M131 - EXAM 2 ANSWERS - Fall 2015

1. $y(0) = 1$

$y(.1) \approx y(0) + [y'(0)](.1) = 1 + [-3(0)(1)+2](.1) = 1.2$

$y(.2) \approx (1.2) + [y'(.1)](.1) \approx (1.2) + [-3(.1)(1.2)+2](.1)$

$= (1.2) + [-.36+2](.1) = (1.2) + [1.64](.1)$

$= (1.2) + .164 = \boxed{1.364}$

2(a) $\frac{dy}{dt} = y^2$

$\frac{dy}{y^2} = dt$

$-\frac{1}{y} = t + C$

at $t=1$, $y=1$; so, $-\frac{1}{1} = 1 + C$

$C = -2$

$-\frac{1}{y} = t - 2$

$\boxed{y = \frac{1}{2-t}}$

(b) $\boxed{b = 2}$; solution is valid on $[1, 2)$, with vert. asymptote at $t=2$

3 (a) TRUE (b) FALSE (c) FALSE (d) FALSE

4 (a) $m = 3$ $n = 5$

(b) infinitely many solutions

(c) three free variables (x_1, x_3, x_5)

(d) $[0 \; 1]$ (corresponds to system $0x_1 = 1$)

5 (a) $\left.\begin{pmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 \end{pmatrix}\right|_{(1,-1)} = \left.\begin{pmatrix} 2x_1x_2 & (x_1)^2 \\ 3x_2 & 3x_1 \end{pmatrix}\right|_{(1,-1)} = \boxed{\begin{pmatrix} -2 & 1 \\ -3 & 3 \end{pmatrix}}$

(b) $\Delta y \approx \begin{pmatrix} -2 & 1 \\ -3 & +3 \end{pmatrix}\begin{pmatrix} .1 \\ .2 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ .3 \end{pmatrix}}$

6 (a) $k = -\frac{8}{100} = -.08$

$b = 16$

(b) $\frac{dx}{dt} = -.08x + 16 = -.08(x-200)$

$w = x - 200$

$\frac{dw}{dt} = -.08w$

$w = w_0 e^{-.08t}$

$x - 200 = (x_0 - 200)e^{-.08t}$

$x = 200 + (20-200)e^{-.08t}$

$$\boxed{x(t) = 200 - 180e^{-.08t}}$$

(c) As $t \to \infty$, $x(t) \longrightarrow \boxed{200}$

7 (a) $\frac{dx_1}{dt} = x_1(3 - 2x_2)$

$\frac{dx_2}{dt} = x_2(-4 + 5x_1)$ $\to$ eq. pt is $\boxed{\left(\frac{4}{5}, \frac{3}{2}\right)}$

(b) $\frac{dx_1}{dx_2} = \frac{x_1(3-2x_2)}{x_2(-4+5x_1)}$

$$\int \left(\frac{-4+5x_1}{x_1}\right) dx_1 = \int \left(\frac{3-2x_2}{x_2}\right) dx_2$$

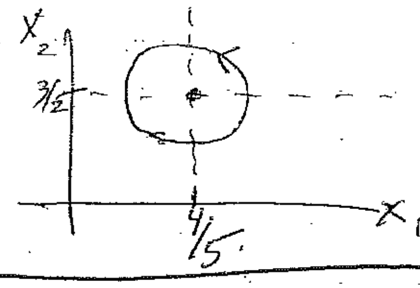
$5x_1 - 4\ln x_1 = -2x_2 + 3\ln x_2 + C$

Solve for C: $5(1) - 4\ln 1 = -2(1) + 3\ln 1 + C$

$5 = -2 + C \Rightarrow C = 7$

$$\boxed{5x_1 - 4\ln x_1 = -2x_2 + 3\ln x_2 + 7}$$

7c) $(x_1(t), x_2(t))$ moves periodically around a closed curve around the eq. point =



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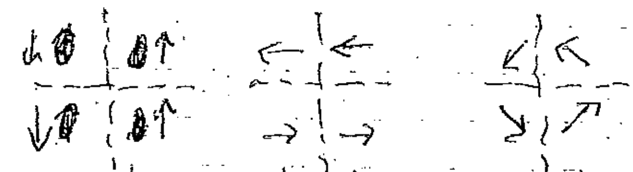
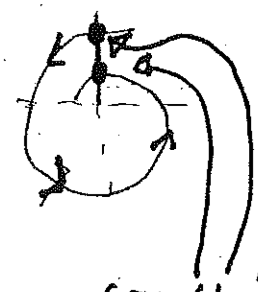
explanation:



(signs of x_1' and x_2' show direction of vertical and horizontal motion)

Why closed?
The pattern of motion shows that $x(t)$ will repeatedly visit the half line $x_1 = \frac{4}{5}$, $x_2 > \frac{3}{2}$.
The function $g(x) = -2x_2 + 3\ln x_2$ is a strictly decreasing function of x_2 on the interval $\frac{3}{2} < x_2 < \infty$.
Therefore there is only one value of x_2 such that the curve $x(t)$ can return to this half line at $(\frac{4}{5}, x_2)$ and satisfy the equation:



(can't hit at different points)

$$5x_1 - 4\ln x_1 = -2x_2 + \ln x_2 + 7$$

$$5\left(4/5\right) - 4\ln\left(\frac{4}{5}\right) = -2x_2 + \ln x_2 + 7$$

8 (a) $x' = Mx$, $M = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$

(b) $\det|tI - M| = (t-1)(t-2) - (3)(4) = t^2 - 3t - 10$
$= (t-5)(t+2)$

$$\boxed{D = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}}$$

columns of P are eigenvectors.

$\underline{\lambda = 5:}$ $\begin{pmatrix} 1-5 & 4 \\ 3 & 2-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{\lambda = -2}$ $\begin{pmatrix} 1+2 & 4 \\ 3 & 2+2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

$$\boxed{P = \begin{pmatrix} 1 & 4 \\ 1 & -3 \end{pmatrix}}$$

(c) $x = Py$ $\begin{pmatrix} dy_1/dt \\ dy_2/dt \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{5t} \\ c_2 e^{-2t} \end{pmatrix}$

$x = \begin{pmatrix} 1 & 4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} c_1 e^{5t} \\ c_2 e^{-2t} \end{pmatrix} = \boxed{\begin{pmatrix} c_1 e^{5t} + 4c_2 e^{-2t} \\ c_1 e^{5t} - 3c_2 e^{-2t} \end{pmatrix}}$

(d) $P^{-1} = -\frac{1}{7}\begin{pmatrix} -3 & -4 \\ -1 & 1 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix}$

(e) At $t=0$, $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $y = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$; so, $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = P^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = +\frac{1}{7}\begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 11/7 \\ -1/7 \end{pmatrix}}$