The topological full group of a Cantor minimal system is dense in the full group

Sergey Bezuglyi*

Institute for Low Temperature Physics and Engineering

47, Lenin ave., 310164 Kharkov, Ukraine

Jan Kwiatkowski†

Nicholas Copernicus University

Chopina 12/18, 87-100 Torun, Poland

Abstract

To every homeomorphism T of a Cantor set X one can associate the full group $[T]$ formed by all homeomorphisms γ such that $\gamma(x) = T^{n(x)}(x)$, $x \in X$. The topological full group $[[T]]$ consists of all homeomorphisms whose associated orbit cocycle $n(x)$ is continuous. The uniform and weak topologies, τ_u and τ_w, as well as their intersection τ_{uw} are studied on Homeo(X). It is proved that $[[T]]$ is dense in $[T]$ with respect to τ_u. A Cantor minimal system (X, T) is called saturated if any two clopen sets of "the same measure" are $[[T]]$-equivalent. We describe the class of saturated Cantor minimal systems. In particular, (X, T) is saturated if and only if the closure of $[[T]]$ in τ_{uw} is $[T]$ and if and only if every infinitesimal function is a T-coboundary. These results are based on a description of homeomorphisms from $[[T]]$ related to a given sequence of Kakutani-Rokhlin partitions. It is shown that the offered method works for some symbolic Cantor minimal systems. The tool of Kakutani-Rokhlin partitions is used to characterize $[[T]]$-equivalent clopen sets and the subgroup $[[T]]_x \subset [[T]]$ formed by homeomorphisms preserving the forward orbit of x.

1 Introduction and preliminaries

1.1. Introduction. A Cantor minimal (C. m.) system (X, T) consists of a Cantor set X and a minimal homeomorphism T of X. Lately, such systems have been studied, first of all, from the point of view of orbit equivalence theory (see, e.g., [GPS1, GW, HPS, O]). It turns out that they can be classified up to orbit equivalence in terms of dimension groups and C^*-algebras. A similar problem had been earlier solved for measurable dynamical systems. In this connection, it seems natural to introduce and study some well known concepts of

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measurable dynamics in context of Cantor minimal systems. T. Giordano, I. Putnam, and C. Skau showed in [GPS2] that the notions of the full group $[T]$ and topological full group $[[T]]$ play the very important role as they do in measurable dynamics. In particular, they proved that two minimal homeomorphisms of Cantor sets are orbit equivalent if and only if their full groups are isomorphic.

In this paper, we are mainly interested in the following problems: (1) find a description and structure of a homeomorphism from $[[T]]$; (2) define a natural topology on Homeo(X) and investigate the topological properties of $[[T]]$ and $[T]$.

It is known that every C. m. system can be represented as a Bratteli-Vershik system [HPS], i.e. (in other terms) as a nested sequence of Kakutani-Rokhlin partitions. If ξ is a Kakutani-Rokhlin partition, then T generates two other partitions α and α' defined on the finite set of ξ-towers. We associate to ξ a finite set Γ of homeomorphisms from the topological full group preserving, in some sense, α and α'. For this, we determine by ξ a finite family of partitions $\mathcal{E}$ (actually, they define some homeomorphisms from $[[T]]$) such that $\xi \succ \mathcal{E}$ and every $\mathcal{E}$ satisfies the so called level condition (see Definition 2.1). The problem is whether every homeomorphism from $[[T]]$ eventually gets into some Γ when we take a refining sequence of Kakutani-Rokhlin partitions. We prove such a statement for a wide class of sequences of Kakutani-Rokhlin partitions (Theorem 2.2). In fact, every C. m. system (X,T) is topologically conjugate to a system that satisfies the conditions of Theorem 2.2. To see this, it suffices to take a Bratteli-Vershik minimal system (Z,ϕ) conjugated to (X,T) [GPS1]. The proposed method allows us to find the structure of homeomorphisms from Γ in simple terms related to a Kakutani-Rokhlin partition.

In the next section, we describe natural sequences of Kakutani-Rokhlin partitions satisfying the conditions of Theorem 2.2 for odometers, Toeplitz-Morse, Chacon, and Grillenberger flows. For these C. m. systems, the found description looks very simple. In particular, the homeomorphisms from the topological full group of an odometer are mainly defined by finite permutations. It was also proved in [GPS1] that the orbit equivalence class of a Cantor strictly ergodic dynamical system (X,T) is completely determined by $\Lambda(X,T)$, the set of values of a unique T-invariant measure on clopen subsets. We find this set for all examples mentioned above.

The topological full group $[[T]]$ contains a very interesting subgroup $[[T]]_x$ that is formed by homeomorphism preserving the forward orbit of x. It was proved that $[[T]]_x$ does not depend on x and two such subgroups of different C. m. systems are isomorphic if and only if the two systems are strongly orbit equivalent [GPS2]. As a consequence of presented results, we prove that $[[T]]_x$ can be obtained as an increasing sequence of finite groups found by a sequence of Kakutani-Rokhlin partitions.

Because the topological full group is a countable subgroup in the full group, it would be interesting to find out whether homeomorphisms from $[T]$ can be approximated (in some sense) by elements from $[[T]]$. To do this, we introduce the uniform topology, τ_u, on Homeo(X). This topology has the well known analog in measurable dynamics: the (uniform) distance between two automorphisms is the measure of the set where they are different. If one considers a C. m. system (X,T), then the full group $[T]$ is closed but not complete with respect to τ_u in contrast to the weak topology τ_w studied in [GPS2, GW]. One of the main results of Section 4 is Theorem 4.5: the full group of a Cantor minimal system is the closure of the topological full group in the uniform topology. To prove this result, we again use the description of homeomorphisms from $[[T]]$ found in Section 2. If we consider the topology τ_{uw} on Homeo(X) which is the intersection of τ_u and τ_w, then $[T]$ becomes complete and closed in τ_{uw}. But, in general, the density property of $[[T]]$ in $[T]$ can be lost. Nevertheless there exists a class of C. m. systems, called saturated ones, such that this property holds. A C. m. system is called saturated if for any two clopen subsets A and B with $\mu(A) = \mu(B)$ for every T-invariant measure μ there is γ in $[[T]]$ such that $\gamma(A) = B$. We prove that (X,T) is saturated if and only if $\overline{[[T]]}^{\tau_{uw}} = [T]$ and if and only if every infinitesimal function is a T-coboundary. We also show that this class contains odometers and does not contain the Chacon flow.

1.2. Kakutani-Rokhlin partitions. The notion of a Kakutani-Rokhlin (K-R) partition is one of the most useful in the study of C. m. systems. Here we recall the corresponding definitions and facts.

Let (X,T) be a C. m. system. A *Kakutani-Rokhlin partition* is a partition $\mathcal{P}$ of X into clopen sets of the form

$$\mathcal{P} = \{T^k(Z_j) \mid j \in I,\ 0 \le k \le h(j) - 1\}$$

where I is a finite set, $|I| < \infty$. In other words, X is partitioned into $|I|$ disjoint clopen T-towers. The clopen set Z_j is the base and $h(j)$ is the height of the j-th tower. $B(\mathcal{P}) = \cup_j Z_j$ is called the base of $\mathcal{P}$. If a subset of X is a union of some atoms of $\mathcal{P}$, then it is called a $\mathcal{P}$-set.

In the sequel, we will use the following property of K-R partitions: $T^{h(j)}(Z_j) \subset \cup_{i\in J} Z_i,\ j \in I$, i.e. the top of every tower is mapped by T into the base of the K-R partition.

A natural method to construct a K-R partition is as follows. Let Z be a clopen subset of X; consider $Z_n = \{x \in Z : T^n(x) \in Z,\ T^i x \notin Z,\ i = 0, 1, ..., n-1\}, n \in \mathbb{N}$. In fact, there is a finite number of non-empty Z_n's only. It is easily seen (because T is minimal) that T-orbits of such Z_n's form a K-R partition of X [P].

We will consider the sequences $\mathcal{P}_n,\ n \in \mathbb{N}$, of K-R partitions satisfying the following conditions:

(i) $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$, $\mathcal{P}_{n+1} \succ \mathcal{P}_n$ and $B(\mathcal{P}_{n+1}) \subset B(\mathcal{P}_n),\ n \in \mathbb{N}$;

(ii) $(\mathcal{P})_n$ spans the clopen topology on X.

If $(\mathcal{P}_n)$ satisfies additionally the condition

(iii) $\cap_n B(\mathcal{P}_n)$ consists of one point,

then it is called a *nested* sequence. It follows from [P] that to any C. m. system one can associate a nested sequence of K-R partitions. Based on this fact, one can find the very useful Bratteli-Vershik realization of a C. m. system [HPS].

Let $\mathcal{P}$ be a K-R partition with towers $\mathcal{P}(i) = (D_{0,i}, ..., D_{h(i)-1,i}),\ i = 1, ..., k$, where $D_{j+1,i} = T(D_{j,i}),\ j = 0, ..., h(i) - 2$. Define two partitions $\alpha = \alpha(\mathcal{P})$ and $\alpha' = \alpha'(\mathcal{P})$ of $\{1, 2, ..., k\}$ as follows. Say that J is an atom of α if there exists a subset J' of $\{1, ..., k\}$ such that

$$T(\bigcup_{i\in J} D_{h(i)-1,i}) = \bigcup_{i'\in J'} D_{0,i'} \tag{1.1}$$

and for every non-empty proper subset J_0 of J the T-image of $\cup_{i\in J_0} D_{h(i)-1,i}$ is not a $\mathcal{P}$-set. It follows from (1.1) that J' is uniquely defined by J and T; J' will be denoted by $T(J)$. It is easily seen that all such subsets J' form a partition α' of $\{1, ..., k\}$. The map $J \to T(J)$

establishes a 1-1 correspondence between atoms of α and α'. As a particular case, it may be that α (and therefore α') is the trivial partition.

To clarify this definition, let us introduce a matrix $M = (m(i,i') : i, i' = 1, ..., k)$ with

$$m(i,i') = \begin{cases} 1 & \text{if } T(D_{h(i)-1,i}) \cap D_{0,i'} \neq \emptyset \\ 0 & \text{if } T(D_{h(i)-1,i}) \cap D_{0,i'} = \emptyset. \end{cases}$$

Take some $i \in \{1, ..., k\}$ and find J_1 such that $m(i, j_1) = 1,\ j_1 \in J_1$. Then find $I_1 = \{i_1 : m(i_1, j_1) = 1 \text{ for some } j_1 \in J_1\}$. Knowing I_1, we define J_2 such that $m(i_1, j_2) = 1,\ j_2 \in J_2$ where $i_1 \in I_1$. Next, define I_2 using J_2. We get two increasing finite sequences (I_p) and (J_p) that are stabilized in several of steps. The latest members of these sequences are the atoms J and $J' = T(J)$ of α and α' respectively. Similarly, one can define a matrix $M = \{m(J, J') : J \in \alpha,\ J' \in \alpha'\}$ by setting $m(J, J') = 1$ if and only if there exist $i \in J$ and $i' \in J'$ such that $m(i, i') = 1$ or, equivalently, if and only if $T(J) = J'$. Otherwise, $m(J, J') = 0$.

