *The symplectic form is given by*

$$\widetilde{\omega} = \sum_{i=1}^{n} \omega_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left(Ad_{g_1 \cdots g_{i-1}} \bar{\theta}_i \wedge_b Ad_{g_1 \cdots g_{j-1}} \bar{\theta}_j \right).$$

where ω_i *is the quasi-Hamiltonian 2-form on the conjugacy class* $C_i \subset SU(2)$, *see* [AMM1], *and* $\bar{\theta}_i$ *is the right-invariant Maurer-Cartan form on* $C_i \subset SU(2)$. *We denote by* $\wedge_b$ *the wedge product together with the killing form on* G.

4. Bending Hamiltonians

4.1. **Hamiltonian vector fields.** Recall, $K = SU(2)$ and $C_r^n = C_{r_1} \circledast \cdots \circledast C_{r_n}$, where $C_{r_i} \subset P$ is a conjugacy class in $P \simeq SU(2)$. Let $(x, y) = -\frac{1}{2} Tr(xy)$. In this section we will compute the Hamiltonian vector fields X_{f_j} associated to the functions $f_i \in C^\infty(C_r^n)^K$ given by

$$f_j(g) = tr(g_1 \cdots g_j), \; 1 \leq j \leq n.$$

See §2.4 for the definition of the Poisson bracket on $C^\infty(C_r^n)^K$. We leave it to the reader to verify the following lemma.

Lemma 4.1.

$$\begin{aligned} D_{i+1} f_j(g) &= D'_i f_j(g), \; 1 \leq i \leq j-1 \\ D_1 f_j(g) &= D'_j f_j(g) \end{aligned}$$

for all $1 \leq j \leq n$.

We define $F_j : P \to \mathfrak{k}$ by

$$F_j(g) = \Big((g_1 \cdots g_j) - (g_1 \cdots g_j)^{-1} \Big).$$

We then have the following lemma.

Lemma 4.2. $F_j(g) = D_1 f_j(g)$

Proof: For $g \in C_r^n$ and $X \in \mathfrak{k}$

$$\begin{aligned}(D_1 f_j(g), X) &= \frac{d}{dt}\Big|_{t=0} tr(e^{tX} g_1 g_2 \cdots g_j) \\ &= tr(X g_1 g_2 \cdots g_j) \\ &= tr(g_1 g_2 \cdots g_j X)\end{aligned}$$

but since

$$tr((g_1 g_2 \cdots g_j)^{-1} X) = tr((g_1 \cdots g_j)^* X) = tr(X^* g_1 \cdots g_j) = -tr(g_1 \cdots g_j X)$$

it follows that

$$\begin{aligned}tr(g_1 g_2 \cdots g_j X) &= \frac{1}{2} tr\Big(((g_1 g_2 \cdots g_j) - (g_1 \cdots g_j)^{-1}) X\Big) \\ &= \Big(-((g_1 \cdots g_j) - (g_1 \cdots g_j)^{-1}), X\Big).\end{aligned}$$

Since $-\Big((g_1 \cdots g_j) - (g_1 \cdots g_j)^{-1}\Big) \in \mathfrak{k}$ and $(,)$ is a nondegenerate bilinear form, we have $D_1 f_j(g) = -\Big((g_1 \cdots g_j) - (g_1 \cdots g_j)^{-1}\Big) = -F_j(g)$. □

We have the following formula of the Hamiltonian vector fields X_{f_i}.

Theorem 4.3. *The Hamiltonian vector field X_{f_i} is has an i-th component given by*

$$(X_{f_j}(g))_i = dR_{g_i} F_j(g) - dL_{g_i} F_j(g),\ 1 \le i \le j,$$

$$(X_{f_j}(g))_i = 0,\ j < i \le n$$

Proof: Recall from Corollary 2.13 that for $\psi \in C^\infty(C_r^n)^K, X_\psi(g)$ is given by

$$(X_\psi(g))_i = dL_{g_i} \Psi_i(g) - dR_{g_i} \Psi_i(g)$$

where $\Psi_i(g) = D_1\psi(g) - D_1'\psi(g) + D_2\psi(g) - \cdots - D_{i-1}\psi(g) + D_i\psi(g)$. This together with Lemma 4.1 gives us

$$(X_{f_j}(g))_i = dL_{g_i} D_1 f_j(g) - dR_{g_i} D_1 f_j(g),\ 1 \le i \le j$$

and

$$(X_{f_j}(g))_i = 0,\ j < i \le n.$$

But from Lemma 4.2, $-F_j(g) = D_1 f_j(g)$, completing the proof.

□

4.2. **Commuting flows.** In this section we will show the family of Hamiltonians $\{f_j\}_{j=1}^n$ Poisson commute for $1 \le j \le n$.

Proposition 4.4. $\{f_i, f_j\} \equiv 0$ *for all* i, j.

