Math 131 – Fall 2015 – Boyle –Exam 2

- NO CALCULATORS OR ELECTRONIC DEVICES ALLOWED.
- Use a separate answer sheet for each question; use the back side of an answer sheet if you need more space to answer a question.
- Give your pledge on page 1 only, covering the whole test.
- Draw a box around a final answer to a problem.

1. (10 points) For the initial value problem $dy/dx = -3xy + 2$, $y(0) = 1$ use Euler's method with step size 0.1 to estimate $y(0.2)$.

2. (13 points)
(a) (10 pts) Solve the initial value problem $dy/dt = y^2$, $y(1) = 1$.
(b) (3 pts) What is the largest b (either a number or ∞) such that the solution is valid on the interval $[1, b)$?

3.(12 points) Answer True or False. No comment required.
(a) If A and B are 2×3 matrices, then $A + B$ is well defined.
(b) If A and B are 2×3 matrices, then AB is well defined.
(c) If A and B are 2×2 matrices, then $AB = BA$.
(d) If A, B, C are matrices such that $AB = AC$, then $B = C$.

4. (12 points) Suppose the following matrix is the augmented matrix of a system of linear equations in m equations in n variables:

$$A = \begin{pmatrix} 0 & -2 & 3 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

(a) (4 pts) What is m? What is n?
(b) (3 pts) How many solutions does the system have?
(c) (2 pts) How many free variables are there?
(d) (3 pts) Give an example of an augmented matrix for a system of linear equations with no solution.

5. (12 points) Suppose $y = (y_1, y_2), x = (x_1, x_2)$ and $y = f(x)$ is defined by

$$y_1 = (x_1)^2 x_2 \quad \text{and} \quad y_2 = 3x_1 x_2 .$$

(a) (7 pts) Compute the matrix which is the derivative of y with respect to x at the input $(x_1, x_2) = (1, -1)$.
(b) (5 pts) Use the derivative to approximate $f(1.1, -0.8) - f(1, -1)$.

6. (13 points) At time $t = 0$, a tank holds 100 gallons of water that contain 20 pounds of salt. A salt solution (2 pounds of salt per gallon) flows into the tank at the rate of 8 gallons per hour, and the solution in the tank flows out at the same rate. The amount x of salt (in pounds) at time t (in hours) is assumed to satisfy a differential equation of the form $dx/dt = kx + b$, where k and b are constants.
(a) (4 pts) What are k and b?
(b) (7 pts) Find a formula for x as a function of t.
(c) (2 pts) As $t \to \infty$, what value does $x(t)$ approach?

7. (13 points) The system of differential equations

$$\frac{dx_1}{dt} = 3x_1 - 2x_1x_2 \qquad \text{and} \qquad \frac{dx_2}{dt} = -4x_2 + 5x_1x_2$$

is chosen such that $x_1(t)$ and $x_2(t)$ model the sizes of two populations as a function of time.
(a) (2 pts) What is the equilibrium point of the system at which $x_1 \neq 0$ and $x_2 \neq 0$?
(b) (8 pts) Suppose at $t = 0$ that $x_1 = 1 = x_2$. Find an equation (expressed in terms of x_1 and x_2, without using derivatives or t) satisfied by the solution $(x_1(t), x_2(t))$ for all t.
(c) (3 pts) For the initial condition $x_1(0) = 0.1$ and $x_2(0) = 0.1$, what is the long term behavior of $x(t)$ as t increases?

8. (19 points) For the system of differential equations

$$\begin{aligned} x_1 + 4x_2 &= \frac{dx_1}{dt} \\ 3x_1 + 2x_2 &= \frac{dx_2}{dt} \end{aligned}$$

do the following.
(a) (2 pts) Presented as a matrix equation, this system takes the form $Mx = dx/dt$, where $x = \binom{x_1}{x_2}$ and $dx/dt = \begin{pmatrix} dx_1/dt \\ dx_2/dt \end{pmatrix}$. What is the matrix M?
(b) (8 pts) Find a matrix P and a diagonal matrix D such that $P^{-1}MP = D$.
(c) (3 pts) Find the general solution $x(t)$ to the given linear system. (It should include two undetermined constants, C_1 and C_2.)
(d) (3 pts) Compute P^{-1}.
(e) (3 pts) Assuming the initial condition $x_1 = 1$ and $x_2 = 2$ at $t = 0$, compute the constants C_1 and C_2 in your general solution.