The 1-1 correspondence $T : J \to J'$ between α and α' allows us to introduce cycles on $\{1, ..., k\}$. Denote by $J(i)$ and $J'(i)$ the atoms of α and α' containing i. We say that $l = (i_1, ..., i_s)$ is a *cycle* on $\{1, ..., k\}$ if $i_1 \in J'(i_1), i_2 \in J'(i_2) = T(J(i_1)), ..., i_s \in J'(i_s) = T(J(i_{s-1}))$, and $T(J(i_s)) = J'(i_1)$.

2 Structure of homeomorphisms from the topological full group

2.1. Compatible partitions and level condition. Let (X, T) be a C. m. system. We remind that, by definition, a homeomorphism γ belongs to the *full group* $[T]$ if $\gamma(x) = T^{n_\gamma(x)}(x),\ x \in X$. The function $n_\gamma : X \to \mathbb{Z}$ is called the orbit cocycle. The *topological full group* $[[T]]$ is the subgroup of all homeomorphisms $\gamma \in [T]$ whose associated orbit cocycle n_γ is continuous.

To every $\gamma \in [[T]]$, one can associate a clopen finite partition $\mathcal{E} = \mathcal{E}(\gamma)$ of X such that $\mathcal{E} = \{E_l : l \in K\},\ |K| < \infty,\ K \subset \mathbb{Z}$, where $E_l = \{x \in X : \gamma(x) = T^l(x)\}$. Evidently, the sets $T^l(E_l),\ l \in K$, also form a clopen partition of X. Denote it by $\mathcal{E}(K)$. It is well known (see, e.g. [GPS2]) that if $\mathcal{E}$ and $\mathcal{E}(K)$ are given clopen finite partitions of X, then they define the homeomorphism $\gamma = \gamma(\mathcal{E}, K) \in [[T]]$ as follows:

$$\gamma(x) = T^l(x), \quad x \in E_l, \quad l \in K. \tag{2.1}$$

We will use the notation $(\mathcal{E}, K)$ for such a pair of partitions $\mathcal{E}$ and $\mathcal{E}(K)$ and call it a *compatible pair*.

We will prove (Theorem 2.2) that under some natural assumptions about a sequence of K-R partitions one can describe all elements from the topological full group. In the next section, it will be shown that Theorem 2.2 can be used for the study of some symbolic minimal Cantor systems.

For a given C. m. system (X, T), consider a K-R partition ξ such that ξ is the union of $k = k(\xi)$ disjoint T-towers $\xi(i) = \{T^j(D_{0,i}) : 0 \leq j \leq h(i) - 1\},\ i = 1, ..., k$, where $h(i)$ is the height of the i-th tower. Let

$$h = \min\{h(i) : 1 \leq i \leq k\}$$

and suppose that $i_{\min}$ is taken such that $h = h(i_{\min})$. Denote $D_{j,i} = T^j(D_{0,i})$. Let $U = U(\xi)$ be the set of all pairs (j,i) where $i = 1, ..., k,\ j = 0, 1, ..., h(i) - 1$; then every atom of ξ is enumerated by a pair from U.

Take a compatible pair of partitions $(\mathcal{E}, K)$ (in other words, we take $\gamma = \gamma(\mathcal{E}, K) \in [[T]]$). Assume that the K-R partition ξ refines both partitions $\mathcal{E}$ and $\mathcal{E}(K)$

$$\xi \succ \mathcal{E}, \quad \xi \succ \mathcal{E}(K), \quad \text{and} \quad K \subset (-h, ..., h). \tag{2.2}$$

This means that every E_l and $T^l(E_l)$, $l \in K$, are ξ-sets, i.e. for every $(j,i) \in U(\xi)$ there is $l = l(j,i) \in K$ such that

$$\gamma(D_{j,i}) = T^l(D_{j,i}). \tag{2.3}$$

To formulate another additional condition for $(\mathcal{E}, K)$, we divide $U(\xi)$ into three disjoint subsets $U_{\text{in}}, U_{\text{top}}$ and U_{bot} (this decomposition depends on $(\mathcal{E}, K)$). We say that

(A) $(j,i) \in U_{\text{in}}$ if $\gamma(D_{j,i}) \subset \xi(i)$, i.e. $0 \le l + j \le h(i) - 1$;

(B) $(i,j) \in U_{\text{top}}$ if $\gamma(D_{j,i})$ goes through the top of $\xi(i)$, i.e. $l + j \ge h(i)$;

(C) $(j,i) \in U_{\text{bot}}$ if $\gamma(D_{j,i})$ goes through the bottom of $\xi(i)$, i.e. $l + j < 0$

(here $l = l(j,i)$ is taken from (2.3)).

Let α, α' be the partitions defined by (X,T) and ξ as in **1.2.** Take $J \in \alpha$ and $J' \in \alpha'$. For $r = 0, 1, ..., h_J - 1$ where $h_J = \min\{h(i) : i \in J\}$, denote

$$V_1(r,J) = \{(h(i) - h_J + r, i) : i \in J\}, \quad V_2(r,J') = \{(r,i) : i \in J'\},$$

$$F_1(r,J) = \bigcup_{(j,i) \in V_1(r,J)} D_{j,i}, \quad F_2(r,J') = \bigcup_{(j,i) \in V_2(r,J')} D_{j,i}.$$

Assuming that (2.2) holds, we introduce

Definition 2.1. We say that a pair $(\mathcal{E}, K)$ (or the homeomorphism $\gamma(\mathcal{E}, K)$) satisfies level condition (L) if either $U = U_{\text{in}}$ or:

(L^+) whenever $(j,i) \in U_{\text{top}}$ and $D_{j,i} \subset E_l$, then $F_1(h_J - h(i) + j, J) \subset E_l$ where $J = J(i)$;

(L^-) whenever $(j,i) \in U_{\text{bot}}$ and $D_{j,i} \subset E_l$, then $F_2(j, J') \subset E_l$ where $J' = J'(i)$.

In other words, condition (L^+) says that whenever the set $D_{j,i}$ goes through the top of the i-th tower under the action of γ, then the entire level $F_1(r, J(i))$, $r = h_j - h(i) + j$ (containing $D_{j,i}$) also goes through the top of ξ. Similarly, one can clarify condition (L^-) by taking the level $F_2(j, J'(i))$ and $D_{j,i}$ that goes through the bottom of ξ.

Define a subset $\Gamma(\xi)$ in $[[\Gamma]]$: we say that a homeomorphism $\gamma(\mathcal{E}, K)$ defined by (2.1) belongs to $\Gamma(\xi)$ if the corresponding compatible pair $(\mathcal{E}, K)$ satisfies (2.2) and condition (L). Clearly, $\Gamma(\xi)$ is a finite set.

Let now (ξ_t), $t \ge 0$, be a sequence of K-R partitions (we will use the same notations as above with an additional index t). The next theorem gives the sufficient conditions under which for every $\gamma \in [[T]]$ one can find some t such that $\gamma \in \Gamma(\xi_t)$, i.e. $\gamma = \gamma(\mathcal{E}, K)$.

Theorem 2.2. *Let a sequence (ξ_t) of Kakutani-Rokhlin partitions be chosen such that:*

(1) $h(t) = \min\ \{h(i,t) : 1 \le i \le k_t\} \to \infty \quad (t \to \infty)$,

(2) (ξ_t) *satisfies the properties (i) and (ii) from* **1.2**.

Then $\cup_t \Gamma(\xi_t) = [[T]]$.

Proof. Let $\gamma \in [[T]]$, then $\gamma(x) = T^{n_\gamma(x)}(x)$ where $n_\gamma : X \to \mathbb{Z}$ is a continuous function. Denote by K the finite subset of $\mathbb{Z}$ formed by the values of n_γ. Then $E_l = \{x \in X : n_\gamma(x) = l\}$, $l \in K$, is a non-empty clopen subset in X. According to the theorem assumption, one can find t so large that $|n_\gamma(x)| < h(t)$ and moreover both E_l and $\gamma(E_l) = T^l(E_l)$ are ξ_t-sets, $l \in K$. In such a way, we have defined a compatible pair $(\mathcal{E}, K)$ where $\mathcal{E}$ is the partition $\{E_l : l \in K\}$. We will show that this pair satisfies condition (L).

If $U = U_{\rm in}$, then there is nothing to prove. Take $l \in U_{\rm top}$ and suppose that $D^t_{j_0,i_0}$ is chosen as in (B), i.e. $j_0 + l \geq h(i_0, t)$. Let $J = J(i_0)$ and let $J' = T(J)$ be the element of α' corresponding to J by (1.1). Denote by J_1 the subset of J that consists of all i such that $D^t_{j(i),i} \cap E_l = \emptyset$ where $j(i) = j_0 + h(i,t) - h(i_0,t)$. Set $J_2 = J \setminus J_1$; then $i_0 \in J_2$. Assume that $J_1 \neq \emptyset$. Then J_2 is a proper non-empty subset of J. By the definition of a K-R partition, we get

$$\begin{aligned} T^l(\bigcup_{i\in J_2} D^t_{j(i),i}) &= \bigcup_{i\in J_2} T^{l-h(i,t)+1+j(i)}(D^t_{h(i,t)-1,i}) \\ &= T^{l-h(i_0,t)+j_0}(\bigcup_{i\in J_2} T(D^t_{h(i,t)-1,i})). \end{aligned}$$

Due to the choice of K, we have $0 \leq l - (h(i_0,t) - j_0) < h(t)$. It follows from the definition of the partition α that $Q = T^l(\cup_{i\in J_2} D^t_{j(i),i})$ is not a ξ_t-set.

On the other hand, we have that

$$Q \subset T^{l-h(i_0,t)+j_0}(\bigcup_{i\in J'} D^t_{0,i})$$

At that time, Q is a subset of the ξ_t-set $T^l(E_l)$ and moreover, in view of the evident equality

$$Q = T^l(E_l) \bigcap T^{l-h(i_0,t)+j_0}(\bigcup_{i\in J'} D^t_{0,i}),$$

we get that Q is a ξ_t-set. This contradiction shows that $J_1 = \emptyset$ and, therefore, $(\mathcal{E}, K)$ satisfies condition (L^+). The same method can be used to prove that $(\mathcal{E}, K)$ also satisfies condition (L^-). The proof is completed. $\square$

Remark 2.3. 1) We observe that every C. m. system (X, T) is topologically conjugate to a system that satisfies the conditions of Theorem 2.2. For this, take a Bratteli-Vershik model (Ω, φ) for (X, T) [HPS]. (We use the notations from [HPS]). Note that it is sufficient to deal with an improper Bratteli-Vershik model. Here Ω is the set of all infinite paths in a simple Bratteli diagram $B = (V, E, >)$, and φ is the Vershik homeomorphism defined on Ω. To determine the K-R partitions (ξ_t), $t \geq 0$, it suffices to define ξ_t-towers. Fix some $i \in V_t$ and consider the set of all finite paths from the initial point V_0 into i. In other words, every such path corresponds to an atom in $\xi_t(i)$-tower and, therefore $\xi_t(i)$ is formed by the family of such atoms. It is easily checked that (ξ_t) satisfies the conditions of Theorem 2.2.