Proof: Without loss of generality we may assume $i < j$, then by Proposition 2.12

$$\begin{aligned}
\{f_i, f_j\}(g) &= \sum_{k=1}^{j} \Big(D'_k f_i(g) - D_k f_i(g), F_j(g)\Big) \\
&= -\Big(\sum_{k=1}^{j} (D'_k f_i(g) - D_k f_i(g)), F_j(g)\Big) \\
&= \Big(0, F_j(g)\Big) \\
&= 0
\end{aligned}$$

Here we used $\sum_{k=1}^{i}(D_k f_i - D'_k f_i) = 0$. □

4.3. **Hamiltonian flow.** In this section we will calculate the Hamiltonian flow, Φ_j^t, associated to f_j. Recall that the Hamiltonian flow is the solution to the ODE

$$(*)\begin{cases} \frac{dg_i}{dt} = dR_{g_i}F_j(g) - dL_{g_i}F_j(g),\ 1 \leq i \leq j \\ \frac{dg_i}{dt} = 0, j < i \leq n \end{cases}$$

Lemma 4.5. *$F_j(g)$ is invariant along solution curves of (*).*

Proof: To prove the lemma, it suffices to show that $\psi_j(g) = g_1 \cdots g_j$ is invariant along solution curves of (*).

$$\begin{aligned}
\frac{d}{dt}\psi_j(g(t)) &= \frac{d}{dt}(g_1(t)g_2(t)\cdots g_j(t)) \\
&= \frac{dg_1}{dt}(t)g_2(t)\cdots g_j(t) + k_1(t)\frac{dg_2}{dt}(t)\cdots g_j(t) + \cdots + g_1(t)g_2(t)\cdots\frac{dg_j}{dt}(t) \\
&= [F_j(g(t))g_1(t) - g_1(t)F_j(g(t))]g_2(t)\cdots g_j(t) + g_1(t)[F_j(g(t))g_2(t) - g_2(t)F_j(g(t))]\cdots g_j(t) \\
&= g_1(t)g_2(t)\cdots[F_j(g(t))g_j(t) - g_j(t)F_j(g(t))] \\
&= F_j(g(t))g_1(t)\cdots g_j(t) - g_1(t)\cdots g_j(t)F_j(g(t)) \\
&= 0
\end{aligned}$$

□

Lemma 4.6. *The curve* $\exp\big(tF_j(g)\big)$ *is periodic with period* $2\pi/\sqrt{4 - f_j^2}$.

Proof: Left to reader. □

We are now able to find the Hamiltonian flow Φ_j^t.

Theorem 4.7. *The Hamiltonian flow,* Φ_j^t*, associated to the Hamiltonian* f_j *given by* $\Phi_j^t(g) = \big(\widetilde{g}_1(t), ..., \widetilde{g}_n(t)\big)$ *where*

$$\widetilde{g}_i(t) = \begin{cases} Ad\Big(\exp(tF_j(g))\Big)g_i,\ 1 \leq i \leq j \\ g_i, j < i \leq n. \end{cases}$$

The flow is periodic with period $2\pi/\sqrt{4 - f_j^2}$.

The flows $\{\Phi_j^t\}$ do not give rise to a torus action on M_r since they do not have constant period. We now look at the length functions $\ell_j(g) = \cos^{-1}(-\frac{1}{2}f_j(g))$. Then

$$d\ell_j = \frac{1}{\sqrt{4-f_j^2}}df_j$$

and

$$X_{\ell_j} = \frac{1}{\sqrt{4-f_j^2}}X_{f_j}.$$

It is not difficult to see that the family of functions $\{\ell_j\}_{j=2}^{n-1}$ also Poisson commute, but their Hamiltonian flows are not everywhere defined. If we restrict to the space M_r' such $\ell_j \neq 0$ or $\ell_j \neq \pi$ for all diagonals in M_r. The Hamiltonian flows $\{\Psi_j^t\}$ on M_r' associated to $\{\ell_j\}$ are periodic with constant period 2π and constant angular velocity 1. These flows define a Hamiltonian $(n-3)$-torus action on the space M_r'

5. Braid action on M_r

There exists an action of the pure braid group P_n on the manifold M_r which preserves the symplectic structure. In this section, we show that the generators of the pure braid group arise as the time 1 Hamiltonian flows of the family of functions $h_{ij}, 1 \leq i < j \leq n-1$ where $h_{ij} \in C^\infty(M_r)^K$ is defined by,

$$h_{ij}(g) = \frac{1}{2}\Big(\cos^{-1}\big(-\frac{1}{2}tr(g_ig_j)\big)\Big)^2.$$

Let C_{12} denote $C_1 \circledast C_2$, where $C_i \subset P$ is a conjugacy class. Let w_{12} denote the quasi-Poisson bivector on C_{12}. We have the following proposition.

