THE NORMAL DISTRIBUTIONS $\mathcal{N}(\mu, \sigma)$

Let μ and σ be real numbers, with $\sigma > 0$, and suppose X is a random variable with mean μ and standard deviation σ. To say that X has the $\mathcal{N}(\mu, \sigma)$ distribution is the same thing as saying that its standardized version

$$Z = \frac{X - \mu}{\sigma}$$

has the standard normal distribution $\mathcal{N}(0, 1)$, that is, Z has the distribution given by the probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \ .$$

Note, we have $X = \mu + \sigma Z$, and for any numbers a, b with $a \leq b$, each of the following conditions is equivalent:

$$\begin{aligned}
a &\leq Z &&\leq b \ , \\
a\sigma &\leq \sigma Z &&\leq b\sigma \ , \\
\mu + a\sigma &\leq \mu + \sigma Z &&\leq \mu + b\sigma \ , \\
\mu + a\sigma &\leq X &&\leq \mu + b\sigma \ , \\
a\sigma &\leq X - \mu &&\leq b\sigma \ .
\end{aligned}$$

Because these conditions define the same event, they have the same probability, in particular

$$\text{Prob}(a \leq Z \leq b) = \text{Prob}(a\sigma \leq X - \mu \leq b\sigma) = \text{Prob}(\mu + a\sigma \leq X \leq \mu + b\sigma)$$
$$*** \quad \text{Prob}(a \leq Z \leq b) \ = \ \text{Prob}(\mu + a\sigma \leq X \leq \mu + b\sigma) \ *** \, .$$

For example, if $\mu = 7$ and $\sigma = 3$, and we take $a = -2, b = 2$, then

$$\begin{aligned}
\text{Prob}(-2 \leq Z \leq 2) &= \text{Prob}(-2(3) \leq X - 7 \leq 2(3)) \\
&= \text{Prob}(7 - 2(3) \leq X \leq 7 + 2(3)) = \text{Prob}(1 \leq X \leq 13) \ .
\end{aligned}$$

SO: the distribution $\mathcal{N}(\mu, \sigma)$ looks just like $\mathcal{N}(0, 1)$, except it is recentered at μ and rescaled by σ.

If the standardized version Z of X is approximately $\mathcal{N}(0, 1)$, then "=" in the last four equations becomes "approximately equals".

It's important to understand that for the normal distribution, the area under the curve (the graph of the p.d.f.) outside $[-, t, t]$ falls off vary rapidly as t gets large.

Let Φ denote the cumulative distribution function for the standard normal distribution. I.e., if $Z \sim \mathcal{N}(0,1)$, then $\text{Prob}(Z \leq t) = \phi(t)$. Then (using = to mean = to the indicated decimal places (where the rightmost digit might be off by at most 1): when Z is standard normal, here is a table:

k	$\text{Prob}(\lvert Z\rvert > k)$
1	.32
2	.046
3	.002 7
4	.000 063
5	.000 000 57
6	.000 000 002 0
7	.000 000 000 002 6
8	.000 000 000 000 001 2
9	.000 000 000 000 000 000 23
10	.000 000 000 000 000 000 000 015 $= (1.5)10^{-23}$

The distribution of ANY normal random variable about its mean, measured in units of standard devitations, gives the exact same numbers.

For example, if X is a normally distributed random variable with mean μ and standard deviation σ, then
$\text{Prob}(|X - \mu| > 2\sigma) = .046$ and $\text{Prob}(|X - \mu| > 4\sigma) = .000\,063$.