2) Let $\pi : (X, T) \to (Y, S)$ be a factor map of C. m. systems. If $\gamma : y \mapsto S^{n(y)}(y) \in [S]$, then $P_\pi(\gamma) : x \mapsto T^{n(\pi(x))}(x)$ belongs to $[T]$. Thus, π generates a group monomorphism

$P_\pi : [S] \to [T]$ such that $P_\pi([[S]]) \subset [[T]]$.

2.2. Structure of homeomorphisms from $[[T]]$. We first analyze the proof of Theorem 2.2 to distinguish the most essential elements. We have proved that if a sequence (ξ_t) of K-R partitions (satisfying conditions of Theorem 2.2) is given, then to every homeomorphism $\gamma \in [[T]]$ one can associate a finite clopen partition $\mathcal{E}$ and a finite set $K \subset \mathbb{Z}$ so that the pair $(\mathcal{E}, K)$ satisfies (2.2) and condition (L). Moreover, there is a K-R partition ξ taken from the given sequence such that $\xi \succ \mathcal{E}$ and $\xi \succ \mathcal{E}(K)$. (We use our notations from **2.1**, and, for brevity, the index t is omitted when it does not lead to misunderstanding). Now we will show that if ξ and $(\mathcal{E}, K)$ are given, then one can define another partition $\hat{\mathcal{E}}$ (also induced by γ) such that $\xi \succ \hat{\mathcal{E}} \succ \mathcal{E},\ \xi \succ \hat{\mathcal{E}} \succ \mathcal{E}(K)$.

Set for $J \in \alpha,\ J' \in \alpha'$,

$$L^+(J) = \{0 \le r \le h_J - 1 : F_1(r, J) \subset E_l \text{ and } \ l + r \ge h_J\},$$

$$L^-(J') = \{0 \le r \le h_{J'} - 1 : F_2(r, J') \subset E_l \text{ and } l + r < 0\}. \tag{2.4a}$$

The homeomorphism γ induces two one-to-one maps $\rho^+(J)$ and $\rho^-(J')$ determined on $L^+(J)$ and $L^-(J')$, respectively:

$$\rho^+(J)(r) = l + r - h_J,$$

$$rho^-(J')(r) = r + l + h_{T^{-1}(J')}. \tag{2.4b}$$

Let $M^+(J) = \rho^+(J)(L^+(J))$ and $M^-(J') = \rho^-(J')(L^-(J'))$. It follows from the proof of Theorem 2.2 that the families of sets $\mathcal{F}^+(J) = \{F_1(r, J) : r \in L^+(J)\}\ (J \in \alpha),\ \mathcal{F}^-(J') = \{F_2(r, J') : r \in L^-(J')\}\ (J' \in \alpha')$ and $\mathcal{F} = \{D_{j,i} : (j, i) \notin V_1(r, J) \cup V_2(r, J'), J \in \alpha, J' \in \alpha'\}$ form a partition $\hat{\mathcal{E}}$ such that $\xi \succ \hat{\mathcal{E}} \succ \mathcal{E}$. The proved theorem asserts that

$$\gamma(F_1(r, J)) = F_2(\rho^+(J)(r), T(J)),\ r \in L^+(J),$$

$$\gamma(F_2(r, J')) = F_1(\rho^-(J')(r), T^{-1}(J')),\ r \in L^-(J'). \tag{2.5}$$

Denote

$$V_1(J) = \bigcup_{r \in L^+(J)} V_1(r, J), \qquad V_2(J') = \bigcup_{r \in L^-(J')} V_1(r, J').$$

Take $(j, i) \notin V_1(J) \cup V_2(J')$ where $J = J(i),\ J' = J'(i)$. Then γ induces also the map $\rho(i)(j) = l + j$ where l is determined in (2.3). If we use the notations $L_i^+(J) = L^+(J) + h(i, t) - h_J,\ L_i^-(J') = L^-(J')$ and $M_i^+(J)(= M^+(J)),\ M_i^-(J')$ for images of $L_i^+(J)$ and $L_i^-(J')$ with respect to the maps defined in (2.5), then we see that γ generates a one-to-one map

$$\rho(i) : P(i) \to R(i),\ i = 1, ..., k \tag{2.4c}$$

where $P(i) = \{0, 1, ..., h(i, t) - 1\} - (L_i^+(J) \cup L_i^-(J'))$ and $R(i) = \{0, 1, ..., h(i, t) - 1\} - (M_i^+(T(J)) \cup M_i^-(T^{-1}(J')))$, $J = J(i), J' = J'(i)$.

Thus, we get that the image of $\hat{\mathcal{E}}$ under action of γ is the partition $\hat{\mathcal{E}}(K)$ defined by the families of sets $\mathcal{G}^+(J) = \{F_2(r', T(J)) : r' \in M^+(J)\},\ \mathcal{G}^-(J') = \{F_1(r, T^{-1}(J')) : r \in M^-(J')\}$ and $\mathcal{G} = \{D^t_{\rho(i)(j),i} : j \in P(i), i = 1, ..., k\}$.

To summarize, we observe that every homeomorphism γ from $[[T]]$ (or, otherwise, the pair $(\mathcal{E}, K)$) defines the following objects (recall that ξ is fixed):

the subsets $L^+(J), L^-(J')$ and one-to-one maps $\rho^+(J),\ \rho^-(J'),\ \rho(i)$ satisfying (2.4a), (2.4b), (2.4c) where $J \in \alpha, J' \in \alpha',\ i = 1, ..., k.$ $(*)$

the families of disjoint sets $\mathcal{F}^+(J),\ \mathcal{F}^-(J'),\ J \in \alpha, J' \in \alpha'$ and $\mathcal{F}$ form a partition $\hat{\mathcal{E}}$ satisfying (2.5). $(**)$

It is clear that the other objects introduced above are completely determined by those from $(*)$ and $(**)$.

It is easily seen that the described procedure is reversible. This means that if $L^+(J),\ L^-(J'),\ \rho^+(J),\ \rho^-(J),\ \rho(i)$ and $\mathcal{F}^+(J),\ \mathcal{F}^-(J'), \mathcal{F}$ are taken as in $(*)$ and $(**)$, then we can restore a homeomorphism $\gamma = \gamma(\mathcal{E}, K) \in [[T]]$. To do this, we determine the orbit cocycle $n_\gamma : X \to \mathbb{Z}$ as follows:

$$n_\gamma(x) = \begin{cases} h_J - r + \rho^+(J)(r), & \text{if } x \in F_1(r, J),\ r \in L^+(J) \\ -h_{T^{-1}(J')} - r + \rho^-(J'), & \text{if } x \in F_2(r', J'),\ r' \in L^-(J') \\ \rho(i)(j) - j, & \text{if } x \in D_{j,i},\ (j, i) \notin V_1(J) \cup V_2(J) \end{cases} \tag{2.6}$$

In fact, this function is piecewise constant, $n_\gamma(x) = n_i(j),\ x \in D_{j,i}$, with the same values along the sets $F_1(r, J)$ and $F_2(r', J')$. We leave other details to the reader.

Now we consider a few particular cases of the described structure.

2.2(a). We first give a simple example confirming the existence of subsets $L^+(J), L^-(J')$, maps $\rho^+(J),\ \rho^-(J'),\ \rho(i)$ and $\mathcal{F}^+(J),\ \mathcal{F}^-(J'),\ \mathcal{F}$ satisfying $(*)$ and $(**)$. For simplicity, let us assume that α and α' are trivial partitions. Let h be the height of the lowest tower. Suppose that L^+, L^-, M^+, M^- are chosen in $\{0, 1, ..., h-1\}$ such that L^+ and M^- are in $([\frac{1}{2}h] + 1, ..., h-1)$, and L^- and M^+ are in $(0, ..., [\frac{1}{2}h])$. Moreover, $|L^+| = |M^+|,\ |L^-| = |M^-|$. Take some one-to-one maps $\rho^+ : L^+ \to M^+$ and $\rho^- : L^- \to M^-$. The "level sets" $F_1(r),\ r \in L^+$ and $F_2(r'),\ r' \in L^-$ are defined as above. Then all conditions in $(*)$ and $(**)$ are fulfilled and such a choice gives us a homeomorphism γ from $[[T]]$.

2.2(b). Suppose now that for a sequence of K-R partition ξ_t, we take $\gamma \in [[T]]$ such that the condition $L^+(J) = L^-(J') = \emptyset$ is realized in our construction. This means that both $U_{\rm top}$ and $U_{\rm bot}$ are empty. Then it follows from (2.4c) that for every $(j, i) \in U_t$ one has $0 \le j + n_i(j) \le h(i, t) - 1$ and $\rho(i)(j) = j + n_i(j)$ is a permutation of $\{0, 1, ..., h(i, t) - 1\}$. Therefore $\gamma(x) = T^{\rho(i)(j)-j}(x),\ x \in D_{j,i}$, and the set of all such γ generates a subset $\Gamma_t^0 \subset \Gamma(\xi_t)$. Moreover, Γ_t^0 is a subgroup in $[[T]]$ that is isomorphic to $\oplus_{1 \le i \le k_t} S_{h(i,t)}$ where S_n is the group of all permutations of n elements. Because ξ_{t+1} refines ξ_t, we get that $\Gamma_t^0 \subset \Gamma_{t+1}^0$. It is clear that $\Gamma^0 = \cup_t \Gamma_t^0$ is a proper subgroup in $[[T]]$.

2.2(c). The next interesting case we get if assume that all towers in ξ_t have the same height, $h(i, t) = h(t),\ i = 1, ..., k_t$. Then we obtain that $L_i^+(J) = L^+(J),\ M_i^-(J') = M^-(J')$ where $L^+(J), L^-(J'), \rho^+(J), \rho^-(J')$ satisfy $(*)$. Note that in this case the condition $(*)$ implies $(**)$. The map $\rho(i),\ 1 \le i \le k_t$, sends $\{0, 1, ..., h(t)-1\} \setminus (L^+(J) \cup L^-(J'))$ onto $\{0, 1, ..., h(t) - 1\} \setminus (M^+(J) \cup M^-(J'))$.