Proposition 5.1. *The diffeomorphism $R : C_1 \circledast C_2 \to C_2 \circledast C_1$ given by $R(g_1, g_2) = (Ad_{g_1}g_2, g_1)$ is a bivector map taking w_{12} to w_{21}.*

Remark 5.2. *The diffeomorphism $R' : C_1 \circledast C_2 \to C_2 \circledast C_1$ given by $R'(g_1, g_2) = (g_2, Ad_{g_2^{-1}}g_1)$ is also a bivector map taking w_{12} to w_{21}.*

Remark 5.3. $R \circ R' = Id_{C_1 \circledast C_2} = R' \circ R$

We now define $R_i : C_1 \circledast \cdots \circledast (C_i \circledast C_{i+1}) \circledast \cdots \circledast C_n \to C_1 \circledast \cdots \circledast (C_{i+1} \circledast C_i) \circledast \cdots \circledast C_n$ to be the map given by

$$R_i(g_1, ..., g_i, g_{i+1}, ...g_n) = (g_1, ..., Ad_{g_i}g_{i+1}, g_i, ..., g_n)$$

that is, R applied to the ith and $(i+1)$th term of M_r. R_i' can be defined in a similar way.

Lemma 5.4. *The full braid group B_n has a faithful representation as a group of automorphism of the closed n-gons in $\mathbb{S}^3$ in which side-lengths are fixed but the order of the sides is not fixed. The generators of B_n are given by R_i, $1 \leq i \leq n-1$.*

We now restrict B_n to P_n to get an action of the pure braid group on C_r^n. This action induces a symplectomorphism on the moduli space M_r.

Corollary 5.5. *Let $A_{ij} = R_{j-1} \circ \cdots \circ R_{i+1} \circ R_i^2 \circ R_{i+1}' \circ \cdots \circ R_{j-1}'$, $1 \leq i < j \leq n$. A_{ij} induces a symplectomorphism from M_r to itself. A_{ij}, $1 \leq i < j \leq n$ are the generators of P_n which has a faithful representation as a group of automorphisms of M_r.*

We will now show that the braid group actions A_{ij} can be realized as the time one Hamiltonian flows of the Hamiltonians h_{ij} given at the start of the section. We begin by studying the Hamiltonian flows associated to the functions $f_{ij} \in C^{\infty}(C_r^n)^K$ given by $f_{ij}(g) = tr(g_i g_j)$. Define $F_{ij} : C_r^n \to \mathfrak{k}$ by $F_{ij}(g) = \big((g_i g_j) - (g_i g_j)^{-1}\big)$.

The Hamiltonian flow associated to f_{ij} is given by $\Phi_{ij}^t(g) = (\widehat{g_1}(t), ..., \widehat{g_n}(t))$ where

$$\widehat{g_k}(t) = \begin{cases} g_k,\ 0 < k < i \text{ and } j < k < n+1 \\ Ad\Big(\exp\big(tF_{ij}(g)\big)\Big)g_k,\ k = i, j \\ Ad\Big(\exp\big(tF_{ij}(g)\big)g_j \exp\big(-tF_{ij}(g)\big)g_j^{-1}\Big)g_k,\ i < k < j. \end{cases}$$

The following formula is used to relate Φ_{ij}^t to A_{ij}.

Lemma 5.6.

$$\exp\left(\frac{\cos^{-1}(-\frac{1}{2}tr(g))}{\sqrt{4 - tr^2(g)}}(g - g^{-1})\right) = g$$

We now notice that for time $t = \frac{\cos^{-1}(-\frac{1}{2}f_{ij}(g))}{\sqrt{4 - f_{ij}^2(g)}}$,

$$\Phi_{ij}^t = A_{ij}.$$

The time for which the Φ_{ij}^t flows depends on the point in M_r at which flow begins. We would like time to be independent on the starting point. We can achieve this by taking the Hamiltonian flows of the functions $h_{ij} = \frac{1}{2}\left(\cos^{-1}(-\frac{1}{2}f_{ij})\right)^2$. The Hamiltonian flow $\widetilde{\Phi}_{ij}^t$ associated to h_{ij} is the renormalization of the flow Φ_{ij}^t so that

$$\widetilde{\Phi}_{ij}^1 = A_{ij}$$

on M_r. We can see the pure braid group as the integer points in the Hamiltonian flows $\widetilde{\Phi}_{ij}^t$, $1 \leq i < j \leq n$.

6. Connection with symplectic forms on relative character varieties of n-punctured 2-spheres

In this section, we relate the symplectic form on $M_r(\mathbb{S}^3)$ given in Remark 3.8 to the symplectic form of Goldman type obtained from the description of $M_r(\mathbb{S}^3)$ as the moduli space of flat connections on an n-punctured 2-sphere. We follow the arguments of Kapovich and Millson [KM1, §5] which considers the analogous question for $M_r(\mathbb{E}^3)$. We begin with the general case in which G is any Lie group with Lie algebra $\mathfrak{g}$ which admits a nondegenerate, G-invariant, symmetric, bilinear form.

6.1. **Relative characteristic varieties and parabolic cohomology.** Let $\Sigma = \mathbb{S}^2 - \{p_1, ..., p_n\}$ denote the n-punctured 2-sphere and $U_1, ..., U_n$ be disjoint disc neighborhoods of $p_1, ..., p_n$, repectively. Further, Γ is the fundamental group of Σ with generators γ_i, $T = \{\Gamma_1, ..., \Gamma_n\}$ is the collection of subgroups of Γ with Γ_i the cyclic subgroup generated by γ_i, and $U = U_1 \cup \cdots \cup U_n$.

Fix $\rho_0 \in \text{Hom}(\Gamma, G)$ a representation. In [KM2] the relative representation variety $\text{Hom}(\Gamma, T; G)$ is defined as the representations $\rho : \Gamma \to G$ such that $\rho|_{\Gamma_i}$ is contained in the closure of the conjugacy class of $\rho_0|_{\Gamma_i}$.

Remark 6.1. *If $G = SU(2)$, there exists a ρ_0 such that the relative character variety $\mathrm{Hom}(\Gamma, T; G)/G$ is isomorphic to $M_r(\mathbb{S}^3)$. We will make this isomorphism explicit later on.*

Let $\rho \in \mathrm{Hom}(\Gamma, T; G)$. Then ρ induces a flat principal G-bundle over Σ. The associated flat Lie algebra bundle will be denoted by $ad\,P$.

We define the parabolic cohomology, $H^1_{par}(\Sigma, ad\ P)$ to be the subspace of the de Rham cohomology classes in $H^1_{DR}(\Sigma, ad\ P)$ whose restrictions to each U_i are trivial.

6.2. **Gauge theoretic description of the symplectic form.** Let b be the nondegenerate, G-invariant, symmetric, bilinear form on $\mathfrak{g}$. A skew symmetric bilinear form

$$B : H^1_{par}(\Sigma, ad\,P) \times H^1_{par}(\Sigma, ad\,P) \to H^2(\Sigma, U; \mathbb{R})$$

is defined by taking the wedge product together with the bilinear form b. Evaluating on the relative fundamental class of Σ gives the skew symmetric form,

$$A : H^1_{par}(\Sigma, ad\,P) \times H^1_{par}(\Sigma, ad\,P) \to \mathbb{R}.$$

Poincare duality give us nondegeneracy of A, so A is a symplectic form on $\mathrm{Hom}(\Gamma, T; G)$. We will show A corresponds to the symplectic form $\widetilde{\omega}$ given in Remark 3.8.

We first pass through the group cohomology description of $H^1_{par}(\Sigma, ad\,P)$ to make this correspondence explicit.

We identify the universal cover of Σ, denoted $\widetilde{\Sigma}$, with the hyperbolic plane, $\mathbb{H}^2$. Let $p : \widetilde{\Sigma} \to \Sigma$ by the covering projection. We define the $\mathcal{A}^\bullet(\widetilde{\Sigma}, p^* Ad\,P)$ with $\mathcal{A}^\bullet(\widetilde{\Sigma}, \mathfrak{g})$ by parallel translation from a point x_0. Given $[\eta] \in H^1(\Sigma, ad\,P)$ choose a representing closed 1-form $\eta \in \mathcal{A}^1(\Sigma, ad\,P)$. Let $\widetilde{\eta} = p^*\eta$. Then there is a unique function $f : \widetilde{\Sigma} \to \mathfrak{g}$ satisfing:

- $f(x_0) = 0$
- $df = \widetilde{\eta}$

A 1-cochain $h(\eta) \in C^1(\Gamma, \mathfrak{g})$ is defined by

$$h(\eta)(\gamma) = f(x) - Ad_\rho(\gamma) f(\gamma^{-1}x).$$

This induces an isomorphism from $H^1(\Sigma, ad\,P)$ to $H^1(\Gamma, \mathfrak{g})$. It can be seen that $[\eta] \in H^1_{par}(\Sigma, ad\,P)$ if and only if $h(\eta)$ restricted to Γ_i is exact for all i. That is, there exists an $x_i \in \mathfrak{g}$ such that $h(\eta)(\gamma_i^k) = x_i - Ad_\rho(\gamma_i^k)x_i$ for each γ_i a generator of Γ.