2.2(d). The simplest case in the described class of Cantor minimal systems we get when $k_t = 1$, i.e. (X, T) is conjugate to an odometer. Then each $\gamma \in [[T]]$ defines a permutation ρ of $\{0, 1, ..., h(t) - 1\}$ and a vector $\varepsilon = (\varepsilon_l : l = 0, 1, ..., h(t) - 1),\ \varepsilon_l \in$

$(-1,0,1)$, as follows:

$$\rho(l) = \begin{cases} \rho^+(l), & l \in L^+ \\ \rho^-(l), & l \in L^- \\ \rho(1)(l), & l \notin L^+ \cup L^- \end{cases}$$

and

$$e_l = \begin{cases} 1, & l \in L^+ \\ -1, & l \in L^- \\ 0, & \rho(1)(l) = l \\ \mathrm{sign}(\rho(1)(l) - l)), & \text{otherwise.} \end{cases}$$

Here ρ^+, ρ^- and $\rho(1)$ are taken from the above construction. Conversely, consider a permutation, ρ, of $\{0,1,...,h(t)-1\}$ and a vector $\varepsilon = (\varepsilon_0, \varepsilon_1, ..., \varepsilon_{h(t)-1})$ such that ε_i takes values in $(-1,0,1)$ and ε compatible with ρ. This means that $\varepsilon_l = 0$ if and only if $\rho(l) = l$. Then the function

$$n(j) = \begin{cases} 0 & \text{if } \rho(j) = j \\ \rho(j) - j & \text{if } (\rho(j) - j)\varepsilon_j > 0 \\ \rho(j) - j + \varepsilon_j h(t) & \text{if } (\rho(j) - j)\varepsilon_j < 0 \end{cases}$$

defines a homeomorphism $\gamma = \gamma(\rho, \varepsilon) \in [[T]]$ as follows: $\gamma(x) = T^{n(j)}$, $x \in D_j$, $j = 0,1,...,h(t)-1$ where D_j's are atoms of the unique tower. Evidently this correspondence between homeomorphisms from $[[T]]$ and the set of all compatible pairs (ρ, ε) is one-to-one.

2.3. The subgroup Γ^0. Let (X,T) be a C. m. system. For all $x \in X$, let $O_T^+(x) = \{T^k(x) : k \geq 1\}$ denote the forward T-orbit of x, and let $[[T]]_x$ denote the subgroup of those γ from $[[T]]$ such that $\gamma(O_T^+(x)) = O_T^+(x)$. It is known that $[[T]]_x$ is a countable, locally finite ample group with minimal action on X [GPS2, Kr]. It was also proved in [Kr] that all groups $[[T]]_x$, $x \in X$, are isomorphic. It is important to mention the next result proved in [GPS2]: two C. m. systems (X_1, T_1) and (X_2, T_2) are strong orbit equivalent if and only if $[[T_1]]_{x_1}$ and $[[T_2]]_{x_2}$ are isomorphic as abstract groups for any $x_i \in X_i$, $i = 1,2$.

In **2.2(b)**, we introduced the subgroup $\Gamma^0 \subset [[T]]$. It turns out that this group coincides with $[[T]]_x$ for some $x \in X$.

Theorem 2.4. *Let (X,T) be a C. m. system and let (ξ_t), $t \geq 0$, be a sequence of K-R partitions satisfying the conditions of Theorem 2.2. Then, there exists a point $x \in X$ such that $\Gamma^0 = [[T]]_x$.*

Proof. We can assume without lost of generality that the partitions α_t and α'_t are trivial. Take ξ_t, $t \geq 0$, as in the proof of Theorem 2.2 (we use the notations from **2.1** and **2.2**). In view of property (i) from **1.2**, there is a point $x \in \cap_t T^{-1}(B(\xi_t))$. Suppose now that $\gamma \in [[T]]_x$. Then, as it was shown in **2.2**, we can find some ξ_t and then associate to γ the collection $(L^+, L^-, \rho^+, \rho^-, \rho(i))$, $i = 1, ..., k_t$, satisfying $(*)$ and $(**)$. For definiteness, assume that x belongs to the top of $\xi_t(i(0))$-tower, $x \in D^t_{h(i(0),t)-1,i}$, $1 \leq i(0) \leq k_t$. Let $i(n)$, $n \in \mathbb{N}$, be the numbers of ξ_t-towers which the forward T-orbit of x intersects successively. Then, $O_T^+(x)$ can be divided into subsets $O(n)$, $n \in \mathbb{N}$, where $O(1) = \{T(x), ..., T^{h(i(1),t)}(x)\}$, $O(2) = \{T^{h(i(1),t)+1}(x), ..., T^{h(i(1),t)+h(i(2),t)}(x)\}$ and so on. Thus,

$O(n)$ is the part of $O_T^+(x)$ that is in $\xi(i(n))$. The backward orbit is divided similarly into $O(0), O(-1), ...$ where, for example, $O(0) = \{T^{-h(i(0),t)+1}(x), ..., T^{-1}(x), x\}$.

If we assume that $L^+ \neq \emptyset$, then there is some $T^k(x) \in F_1(l) \cap O(0),\ l \in L^+$, such that $\gamma(T^k(x)) \in O(1)$. This fact contradicts to our assumption that $\gamma \in [[T]]_x$. Analogously, we will come to a contradiction assuming that $L^- \neq \emptyset$.

Conversely, let $\gamma \in \Gamma^0$, i.e. L^+ and L^- are empty for some ξ_t. It follows that $\gamma(O(n)) = O(n),\ n \in \mathbb{Z}$, and therefore $\gamma \in [[T]]_x$. □

Remark 2.5. Let μ be a T-invariant probability measure on X. In [GPS2], the group homomorphism $I_\mu : [[T]] \to \mathbb{R},\ I_\mu(\gamma) = \int_X m_\gamma d\mu$ was introduced and studied where $m_\gamma = \sum_l l\chi_{E_l}$. It was shown that, in fact, I_μ takes values in $\mathbb{Z}$ and $[[\Gamma_\mu]] = [[T]]$ where Γ_μ is the kernel of I_μ [GPS2, Proposition 5.8]. We note that this result follows also from the found description $(*)$ and $(**)$. The point is that every γ from $[[T]]$ is defined by a finite set of permutations acting on the towers of ξ. Because each permutation is the product of some transpositions and every transposition corresponds to a homeomorphism from Γ_μ, we get the above statement.

2.4. $[[T]]$**-equivalent clopen sets**. We will say that two clopen sets A and B are $[[T]]$-*equivalent* (resp. $[T]$-*equivalent*) if there exists $\gamma \in [[T]]$ (resp. $\gamma \in [T]$) such that $\gamma(A) = B$. E. Glasner and B. Weiss proved that if $\mu(A) = \mu(B)$ for every T-invariant probability measure μ, then A and B are $[T]$-equivalent [GW, Proposition 2.6]. It follows from [Lemma 3.3, GPS2] that two clopen sets A and B are $[[T]]$-equivalent if and only if $\chi_A(x) - \chi_B(x)$ is a coboundary. We consider another approach to this problem based on K-R partitions. If ξ is a K-R partition with towers $\xi(i),\ i = 1, ..., k$, then for a clopen ξ-set A denote by $N_i(A) = |\{0 \leq j \leq h(i) - 1 : D_{j,i} \subset A\}|$.

Theorem 2.6. *Let there (X, T) be a C. m. system and (ξ_t) be a sequence of K-R partitions satisfying conditions of Theorem 2.2. Then two clopen sets A and B are $[[T]]$-equivalent if and only if there exists t such that for every cycle $l = (i_1, ..., i_s)$ on $\{1, ..., k_t\}$ one has*

$$\sum_{i \in l} N_i^t(A) = \sum_{i \in l} N_i^t(B). \tag{2.7}$$

(it is assumed that t is already chosen so large that A and B are ξ_t-sets).

Proof. Suppose that for given clopen sets A and B there exists $\gamma \in [[T]]$ such that $\gamma(A) = B$. Find t such that A and B become ξ_t-sets and $\gamma \in \Gamma(\xi_t)$. Let us observe that if $r \in L^+(J),\ J \in \alpha_t$, then either $F_1(r, J) \cap A = \emptyset$ or $F_1(r, J) \subset A$. The same statement holds for B. It follows from the fact that A, B are ξ_t-sets and from the equality $\gamma(F_1(r, J) \cap A) = F_2(r', J') \cap B$ where $J' = T(J),\ r' = \rho_J^+(r),\ r \in L^+(J)$.

Now we define for $J \in \alpha,\ J' \in \alpha'$

$$L^+(J, A) = \{r \in L^+(J) : F_1(r, J) \subset A\},$$

$$L^-(J', A) = \{r \in L^-(J') : F_2(r, J') \subset A\}$$

and

$$L_A(i) = \{0 \leq j \leq h(i, t) : D_{j,i}^t \subset A \text{ and } \gamma(D_{j,i}^t) \subset \xi_t(i)\}.$$

The set $L_B(i)$ is determined analogously.

If $J = J(i)$, $J' = J'(i)$, then the fact that $\gamma(A) = B$ implies $N_i(A) = |L_A(i)| + |L^+(J, A)| + |L^-(J', A)|$ and $N_i(B) = |L_B(i)| + |L^+(T^{-1}(J'), A)| + |L^-(T(J), A)|$. We note that $|L_A(i)| = |L_B(i)|$. Therefore

$$N_i(A) - N_i(B) = |L^+(J, A)| + |L^-(J', A)| - |L^+(T^{-1}(J'), A)| - |L^-(T(J), A)|. \quad (2.8)$$

If $l = (i_1, ..., i_s)$ is a cycle then $i_1 \in J'(i_1), i_2 \in J'(i_2) = T(J(i_1)), ..., i_s \in J'(i_s) = T(J(i_{s-1}))$ and $T(J(i_s)) = J'(i_1)$. We get from (2.8)

$$\begin{aligned}
\sum_{i\in l}(N_i(A) - N_i(B)) &= \sum_{k=1}^{s}(N_{i_k}(A) - N_{i_k}(B)) \\
&= \sum_{k=1}^{s}(|L^+(J(i_k), A)| + |L^-(J'(i_k), A)| - \\
&\quad - |L^+(T^{-1}(J'(i_k)), A)| - |L^-(T(J(i_k)), A)|) \\
&= 0.
\end{aligned}$$

Conversely, suppose that (2.7) holds for every cycle l. Let $B(J') = \cup_{i\in J'} D^t_{0,i}$. Construct a partition ξ'_t that refines ξ_t. For this, take $x \in D^t_{0,i}$, $i \in J' = J'(i)$ and consider $\{x, T(x), ..., T^p(x)\}$ where $T^p(x) \in B(J')$ and $T^n(x) \notin B(J')$, $n < p$. The T-orbit of $x \in B(J')$ will go through the towers $\xi_t(i_1), ..., \xi_t(i_s)$ where $i_1 = i$ and $l(x) = (i_1, ..., i_s)$ will form a cycle in ξ_t. Then define ξ'_t to be such a minimal partition that refines ξ_t and makes $l = l(x)$ $\xi'_(t)$-measurable. In such a way, every l determines a tower in ξ'_t. Then equality (2.7) implies existence of some $\gamma \in \Gamma(\xi'_t)$ such that $\gamma(A) = B$. □

Example 2.7. Let (X, T) be a strictly ergodic C. m. system and let μ be a unique T-invariant measure. Assume that we have a sequence (ξ_t) of K-R partitions such that each tower $\xi_{t+1}(j)$ is a concatenation of exactly k_t parts of equal measures coming from each $\xi_t(j)$-tower, $j = 1, ..., k_t$ (see Theorem 4.7 of [HPS] for details of the Bratteli-Vershik construction). Note that, in this case, the partitions α_t and α'_t are trivial. Then we state that for any two clopen subsets A and B of equal measure μ there exists $\gamma \in [[T]]$ such that $\gamma(A) = B$. In fact, the equality $\mu(A) = \mu(B)$ implies that

$$\sum_{i=1}^{k_t} N^t_i(A) = \sum_{i=1}^{k_t} N^t_i(B).$$

But we get from the construction that $N^{t+1}_j(A) = \sum_{i=1}^{k_t} N^t_i(A)$, $j = 1, ..., k_{t+1}$. Therefore the statement follows from Theorem 2.6.