We construct the fundamental domain $\mathcal{D}$ for Γ operating on $\mathbb{H}^2$ as in [KM1]. Choose x_0 on Σ and make cuts along geodesics from x_0 to the cusps. The resulting fundamental domain $\mathcal{D}$ is a geodesic $2n$-gon with vertices $v_1, ..., v_n$ and cusps $v_1^\infty, ..., v_n^\infty$ ordered so that as we proceed clockwise around $\partial\mathcal{D}$ we see $v_1, v_1^\infty, ..., v_n, v_n^\infty$. The generator γ_i fixes v_i^∞ and satsfies $\gamma_i v_{i+1} = v_i$. Let e_i be the oriented edge joining v_i to v_i^∞ and $\hat{e}_i$ be the oriented edge joining v_i^∞ to v_{i+1}. Then $\gamma_i \hat{e}_i = -e_i$.

Let $\rho \in \mathrm{Hom}(\Gamma, T; G)$ and $c, c' \in T_\rho\left(\mathrm{Hom}(\Gamma, T; G)/G\right) \simeq H^1_{par}(\Gamma, \mathfrak{g})$ be tangent vectors at ρ. The corresponding elements in $H^1_{par}(\Sigma, adP)$ are denoted α and α'. So $f : \Sigma \to \mathfrak{g}$ which satisfies $df = \widetilde{\alpha}$ and $f_i(x_0) = 0$. Let $f(v_i^\infty) = x_i$. Then

$$\begin{aligned} c(\gamma_i) &= f(x) - Ad_{\rho(\gamma_i)} f(\gamma_i^{-1}x) \\ &= f(v_i^\infty) - Ad_{\rho(\gamma_i)} f(\gamma_i^{-1}v_i^\infty) \\ &= f(v_i^\infty) - Ad_{\rho(\gamma_i)} f(v_i^\infty) \\ &= x_i - Ad_{\rho(\gamma_i)} x_i. \end{aligned}$$

There is an equivalent formulas for c', α', and f' with $f'(v_i^\infty) = x_i'$.

Let $B_\bullet(\Gamma)$ be the bar resolution of Γ. Thus $B_k(\Gamma)$ is the free $\mathbb{Z}[\Gamma]$-module on the symbols $[\gamma_1|\gamma_2|\cdots|\gamma_k]$ with

$$\partial[\gamma_1|\gamma_2|\cdots|\gamma_k] = \gamma_1[\gamma_2|\cdots|\gamma_k] + \sum_{i=1}^{k-1}(-1)^i[\gamma_1|\cdots|\gamma_i\gamma_{i+1}|\cdots|\gamma_k] + (-1)^k[\gamma_1|\cdots|\gamma_{k-1}].$$

Let $C_k(\Gamma) = B_k(\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathbb{Z}$ with $\mathbb{Z}[\Gamma]$ acting on $\mathbb{Z}$ by the homomorphism ϵ defined by

$$\epsilon(\sum_{i=1}^{m} a_i\gamma_i) = \sum_{i=1}^{m} a_i.$$

Then $C_k(\gamma)$ is the free abelian group on the symbols $(\gamma_1|\cdots|\gamma_k) = [\gamma_1|\gamma_2|\cdots|\gamma_k] \otimes 1$ with

$$\partial(\gamma_1|\gamma_2|\cdots|\gamma_k) = (\gamma_2|\cdots|\gamma_k) + \sum_{i=1}^{k-1}(-1)^i(\gamma_1|\cdots|\gamma_i\gamma_{i+1}|\cdots|\gamma_k) + (-1)^k(\gamma_1|\cdots|\gamma_{k-1}).$$

A relative fundamental class $F \in C_2(\Gamma)$ is defined by the property

$$\partial F = \sum_{i=1}^{n}(\gamma_i).$$

Let $[\Gamma, \partial\Gamma] = \sum_{i=2}^{n}(\gamma_1\cdots\gamma_{i-1}|\gamma_i) \in C_2(\Gamma)$, then

Lemma 6.2. $[\Gamma, \partial\Gamma]$ *is a relative fundamental class.*

Proof: The proof is left to the reader.

We will now give the symplectic form A in terms of group cohomology. We denote by $\cup_b$ the cup product of Eilenberg-MacLane cochains using the form b on the coefficients.

Proposition 6.3.

$$A(\alpha, \alpha') = \sum_{i=1}^{n}\langle c \cup_b x_i'), (\gamma_i)\rangle - \langle c \cup_b c', [\Gamma, \partial\Gamma]\rangle$$

We will use the next Lemmas to prove Proposition 6.3.