3 Examples

3.1. Odometer (adding machine). Let $\{\lambda_t\}_{t=0}^{\infty}$ be a sequence of integers such that $\lambda_t \geq 2$. Denote by $p_{-1} = 1$, $p_t = \lambda_0\lambda_1\cdots\lambda_t$, $t = 0, 1, ...$. Let Δ be the group of all p_t-adic numbers; then any element of Δ can be represented as an infinite formal series:

$$\Delta = \{x = \sum_{i=0}^{\infty} x_i p_{i-1} \mid x_i \in (0, 1, ..., \lambda_i - 1)\}.$$

It is well known that Δ is a compact metric abelian group. An odometer, σ, is the transformation acting on Δ as follows: $\sigma x = x + 1,\ x \in \Delta$, where $1 = 1p_{-1} + 0p_0 + 0p_1 + \cdots \in \Delta$. From the topological point of view, (Δ, σ) is a strictly ergodic Cantor system.

To an odometer (Δ, σ) defined by a sequence $\{\lambda_t\}_t$, one can associate a so called "generalized number" $\prod_t \lambda_t = r_1^{\alpha_1} r_2^{\alpha_2} \cdots$ where r_i is a prime factor of some λ_t and α_i takes values in $1, ..., \infty,\ i \in \mathbb{N}$. Apart from, we can take $r_1 < r_2 < \cdots$. Let $\{\lambda'_t\},\ \lambda'_t \geq 2$, be another sequence of integers and let (Δ', σ') be the corresponding odometer. It is known that (Δ, σ) and (Δ', σ') are topologically conjugate if and only if $\prod_t \lambda_t = \prod_t \lambda'_t$, i.e. the collection $\{(r_i, \alpha_i) : i \in \mathbb{N}\}$ is a complete invariant of conjugacy. The latter means that these "generalized numbers" have the same prime factors with regard to their multiplicity.

Denoting

$$D_0^t = \{x = \sum_{i=0}^{\infty} x_i p_{i-1} \mid x_0 = x_1 = ... = x_t = 0\},$$

we see that the sets $(D_0^t, D_1^t, ..., D_{p_t-1}^t),\ D_i^t = \sigma^i(D_0^t)$, form a partition ξ_t of Δ into clopen sets. Clearly, $(\xi_t),\ t \geq 0$, is a nested sequence of Kakutani-Rokhlin partitions. By definition, $k_t(\xi_t) = 1$. Therefore the topological full group of an odometer can be described as in **2.2(d)**.

The unique σ-invariant measure μ_σ is completely defined by its values on ξ_t-atoms: $\mu(D_i^t) = p_t^{-1},\ i = 0, 1, ..., p_t - 1$. Therefore the set $\Lambda(\Delta, \sigma)$ of values of μ_σ on clopen subsets is $\{i/p_t^{-1} : i = 0, 1, ..., p_t - 1,\ t \geq 0\}$. It follows from this remark and Theorem 2.2 of [GPS1] that two odometers are orbit equivalent if and only if they are conjugated.

3.2. Toeplitz-Morse flow. We consider here so called Toeplitz-Morse sequences that are a special kind of Toeplitz sequences (see [JK,W]). Let G be a finite alphabet with at least two symbols. A finite sequence $B = (B[0], ..., B[n-1]),\ B[i] \in G$, is called a block. The length n of B is denoted by $|B|$. Denote by Ω the space of all bisequences over G with its natural compact metric topology, and let T be the left shift on Ω. For $\omega = \{\omega[n]\}_{n\in\mathbb{Z}} \in \Omega$, let $O(\omega)$ denote the T-orbit of ω.

Let $\{a_t\}_{t=0}^{\infty}$ be a sequence of blocks over $G \cup \{-\}$ where " $-$ " means a symbol (called a "hole"). Suppose that $|a_t| = \lambda_t \geq 2,\ a_t[i] \in G,\ i = 0, 1, ..., \lambda_t - 2$, and $a_t[\lambda_t - 1] = -$. Denote by $p_t = \lambda_0 \lambda_1 \cdots \lambda_t$. Without loss of generality, we assume that for every $t \geq 0$ and any $g \in G$ there exists $0 \leq i \leq \lambda_t - 2$ such that $a_t[i] = g$. Define inductively a sequence of blocks $(A_t)_{t\geq 0}$ as follows:

$$A_0 = a_0,$$

$$A_{t+1} = A_t a_{t+1}[0] A_t a_{t+1}[1] \cdots a_{t+1}[\lambda_{t+1} - 2] A_t -, \quad t \geq 0, \tag{3.1}$$

Thus, A_{t+1} is obtained as the concatenation of λ_{t+1} copies of A_t with holes filled by the successive elements of a_{t+1} except the latest hole. We will also write down equality (3.1) as $A_{t+1} = A_t * a_{t+1}$. Define a bisequence ω over $G \cup \{-\}$ as follows:

$$\omega[kp_t, (k+1)p_t - 1] = A_t, \quad k \in \mathbb{Z},$$

for every $t \geq 0$. It is evident that all coordinates $\omega[i]$ are filled by elements of G whenever $i \neq -1$, and $\omega[-1] = -$. The sequence ω can be written also in the form $\omega = a_0 * a_1 * \cdots$. We will also use the sequences $\omega_t = a_t * a_{t+1} * \cdots,\ t = 0, 1, ...$ Consider the set

$$X_\omega = \{x \in \Omega : x = \lim_{|i_t|\to\infty} T^{i_t}(\omega)\},$$

i.e. X_ω is the closure of T-orbit of ω in Ω. We get a topological flow (X_ω, T) which is called a Toeplitz-Morse flow. It is well known that a Toeplitz-Morse flow is a strictly ergodic C. m. system [JK,W].

Define a sequence of Kakutani-Rokhlin partitions (ξ_t), $t \geq 0$, for a Toeplitz-Morse flow. Let $A_t g$ be the block obtained from A_t by placing the symbol g on the last position in A_t- (instead of the "hole"). Let E_t be the set of all pairs gh from $G \times G$ such that the block $(A_t g)(A_t h)$ appears in ω infinitely many times. Then for $i = 0, 1, ..., p_t - 1$ and $gh \in E_t$, we set

$$D^t_{i,gh} = \{x \in X : x[-p_t - i, p_t - i - 1] = (A_t g)(A_t h)\}, \quad t \geq 0. \tag{3.2}$$

The sets $D^t_{i,gh}$ are cylinder in X_ω and they generate the clopen topology on X_ω. Each pair $gh \in E_t$ defines the T-tower $\xi_{t,gh} = \{D^t_{i,gh} : i = 0, 1, ..., p_t - 1\}$. For fixed $t \geq 0$, the towers $\xi_{t,gh}$ form the K-R partition ξ_t of X_ω. This sequence is not nested because condition (iii) from **1.2** is not satisfied. Namely, $\cap_t B(\xi_t) = \{\omega_g : g \in G\}$ where $\omega_g[-p_t, p_t - 2] = A_t g A_t$, $t \geq 0$.

Now we apply the results of Section 2 to the Toeplitz-Morse flows. The sequence of K-R partitions (ξ_t) shows that this class of C. m. systems corresponds to the case **2.2(c)**. Therefore we have $k_t = |E_t|$, $h(i,t) = p_t$, $i = 1, 2, ..., k_t$. Find the matrix $M_t = (m_t(gh, g'h') : gh, g'h' \in E_t)$ that was introduced in **1.2**. Note that $m(gh, g'h') = 1$ if and only if $h = g'$ and the block $(A_t g)(A_t h)(A_t h')$ appears infinitely many times in ω or, equivalently, the triple (ghh') appears infinitely many times in ω_{t+1}. We may assume that every block a_{t+1} contains all pairs $gh \in E_t$ and all triples (ghh') from ω_{t+1} because one can take the block $a_t * \cdots * a_{t+k}$ instead of a_t, $t \geq 0$. This means that M_t is completely defined by the block a_{t+1}. It is easily seen that the partitions α_t and α'_t related to (ξ_t) are formed by the atoms $J(h) = Gh$ and $J'(h) = hG$, $h \in G$ respectively. Now we see that Theorem 2.2 can be used, in this case, to describe the topological full group of (X_ω, T).

Let (Δ, σ) be the odometer defined by the sequence (p_t). There exists a factor map $\pi : (X_\omega, T) \to (\Delta, \sigma)$ such that $\pi(D^t_{i,gh}) = D^t_i$ for all $i = 0, 1, ..., p_t - 1$, $t \geq 0$, and $gh \in E_t$ [W]. We showed in **2.2(d)** that every homeomorphism $\gamma \in [[\sigma]]$ is completely defined by a pair (ρ, ε). In Remark 2.3, we introduced the map $P_\pi : [[\sigma]] \to [[T]]$. In this case, $P_\pi(\gamma) \in [[T]]$ is determined by $(L^+, L^-, \rho^+, \rho^-, \rho(i))$ so that $L^+ = \{0 \leq j \leq p_t - 1 : \rho(j) < j,\ \varepsilon_j = 1\}$, $L^- = \{0 \leq j \leq p_t - 1 : \rho(j) > j,\ \varepsilon_j = -1\}$, $\rho^+ = \rho|_{L^+}$, $\rho^- = \rho|_{L^-}$, and $\rho(i)(j) = \rho(j)$, $j \notin L^+ \cup L^-$, $i = 1, ..., k_t$.

It is interesting to know the values of the unique T-invariant measure μ on the cylinder sets $D^t_{i,gh}$ defined in (3.2). Let us denote

$$\mathrm{fr}(g, a_t) = \frac{1}{\lambda_t} |\{0 \leq i \leq \lambda_t - 2 : a_t[i] = g\}|,$$

$$\mathrm{fr}(gh, a_t) = \frac{1}{\lambda_t} |\{0 \leq i \leq \lambda_t - 3 : a_t[i] = g,\ a_t[i+1] = h\}|, \quad t \geq 0.$$

Then for $t \geq 0$ one can compute

$$\mathrm{fr}(g, \omega_t) = \sum_{s=t}^{\infty} \frac{\mathrm{fr}(g, a_s)}{p_t^{(s)}}, \quad p_t^{(s)} = \lambda_t \cdots \lambda_s, \quad s \geq t,$$

$$\mathrm{fr}(gh,\omega_t) = \mathrm{fr}(gh,a_t) + \frac{\mathrm{fr}(h,\omega_{t+1})}{\lambda_t}(\chi_{a_{t+1}[\lambda_{t+1}-2]=g}) + \frac{\mathrm{fr}(g,\omega_{t+1})}{\lambda_t}(\chi_{a_{t+1}[0]=h})$$

where χ is the indicator function. Finally,

$$q_t(gh) = \mu(D^t_{i,gh}) = \frac{1}{p_t}\mathrm{fr}(gh,\omega_{t+1}), \quad gh \in E_t. \tag{3.3}$$

If we denote by $\Lambda_\omega = \Lambda(X_\omega, T)$ the set of values of μ on all clopen subsets in X_ω, then we get that $\Lambda_\omega = \cup_t \Lambda^t_\omega$ where $\Lambda^t_\omega = \{\sum_{gh\in E_t} |I(gh)| q_t(gh) : I(gh)$ is any subset in $(0,1,...,p_t - 1)\}$. It follows from (3.3) and [GPS1, Theorem 2.2] that the set $\{q_t(gh) : t \geq 0, gh \in E_t\}$ defines the class of C. m. systems orbit equivalent to (X_ω, T). If all $q_t(gh)$ are in $\mathbb{Q}$ then the Toeplitz-Morse flow is orbit equivalent to an odometer. Otherwise, if some $q_t(gh)$ is irrational, then the Toeplitz-Morse flow is orbit equivalent to a Denjoy homeomorphism. On the other hand, (Δ, σ) and (X_ω, T) are not conjugate because (Δ, σ) is the maximal equicontinuous factor of (X_ω, T).

3.3. Chacon flow. We remind briefly the definition of Chacon flow. For this, we start with the sequence $\{B_t\}$ of blocks over two symbols $(0, s)$:

$$B_0 = 0, \quad B_{t+1} = B_t B_t s B_t, \quad t > 0.$$

Then $|B_t| = \frac{1}{2}(3^{t+1} - 1) = r_t$. Let ω be a one-sided sequence defined by blocks B_t as follows: $\omega[0, r_t - 1] = B_t,\ t \geq 0$. As in the case of Toeplitz-Morse flows, we take the subset $Y \subset \{0,1\}^{\mathbb{Z}}$ which is the closure of T-orbit of ω with respect to the left shift. The C. m. system (Y, T) is called the *Chacon flow.*

For $t > 0$, we denote

$$D^t_{00} = \{x \in Y : x[-r_t, 2r_t - 1] = B_t B_t B_t\},$$

$$D^t_{s0} = \{x \in Y : x[-r_t - 1, 2r_t - 1] = B_t s B_t B_t\},$$
$$D^t_{0s} = \{x \in Y : x[-r_t, 2r_t] = B_t B_t s B_t\},$$
$$D^t_{ss} = \{x \in Y : x[-r_t - 1, 2r_t] = B_t s B_t s B_t\}. \tag{3.4}$$

Let $D_t = \cup_{p,q=0,s} D^t_{pq}$. Take the K-R partition ξ_t built by the base B_t and the return time function as in **1.2**. Then ξ_t has four towers $\xi_t(pq)$ corresponding the sets D^t_{pq} such that $h(00,t) = h(s0,t) = r_t$ and $h(0s,t) = h(ss,t) = r_t + 1$. It follows from these definitions that the sequence $(\xi_t),\ t \geq 0$, satisfies all conditions of Theorem 2.2. One can see that (ξ_t) is not nested because $\cap_t D_t = \{\omega_1, \omega_2\}$ where

$$\omega_1[-r_t, r_t - 1] = B_t B_t, \qquad \omega_2[-r_t - 1, r_t - 1] = B_t s B_t, \quad t \geq 0.$$

It is known that (Y, T) is strictly ergodic and the values of the unique T-invariant measure ν on the sets $D^t_{ij},\ t \geq 0$ are the following:

$$\nu(D^t_{00}) = \nu(D^t_{ss}) = \frac{1}{3^{t+2}},$$

$$\nu(D^t_{0s}) = \nu(D^t_{s0}) = \frac{2}{3^{t+2}}$$

Therefore the set $\Lambda(Y,T)$ (the set of values of ν on clopen subsets) is a subset in $\mathbb{Q}^+$ and completely determined by $\{3^{-t-2} : t = 0,1,...\}$. Then $\Lambda(Y,T)$ defines the class of odometers which are orbit equivalent to the Chacon flow.

Taking (ξ_t) as above, one can easily point out the matrix M_t and partitions $\alpha_t,\ \alpha'_t$ introduced in **1.2** (in fact, they do not depend on t). We get $J_1 = \{00, s0\}, J_2 = \{0s, ss\}$ and $J'_1 = \{00, 0s\}, J'_2 = \{s0, ss\}$. Then $\alpha_t = (J_1, J_2),\ \alpha'_t = (J'_1, J'_2)$ and $m(J_1, J'_1) = m(J_2, J'_2) = 1,\ m(J_1, J'_2) = m(J_2, J'_1) = 0$. Let us note that there are three cycles for the Chacon flow: $\{00\}, \{ss\}$ of the length 1 and $\{0s, s0\}$ of the length 2.

We are going to show that the Chacon flow (Y,T) does not satisfy the condition of Theorem 2.6. This means that there are two clopen subsets of the same measure which are not $[[T]]$-equivalent. Take (ξ_t) as above and denote the ξ_t-towers by $\xi_t(00), \xi_t(s0), \xi_t(0s), \xi_t(ss)$ according to (3.4). If A is a clopen set, then A a ξ_t-set for some t. One can compute that for $k = 1, 2, ...$

$$\begin{aligned}
N_{00}^{t+k}(A) &= \frac{1}{2}(3^{k-1}+1)N_{00}^t(A) + 3^{k-1}(N_{s0}^t(A) + N_{0s}^t(A)) + \frac{1}{2}(3^{k-1}-1)N_{ss}^t(A),\\
N_{0s}^{t+k}(A) &= \frac{1}{2}(3^{k-1}+1)(N_{00}^t(A) + N_{ss}^t(A)) + (3^{k-1}-1)N_{s0}^t(A) + 3^{k-1}N_{0s}^t(A)),\\
N_{s0}^{t+k}(A) &= 3^{k-1}(N_{00}^t(A) + N_{0s}^t(A)) + (3^{k-1}+1)N_{s0}^t(A) + \frac{1}{2}(3^{k-1}-1)N_{ss}^t(A)),\\
N_{ss}^{t+k}(A) &= \frac{1}{2}(3^{k-1}-1)N_{00}^t(A) + 3^{k-1}(N_{s0}^t(A) + N_{0s}^t(A)) + \frac{1}{2}(3^{k-1}+1)N_{ss}^t(A).
\end{aligned}$$

Take $A = D_{00}^t,\ B = D_{ss}^t$. Then we get that

$$\begin{aligned}
N_{00}^{t+k}(A) &= \frac{1}{2}(3^{k-1}+1),\\
N_{0s}^{t+k}(A) &= \frac{1}{2}(3^{k-1}+1),\\
N_{s0}^{t+k}(A) &= 3^{k-1},\\
N_{ss}^{t+k}(A) &= \frac{1}{2}(3^{k-1}-1)
\end{aligned}$$

and

$$\begin{aligned}
N_{00}^{t+k}(B) = N_{s0}^{t+k}(B) &= \frac{1}{2}(3^{k-1}-1),\\
N_{0s}^{t+k}(B) = N_{ss}^{t+k}(B) &= \frac{1}{2}(3^{k-1}+1).
\end{aligned}$$

Therefore we can see that (2.7) fails for the Chacon flow (Y,T).

3.4. Grillenberger flow. Let $S = \{0,1,...,s-1\},\ s \geq 3$. We define inductively a family $\mathcal{A}_t$ of blocks over S. Every $\mathcal{A}_t$ consists of m_t blocks $B_0^t, ..., B_{m_t-1}^t$ of the same length l_t. For $t = 0$ we set $\mathcal{A}_0 = \{0,1,...,s-1\}$, i.e. $B_i^0 = i,\ i = 0,1,...,s-1$, and $m_0 = s,\ l_0 = 1$. Assume that $\mathcal{A}_t$ is defined. Let ρ be a permutation of the set $\{0,1,...,m_t-1\}$. We set up

$$B_\rho^{t+1} = B_{\rho(0)}^t B_{\rho(1)}^t \cdots B_{\rho(m_t-1)}^t,$$

and then $\mathcal{A}_{t+1}$ is formed by all such blocks B_ρ^{t+1}. We get $m_{t+1} = m_t!,\ l_{t+1} = l_t m_t$. For every $t \geq 0$, take the blocks $L_t,\ F_t \in \mathcal{A}_t$ such that

$$L_{t+1} = L_t B_{\rho_1(1)}^t \cdots B_{\rho_1(m_t-1)}^t, \quad F_{t+1} = B_{\rho_2(0)}^t \cdots B_{\rho_2(m_t-2)}^t F_t. \tag{3.4}$$

Now we can define a two-sided sequence ω over S as follows:

$$\omega[-l_t, l_t - 1] = F_t L_t, \quad t \geq 0.$$

It follows from (3.4) that ω is well-defined. Therefore, one can take the closure Z of T-orbit of ω where T is the left shift on the space of all two-sided sequences over S. Then, we get a strictly ergodic Cantor minimal system (Z, T) which is called a *Grillenberger flow* [Gr].

Define now a sequence ξ_t of K-R partitions for (Z, T). For fixed $t \geq 0$, denote $\bar{j} = (j_1 j_2 j_3)$ where $j_k \in \{0, 1, ..., m_t - 1\}, k = 1, 2, 3$, and at least two numbers from j_1, j_2, j_3 are different. If E_t denotes the number of all such $\bar{j}$, then $|E_t| = m_t^2(m_t - 1)$. The sets

$$D^t_{i,\bar{j}} = \{x \in Z : x[-l_t - i, 2l_t - i - 1] = B^t_{j_1} B^t_{j_2} B^t_{j_3}\}, \quad i = 0, 1, ..., l_t - 1. \tag{3.5}$$

are the atoms of ξ_t. The partitions α_t and α'_t are formed by atoms $J = J(j_2, j_3) = \{\bar{j} = (j_1 j_2 j_3) : 1 \leq j_1 \leq m_t\}$ and $J' = J'(j_1, j_2) = \{\bar{j} = (j_1 j_2 j_3) : 1 \leq j_3 \leq m_t\}$ respectively. The refining partitions $\xi_t,\ t \geq 0$, generate the clopen topology on Z. It is easy to verify that the conditions of Theorem 2.2 are satisfied in this case. The tower $\xi_t(\bar{j})$ defined by (3.5) is enumerated by $\bar{j}$. Therefore all towers have the same height l_t. We conclude that the Grillenberger flow corresponds to the case **2.2(c)**. Remark that the sequence of K-R partitions (ξ_t) is far to be nested. It is easy to construct ω in such a way that the set $\cap_t B(\xi_t)$ would have uncountable many points.