Lemma 6.4.

$$\int_{e_i} B(f, \widetilde{\alpha}') + \int_{\widehat{e}_i} B(f, \widetilde{\alpha}') = b\left(c(\gamma_i), f'(v_i^\infty)\right) - b\left(c(\gamma_i), f'(v_i)\right)$$

Proof: Recall $\gamma_i\widehat{e}_i = -e_i$, so that $\widehat{e}_i = -\gamma_i^{-1}e_i$. We then have

$$\begin{aligned}
\int_{e_i} B(f,\widetilde{\alpha}') + \int_{\widehat{e}_i} B(f,\widetilde{\alpha}') &= \int_{e_i} B(f,\widetilde{\alpha}') + \int_{\widehat{e}_i} B(f,\widetilde{\alpha}') \\
&= \int_{e_i} B(f,\widetilde{\alpha}') + \int_{\gamma_i^{-1}e_i} B(f,\widetilde{\alpha}') \\
&= \int_{e_i} B\left(f,\widetilde{\alpha}'\right) + \int_{e_i} (\gamma_i^{-1})^* B(f,\widetilde{\alpha}') \\
&= \int_{e_i} B\left(f,\widetilde{\alpha}'\right) + \int_{e_i} B\left((\gamma_i^{-1})^* f, (\gamma_i^{-1})^*\widetilde{\alpha}'\right) \\
&= \int_{e_i} B\left(f,\widetilde{\alpha}'\right) + \int_{e_i} B\left(Ad_{\rho(\gamma_i)}(\gamma_i^{-1})^* f, Ad_{\rho(\gamma_i)}(\gamma_i^{-1})^*\alpha'\right) \\
&= \int_{e_i} B\left(f - Ad_{\rho(\gamma_i)}(\gamma_i^{-1})^* f, \widetilde{\alpha}'\right) \\
&= \int_{e_i} B\left(c(\gamma_i), \widetilde{\alpha}'\right) \\
&= b\left(c(\gamma_i), f'(v_i^{\infty})\right) - b\left(c(\gamma_i), f'(v_i)\right)
\end{aligned}$$

□

Lemma 6.5.

$$\sum_{i=1}^{n} b\left(c(\gamma_i), f'(v_i)\right) = \sum_{i=1}^{n} b\left(c(\gamma_i), f'(v_i^{\infty})\right) - \sum_{i=1}^{n} \langle c \cup_b y_i, (\gamma_i)\rangle + \langle c \cup_b c', [\Gamma, \partial\Gamma]\rangle$$

Proof: By definition, for any $x \in \mathbb{H}^2$ and $\gamma \in \Gamma$ we have

$$c'(\gamma) = f'(x) - Ad_{\rho(\gamma)} f'(\gamma^{-1}x)$$

Let $\gamma = \gamma_i$ and $x = v_i$, then

$$c'(\gamma_i) = f'(v_i) - Ad_{\rho(\gamma_i)} f'(v_{i+1})$$

Using $f'(v_1) = 0$, we obtain

$$\begin{aligned}
c'(\gamma_1 \cdots \gamma_i) &= f'(v_1) - Ad_{\rho(\gamma_1\cdots\gamma_i)} f'(\gamma_i^{-1} \cdots \gamma_1^{-1} v_1) \\
&= -Ad_{\rho(\gamma_1\cdots\gamma_i)} f'(v_{i+1}).
\end{aligned}$$

We will also need

$$\begin{aligned}
c'(\gamma_1 \cdots \gamma_i) &= c'(\gamma_1 \cdots \gamma_{i-1}) + Ad_{\rho(\gamma_1\cdots\gamma_{i-1})} c'(\gamma_i) \\
&= c'(\gamma_1) + Ad_{\rho(\gamma_1)} c'(\gamma_2) + \cdots + Ad_{\rho(\gamma_1\cdots\gamma_{i-1})} c'(\gamma_i)
\end{aligned}$$

and, since $\gamma_1 \cdots \gamma_n = 1$,

$$0 = c'(\gamma_1 \cdots \gamma_n) = c'(\gamma_1) + Ad_{\rho(\gamma_1)} c'(\gamma_2) + \cdots + Ad_{\rho(\gamma_1\cdots\gamma_{n-1})} c'(\gamma_n)$$