4 Topologies on $\mathrm{Homeo}(X)$

4.1. Definition and properties of uniform topology. Let Ω be a compact metric space and denote by $\mathrm{Bor}(\Omega)$ the set (group) of all one-to-one Borel maps of Ω onto itself. We will introduce here a new topology τ_u on $\mathrm{Bor}(\Omega)$. Note that our definition is inspired by the notion of uniform metric on the set of all nonsingular automorphisms of a measure space. It follows that the topology τ_u generates the relative topology (denoted again by τ_u) on the set of all homeomorphisms $\mathrm{Homeo}(\Omega) \subset \mathrm{Bor}(\Omega)$.

Let $M_1(\Omega)$ be the set of all Borel probability measures on Ω. It is known that $M_1(\Omega)$ is a convex compact metric space.

Definition 4.1. The *uniform topology* τ_u on $\mathrm{Bor}(\Omega)$ is defined by the family $\mathcal{U} = \{U(T; \mu_1, ..., \mu_p; \varepsilon)\}$ of open neighborhoods (the base of topology): given $\varepsilon > 0,\ \mu_1, ..., \mu_n \in M_1(\Omega)$ and $T \in \mathrm{Bor}(\Omega)$, set

$$U(T; \mu_1, ..., \mu_n; \varepsilon) = \{S \in \mathrm{Bor}(\Omega) : \mu_i(E(S, T)) < \varepsilon,\ i = 1, ..., n\}$$

where $E(S, T) = \{x \in \Omega : Sx \neq Tx\} \cup \{x \in \Omega : S^{-1}x \neq T^{-1}x\}$.

It follows immediately from this definition that a sequence of Borel maps (T_n) is τ_u-converging to a Borel map S if and only if

$$\mu(E(T_n, S)) \to 0, \qquad n \to \infty, \tag{4.1}$$

for every $\mu \in M_1(\Omega)$.

The next statement characterizes converging sequences in the uniform topology.

Proposition 4.2. *A sequence (T_n) τ_u-converges to $S \in \mathrm{Bor}(\Omega)$ if and only if for every $x \in \Omega$ there exists $n(x) \in \mathbb{N}$ such that $T_n(x) = S(x)$ and $T_n^{-1}(x) = S^{-1}(x)$ for all $n \geq n(x)$.*

Proof. Assume that $T_n \xrightarrow{\tau_u} S$ as $n \to \infty$. Therefore, it follows from (4.1) that if we take $\mu = \delta_x,\ x \in \Omega$, then $\delta_x(E(T_n, S)) \to 0$. This means that $x \notin E(T_n, S)$ for all sufficiently large n.

Conversely, suppose that for given $x \in \Omega$ there exists $n(x)$ such that $T_n(x) = S(x),\ T_n^{-1}(x) = S^{-1}(x)$ for all $n \geq n(x)$. Define

$$\Omega_n = \{x \in \Omega : T_m(x) = S(x),\ \ T_m^{-1}(x) = S^{-1}(x),\ m = n, n+1, ...\},\ n \in \mathbb{N}.$$

Clearly, $\Omega_n \subset \Omega_{n+1}$ and $\cup_{n=1}^{\infty}\Omega_n = \Omega$. For each $\mu \in M_1(\Omega)$, we have $\mu(\Omega_n) \to 1$ as $n \to \infty$. Take a neighborhood $U(S; \mu_1, ..., \mu_p; \varepsilon)$ and find n_0 such that $\mu_i(\Omega_n) > 1 - \varepsilon,\ i = 1, ..., p$, when $n \geq n_0$. It is evident that $E(T_n, S) \subset \Omega - \Omega_n$ for all n. Then we get $\mu_i(E(T_n, S)) < \varepsilon$ for all $n \geq n_0$, i.e. $T_n \in U(S; \mu_1, ..., \mu_p; \varepsilon)$. □

Remark 4.3. (1) Show that $(\mathrm{Bor}(\Omega), \tau_u)$ is a *complete nonseparable topological group*. Let $(T_n) \subset \mathrm{Bor}(\Omega)$ be a sequence of Borel maps. It follows from the proof of Proposition 4.2 that the Cauchy condition for (T_n) is equivalent to

(F)
$$\bigcup_n X_n = \Omega \quad \text{and} \quad \bigcup_n T_n(X_n) = \Omega$$

where $X_n = \{x \in \Omega : T_n(x) = T_{n+1}(x) = \cdots\},\ \ n \in \mathbb{N}$. Note that (X_n) and $(T_n(X_n))$ are increasing sequences of Borel subsets. Define

$$T(x) = T_n(x) \ \text{ if } x \in X_n,\ n \in \mathbb{N}. \tag{4.2}$$

Then T=τ_u-$\lim_n T_n$ and obviously T is a one-to-one Borel map. To see that $\mathrm{Bor}(\Omega)$ (and $\mathrm{Homeo}(\Omega)$) is nonseparable, it suffices to consider the set of irrational rotations of the circle. The fact that $\mathrm{Bor}(\Omega)$ is a topological group can be proved straightforward.

(2) $\mathrm{Homeo}(\Omega)$ is *not closed* in $\mathrm{Bor}(\Omega)$ with respect to τ_u. To prove this statement we take a Cantor set X and let $X = E_0 \cup F_0$ be partitioned into two clopen subsets. Suppose that $E_0 = E_1' \cup E_1 \cup E_1''$ and $F_0 = F_1' \cup F_1 \cup F_1''$ are also partitioned into clopen subsets and so on. We get two sequences (E_n) and (F_n) such that $E_{n-1} = E_n' \cup E_n \cup E_n''$ and $F_{n-1} = F_n' \cup F_n \cup F_n''$. Suppose $\cap_n E_n = \{x\}$ and $\cap_n F_n = \{y\}$. Take a homeomorphism α_1 such that $\alpha_1(E_1') = F_1'$ and $\alpha_1^2 = \mathrm{id}$. Define

$$\gamma_1(x) = \begin{cases} \alpha_1(x) & \text{if } x \in E_1' \cup F_1' \\ x & \text{if } x \in E_1 \cup E_1'' \cup F_1 \cup F_1''. \end{cases}$$

Next, let a homeomorphism α_2 be taken such that $\alpha_2(E_2') = F_2'$ and $\alpha_2^2 = \mathrm{id}$. Define

$$\gamma_2(x) = \begin{cases} \gamma_1(x) & \text{if } x \notin E_1 \cup F_1 \\ \alpha_2(x) & \text{if } x \in E_2' \cup F_2' \\ x & \text{if } x \in E_2 \cup E_2'' \cup F_2 \cup F_2''. \end{cases}$$

This procedure allows us to define a sequence of homeomorphisms (γ_n) on X τ_u-converging to a Borel map γ. It follows from the construction that x and y are the points of discontinuity of γ.

Note that if T is a minimal homeomorphism of X, then one can slightly change the construction (more precisely, the choice of E'_n and F'_n) in such a way that $\alpha_n : E'_n \to F'_n$ (and therefore γ_n) would be taken from $[[T]]$, $n \in \mathbb{N}$.

(3) Let a sequence $(T_n) \subset \mathrm{Homeo}(\Omega)$ converges to T in the uniform topology τ_u (i.e. (T_n) satisfies condition (F)). Then $T \in \mathrm{Homeo}(\Omega)$ if and only if the following condition is true:

(C) for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, x' \in X_n$, $n \in \mathbb{N}$, with $d(x, x') < \delta$ one has $d(T_n(x), T_n(x')) < \varepsilon$.

(4) Let (X, T) be a C. m. system and let $\gamma_n \in [T]$, $n \in \mathbb{N}$, be a sequence of homeomorphisms τ_u-converging to $\gamma \in \mathrm{Homeo}(X)$, then $\gamma \in [T]$, i.e. $[T]$ is *closed* in $(\mathrm{Homeo}(X), \tau_u)$. It follows immediately from (4.2).

Example 4.4. Let (Δ, σ) be an odometer such that $\lambda_t > 2$ (see Section 3 for notations). We will construct a sequence $\{S_n\}_{n=1}^{\infty} \subset [[\sigma]]$ that τ_u-converges to a homeomorphism $\gamma \in [\sigma]$. For this, take a permutation $\bar{\rho}_t$ of the set $(0, 1, ..., \lambda_t - 1)$ such that $\bar{\rho}_t(\lambda_t - 1) = \lambda_t - 1$, for all $t \geq 0$. Let us define inductively a sequence $\{\rho_t\}_t$ of permutations of $(0, 1, ..., p_t - 1)$. Set $\rho_0 = \bar{\rho}_0$, and suppose that ρ_i is determined for $i = 1, 2, ..., t$. To define ρ_{t+1}, we note that every $0 \leq j \leq p_{t+1} - 1$ can be uniquely written as $j = qp_t + i$ where $0 \leq q \leq \lambda_{t+1} - 1$ and $0 \leq i \leq p_t - 1$. Set

$$\rho_{t+1}(j) = \begin{cases} qp_t + \rho_t(i) & \text{if } i = 0, 1, ..., p_t - 2 \\ \bar{\rho}_{t+1}(q)p_t + p_t - 1 & \text{if } i = p_t - 1 \end{cases}$$

According to **2.2(d)**, a compatible pair (ρ_t, ε^t) defines a homeomorphism S_t from $[[\sigma]]$. Take a vector $\varepsilon^{t+1} = (\varepsilon_0^{t+1}, ..., \varepsilon_{p_{t+1}-1}^{t+1})$ with $\varepsilon_i^{t+1} \in \{-1, 0, 1\}$, $i = 0, 1, ..., p_{t+1} - 1$, which is compatible with ρ_{t+1} and $\varepsilon_j^{t+1} = \varepsilon_i^t$ where $j = qp_t + i$ and $0 \leq i \leq p_t - 2$. Thus, we have constructed a homeomorphism S_{t+1}. We state that $\{S_t\}_{t=0}^{\infty}$ satisfies (F) amd (C) of Remark 4.3 and therefore τ_u-converges to some $\gamma \in [\sigma]$. To check this fact, one can show that if $X_n = \{x \in \Delta : S_n(x) = S_{n+i}(x),\ i = 1, 2, ...\}$, then

$$X_n = \bigcup_{i=0}^{p_n - 2} (D_i^n \cup \{x_0\})$$

where $x_0 = \sum_{t=0}^{\infty}(\lambda_t - 1)p_{t-1}$. In fact, it follows from the construction that $S_n(x_0) = x_0$ for all n, and if $x \in D_i^n$, $i = 0, 1, ..., p_n - 2$, then $S_{n+1}(x) = S_n(x)$. The proof of property (C) is left to the reader.