We then have,

$$\begin{aligned}
\sum_{i=1}^{n} b\left(c(\gamma_i), f'(v_i)\right) &= -\sum_{i=1}^{n} b\left(c(\gamma_i), Ad_{\rho(\gamma_1\cdots\gamma_i)^{-1}} c'(\gamma_1\cdots\gamma_{i-1})\right) \\
&= -\sum_{i=1}^{n} b\left(Ad_{\rho(\gamma_1\cdots\gamma_{i-1})} c(\gamma_i), c'(\gamma_1) + Ad_{\rho(\gamma_1)} c'(\gamma_2) + \cdots + Ad_{\rho(\gamma_1\cdots\gamma_{i-2})} c'(\gamma_{i-1})\right) \\
&= -\sum_{i=1}^{n}\sum_{j=1}^{i-1} b\left(Ad_{\rho(\gamma_1\cdots\gamma_{i-1})} c(\gamma_i), Ad_{\rho(\gamma_1\cdots\gamma_{j-1})} c'(\gamma_j)\right) \\
&= -\sum_{j=1}^{n}\sum_{i=j+1}^{n} b\left(Ad_{\rho(\gamma_1\cdots\gamma_{i-1})} c(\gamma_i), Ad_{\rho(\gamma_1\cdots\gamma_{j-1})} c'(\gamma_j)\right) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{j} b\left(Ad_{\rho(\gamma_1\cdots\gamma_{i-1})} c(\gamma_i), Ad_{\rho(\gamma_1\cdots\gamma_{j-1})} c'(\gamma_j)\right) \\
&= \sum_{j=1}^{n} b\left(c(\gamma_1\cdots\gamma_j), Ad_{\rho(\gamma_1\cdots\gamma_{j-1})} c'(\gamma_j)\right) \\
&= \sum_{j=1}^{n} b\left(c(\gamma_1\cdots\gamma_{j-1}) + Ad_{\rho(\gamma_1\cdots\gamma_{j-1})} c(\gamma_j), Ad_{\rho(\gamma_1\cdots\gamma_{j-1})} c'(\gamma_j)\right) \\
&= \sum_{j=1}^{n} b\left(c(\gamma_1\cdots\gamma_{j-1}), Ad_{\rho(\gamma_1\cdots\gamma_{j-1})} c'(\gamma_j)\right) + \sum_{j=1}^{n} b\left(c(\gamma_j), c'(\gamma_j)\right) \\
&= \langle c \cup_b c'), [\Gamma, \partial\Gamma]\rangle + \sum_{j=1}^{n} b\left(c(\gamma_j), f'(v_j^{\infty}) - Ad_{\rho(\gamma_j)} f'(v_j^{\infty})\right) \\
&= \langle c \cup_b c', [\Gamma, \partial\Gamma]\rangle + \sum_{j=1}^{n} b\left(c(\gamma_j), f'(v_j^{\infty})\right) - \sum_{j=1}^{n} \langle B(c, y_j'), (\gamma_j)\rangle
\end{aligned}$$

□

Proof of Proposition 6.3:

$$\begin{aligned}
A(\alpha, \alpha') &= \int_{\Sigma} B(\alpha, \alpha') \\
&= \int_{\mathcal{D}} B(\widetilde{\alpha}, \widetilde{\alpha}') \\
&= \int_{\partial\mathcal{D}} B(\widetilde{\alpha}, f') \\
&= \sum_{i=1}^{n} \left(\int_{e_i} B(\widetilde{\alpha}, f') + \int_{\widehat{e}_i} B(\widetilde{\alpha}, f')\right) \\
&= \sum_{j=1}^{n} \langle c \cup_b x_j'), (\gamma_j)\rangle - \langle c \cup_b c', [\Gamma, \partial\Gamma]\rangle
\end{aligned}$$

□

6.3. **Correspondence between $M_r(\mathbb{S}^3)$ and $\mathrm{Hom}\left(\Gamma, T; SU(2)\right)/SU(2)$.** We now restrict to the case $G = SU(2)$. We define the isomorphism

$$\Upsilon : \mathrm{Hom}\left(\Gamma, T; SU(2)\right) \to \widetilde{M_r},$$

where $\widetilde{M_r}$ is the closed polygonal linkages in $\mathbb{S}^3$ based at a point, by

$$\Upsilon(\rho) = \left(\rho(\gamma_1), ..., \rho(\gamma_n)\right).$$

This induces an isomorphism, which we also denote by Υ,

$$\Upsilon : \mathrm{Hom}(\Gamma, T; SU(2))/SU(2) \to M_r.$$

The differential $d\Upsilon_\rho : T_\rho(\mathrm{Hom}(\Gamma, T; SU(2))/SU(2)) \to T_{\Upsilon(\rho)}M_r$ is then defined by

$$d\Upsilon_\rho(c) = \left(dR_{\rho(\gamma_1)}c(\gamma_1), ..., dR_{\rho(\gamma_n)}c(\gamma_n)\right).$$

Here $T_\rho(\mathrm{Hom}(\Gamma, T; SU(2))/SU(2))$ is identified with an element of $\mathbb{Z}^1_{par}(\Gamma, \mathfrak{g})$. We have

$$d\Upsilon_\rho(c) = \left(dR_{g_1}x_1 - dL_{g_1}x_1, ..., dR_{g_n}x_n - dL_{g_n}x_n\right)$$

and

$$d\Upsilon_\rho(c') = \left(dR_{g_1}x'_1 - dL_{g_1}x'_1, ..., dR_{g_n}x'_n - dL_{g_n}x'_n\right).$$

Recall, the symplectic form on M_r is given by

$$\widetilde{\omega} = \sum_{i=1}^{n} \omega_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(Ad_{g_1\cdots g_{i-1}}\bar{\theta}_i \wedge_b Ad_{g_1\cdots g_{j-1}}\bar{\theta}_j\right).$$

We can now prove the main result of this section

Theorem 6.6. $\Upsilon^*\widetilde{\omega} = A$

Proof:

First we note that

$$\Upsilon^*\bar{\theta}_i(c) = c(\gamma_i)$$

and

$$\begin{aligned}
(\Upsilon^*\omega_i)(c, c') &= \omega_i\left(dR_{g_i}c(\gamma_i), dR_{g_i}c'(\gamma_i)\right) \\
&= -\frac{1}{2}\left(Ad_{g_i^{-1}}c(\gamma_i) + c(\gamma_i), x'_i\right) \\
&= -\frac{1}{2}\left(c(\gamma_i), Ad_{g_i}x'_i + x'_i\right) \\
&= -\frac{1}{2}\left(c(\gamma_i), c'(\gamma_i)\right) - \left(c(\gamma_i), Ad_{g_i}x'_i\right) \\
&= -\frac{1}{2}\left(Ad_{g_1\cdots g_{i-1}}c(\gamma_i), Ad_{g_1\cdots g_{i-1}}c'(\gamma_i)\right) + \langle c \cup_b x'_i), (\gamma_i)\rangle
\end{aligned}$$

It follows that

$$
\begin{aligned}
(\Upsilon^*\widetilde{\omega})(c,c') &= \sum_{i=1}^{n}(\Upsilon^*\omega_i)(c,c') + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\Upsilon^*\left(Ad_{g_1\cdots g_{i-1}}\bar{\theta}_i \wedge_b Ad_{g_1\cdots g_{j-1}}\bar{\theta}_j\right)(c,c') \\
&= \sum_{i=1}^{n}\langle c\cup_b x_i'), (\gamma_i)\rangle - \sum_{i=1}^{n}\frac{1}{2}\left(Ad_{g_1\cdots g_{i-1}}c(\gamma_i), Ad_{g_1\cdots g_{i-1}}c'(\gamma_i)\right) \\
&\quad + \sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(Ad_{g_1\cdots g_{i-1}}c(\gamma_i), Ad_{g_1\cdots g_{j-1}}c'(\gamma_j)\right) \\
&\quad - \sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(Ad_{g_1\cdots g_{i-1}}c'(\gamma_i), Ad_{g_1\cdots g_{j-1}}c(\gamma_j)\right) \\
&= \sum_{i=1}^{n}\langle c\cup_b x_i', (\gamma_i)\rangle - \sum_{i=1}^{n}\frac{1}{2}\left(Ad_{g_1\cdots g_{i-1}}c(\gamma_i), Ad_{g_1\cdots g_{i-1}}c'(\gamma_i)\right) \\
&\quad + \sum_{j=2}^{n}\sum_{i=1}^{j-1}\left(Ad_{g_1\cdots g_{i-1}}c(\gamma_i), Ad_{g_1\cdots g_{j-1}}c'(\gamma_j)\right) \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{i}\left(Ad_{g_1\cdots g_{i-1}}c'(\gamma_i), Ad_{g_1\cdots g_{j-1}}c(\gamma_j)\right) \\
&= \sum_{i=1}^{n}\langle c\cup_b x_i', (\gamma_i)\rangle + \sum_{j=2}^{n}\sum_{i=1}^{j-1}\left(Ad_{g_1\cdots g_{i-1}}c(\gamma_i), Ad_{g_1\cdots g_{j-1}}c'(\gamma_j)\right) \\
&= \sum_{i=1}^{n}\langle c\cup_b x_i'), (\gamma_i)\rangle + \sum_{j=2}^{n}\left(Ad_{g_1\cdots g_{i-1}}c'(\gamma_i), c(\gamma_1\cdots\gamma_{i-1})\right) \\
&= \sum_{i=1}^{n}\langle c\cup_b x_i', (\gamma_i)\rangle - \langle c\cup_b c', [\Gamma, \partial\Gamma]\rangle \\
&= A(\alpha,\alpha')
\end{aligned}
$$

□

It is easily seen that the functions ℓ_i from §4.2 corresponds to the following Goldman functions. Let $\varphi : G \to \mathbb{R}$ be defined by $\varphi(g) = \cos^{-1}\left(-\frac{1}{2}trace(g)\right)$. We then defined the function $\varphi_\gamma : \mathrm{Hom}\left(\Gamma, T; SU(2)\right)/SU(2) \to \mathbb{R}$ by $\varphi_g a(\rho) = \varphi\left(\rho(-ga)\right)$. We see that

$$\Upsilon^*\ell_i = \varphi_{\gamma_1\cdots\gamma_i}$$

Then choosing an maximal collection of nonintersecting diagonal on M_r corresponds to a pair of pants decomposition on Σ.

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