4.2. $[[T]]$ **is τ_u-dense in** $[T]$. Let (X, T) be a Cantor minimal system. In Remark 4.3 we have shown that a sequence of homeomorphisms γ_n, $n \in \mathbb{N}$, is τ_u-converging to a homeomorphism if and only if (γ_n) satisfies conditions (F) and (C). If in addition γ_n are taken from $[T]$ (or $[[T]]$), then γ belongs to $[T]$. In this subsection, we prove that every homeomorphism from $[T]$ is a limit of a sequence of homeomorphisms from $[[T]]$.

Theorem 4.5. *Let (X,T) be a minimal Cantor system. Given $\gamma \in [T]$, there exists sequence (γ_s), $\gamma_s \in [[T]]$, which τ_u-converges to γ.*

Proof. In the proof, we will use the notations from Section 2. Because γ is taken from $[T]$, then the sets $X_j = \{x \in X : \gamma(x) = T^j(x)\}$, $j \in \mathbb{Z}$, are closed, pairwise disjoint, and satisfy the conditions

$$\bigcup_{j\in\mathbb{Z}} X_j = X, \quad \bigcup_{j\in\mathbb{Z}} T^j(X_j) = X.$$

Denote $Y_s = \cup_{j=-s}^{s} X_j$, $s = 0, 1, \ldots$. Then $Y_s \subset Y_{s+1}$ and $\cup_{s\in\mathbb{N}} Y_s = X$. Let (ξ_t) be a refining sequence of K-R partitions that satisfies the conditions of Theorem 2.2 (we have noted above that such a sequence always exists). Because X_j (resp. $\gamma(X_j)$) is a clopen subset of Y_s (resp. $\gamma(Y_s)$), then for every s one can find sufficiently large $t = t(s)$ such that every $X_j \subset Y_s$ (resp. $\gamma(X_j) \subset \gamma(Y_s)$) is a union of some sets $D^t_{q,p} \cap Y_s$ (resp. $D^t_{q,p} \cap \gamma(Y_s)$) and $|s| \le h(t)$. Here we use the notations of Theorem 2.2 for atoms of ξ_t. Set

$$I_{s,t} = \{(q,p) : D^t_{q,p} \cap Y_s \neq \emptyset\}, \quad \hat{I}_{s,t} = \{(q,p) : D^t_{q,p} \cap \gamma(Y_s) \neq \emptyset\}.$$

Let $Z_{s,t}$ and $\hat{Z}_{s,t}$ be the ξ_t-hull of Y_s and $\gamma(Y_s)$ respectively, i.e.

$$Z_{s,t} = \bigcup_{(q,p)\in I_{k,t}} D^t_{q,p}, \quad \hat{Z}_{s,t} = \bigcup_{(q,p)\in \hat{I}_{k,t}} D^t_{q,p}.$$

Then $I_{s,t}$ and $\hat{I}_{s,t}$ are divided into the disjoint sets

$$I^j_{s,t} = \{(q,p) \in I_{s,t} : D^t_{q,p} \cap Y_s \subset X_j\}, \ -s \le j \le s,$$

and

$$\hat{I}^j_{s,t} = \{(q,p) \in \hat{I}_{s,t} : D^t_{q,p} \cap \gamma(Y_s) \subset \gamma(X_j)\}, \ -s \le j \le s,$$

respectively. Denote by $Z^j_{s,t}$ and $\hat{Z}^j_{s,t}$ the subsets of $Z_{s,t}$ and $\hat{Z}_{s,t}$ corresponding to $I^j_{s,t}$ and $\hat{I}^j_{s,t}$. Clearly, $Z^j_{s,t} \supset X_j$ and $\hat{Z}^j_{s.t} \supset \gamma(X_j)$. Note that for every $(q,p) \in I_{s,t}$ one can find a uniquely defined $j = j(q,p,t)$ such that γ coincides with T^j on $D^t_{q,p} \cap Y_s$, i.e. $D^t_{q,p} \cap Y_s \subset X_j$. Similarly, we can define $j' = j'(q,p,t)$ such that $\gamma^{-1}(x) = T^{j'}(x)$ for $x \in D^t_{q,p} \cap \gamma(Y_s)$.

The sets $T^j(Z^j_{s,t})$ and $T^{-j}(\hat{Z}^j_{s,t})$, $|j| \le s$, are clopen and therefore one can take sufficiently large $\tau = \tau(s) \ge t(s)$ such that they become ξ_τ-sets. To construct a homeomorphism from $[[T]]$ approximating γ, we will use the same arguments as in the proof of Theorem 2.2.

Let α_t and α'_t be the partitions of $\{1, 2, ..., k_t\}$ defined by ξ_t as in Section 1. Define the sets $I^j_{s,\tau}, \hat{I}^j_{s,\tau}$ and $Z^j_{s,\tau}, \hat{Z}^j_{s,\tau}$ as it was done above for ξ_t. Note that every set $Z^j_{s,\tau}$, $|j| \le s$, can be written as a union of atoms from ξ_τ. In fact, if $J \in \alpha$, $J' \in \alpha'$ and $j > 0$, then

$$Z^j_{s,t} = (\bigcup_{(q,p)\in Q_j(0)} D^\tau_{q,p}) \cup (\bigcup_{(r,J)\in Q_j(1)} F_1(r,J)). \tag{4.3}$$

If $j < 0$, then

$$Z^j_{s,t} = (\bigcup_{(q,p)\in Q_j(0)} D^\tau_{q,p}) \cup (\bigcup_{(r',J')\in Q_j(2)} F_2(r',J')) \tag{4.4}$$

where

$$Q_j(0) = \{(q,p) : 0 \le j+q \le h(p,\tau)-1\}, \tag{4.5}$$

$$Q_j(1) = \{(r,J) : \text{there exists } D^\tau_{q,p} \subset F_1(r,J) \cap Z^j_{s,t} \text{ and } j+q \ge h(p,\tau)\}, \tag{4.6}$$

$$Q_j(2) = \{(r',J') : \text{there exists } D^\tau_{q,p} \subset F_2(r',J') \cap Z^j_{s,t} \text{ and } j+q < 0\}. \tag{4.7}$$

In a similar way one can decompose $\hat{Z}^j_{s,t}$. For this, we define $\hat{Q}_j(0), \hat{Q}_j(1)$ and $\hat{Q}_j(2)$ replacing $Z^j_{s,t}$ in (4.5) - (4.7) by $\hat{Z}^j_{s,t}$. Then $\hat{Z}^j_{s,t}$ can be written using formulas analogous to (4.3), (4.4).

Let for definiteness $j > 0$. Assume that $(r,J) \in Q_j(1)$, i.e. $F_1(r,J) \cap Z^j_{s,t} \neq \emptyset$. Then it follows from Theorem 2.6 that $F_1(r,J) \subset Z^j_{s,t}$ because $T^j(Z)^j_{s,t}$ and $Z^j_{s,t}$ are ξ_τ-sets.

Since ξ_τ refines ξ_t, then some atoms $D^\tau_{q,p} \subset Z^j_{s,t}$ can lie in $X - Y_s$. Therefore we have to consider only those atoms of ξ_τ that intersect Y_s. Define

$$W^j_{s,t} = (\bigcup_{(q,p)\in R_j(0)} D^\tau_{q,p}) \cup (\bigcup_{(r,J)\in R_j(1)} F_1(r,J))$$

where $R_j(0) = Q_j(0) \cap I^j_{s,\tau}$ and $R_j(1) = \{(r,J) \in Q_j(1) : \text{there exists } D^\tau_{q,p} \subset F_1(r,J) \text{ such that } D^\tau_{q,p} \cap Y_s \neq \emptyset\}$.

The case $j < 0$ can be considered similarly. If $j = 0$, then $W^0_{s,t} = Z^0_{s,\tau}$. It is clear that

$$Z^j_{s,t} \supset W^j_{s,t} \supset Z^j_{s,\tau} \supset X_j. \tag{4.8}$$

The set $\hat{W}^j_{s,t}$ is determined by the same way as $W^j_{s,t}$ if one takes $\hat{Z}^j_{s,t}$ and $\gamma(Y_s)$ instead of $Z^j_{s,t}$ and Y_s. We have

$$\hat{Z}^j_{s,t} \supset \hat{W}^j_{s,t} \supset \hat{Z}^j_{s,\tau} \supset \gamma(X_j). \tag{4.9}$$

It follows from the construction that $\{W^j_{s,t}\}_j$ and $\{\hat{W}^j_{s,t}\}_j$ are two families of disjoint clopen subsets and $T^j(W^j_{s,t}) = \hat{W}^j_{s,t}$. For $|j| \le s$, set

$$\gamma_s(x) = T^j(x), \quad x \in W^j_{s,t}.$$

In such a way, γ_s maps $Z_s = \cup_{|j|\le s} W^j_{s,t}$ onto $\hat{Z}_s = \cup_{|j|\le s}\hat{W}^j_{s,t}$. By [Lemma 3.3, GPS2], γ_s can be extended to a homeomorphism from $[[T]]$ (denoted again by γ_s) determined on all X. It follows from (4.8) and (4.9) that the sequence (γ_s) converges to γ in the uniform topology. □

4.3. Topologies τ_w and τ_{uw}. The most known and studied topology on Homeo(X) is the weak topology τ_w. It can be defined by the metric

$$p(T,S) = \sup_{x\in X} d(T(x),S(x)) + \sup_{x\in X} d(T^{-1}(x),S^{-1}(x))$$

where $T,S \in$ Homeo(X) [GPS2,GW]. Then (Homeo$(X),\tau_w$) is a complete separable metric space. It was proved in [GPS2] that the full group $[T]$ is not closed in the weak topology (in contrast to the uniform topology τ_u). Generally speaking, the closure of $[[T]]$ in τ_w does not contain $[T]$. Below we give a statement that describes all C. m. systems having the property $\overline{[[T]]}^{\tau_w} \supset [T]$. On the other hand, (Homeo$(X),\tau_u$) is not complete (in contrast to the weak topology). Therefore it would be interesting to find a topology on Homeo(X) such that Homeo(X) is complete and $[T]$ is closed. It is natural to consider the topology τ_{uw} such that its base is formed by intersection of bases for τ_u and τ_w. In other words, a sequence of homeomorphisms (γ_n) is τ_{uw}-converging to a homeomorphism γ if simultaneously $\gamma_n \xrightarrow{\tau_u} \gamma$ and $p(\gamma_n, \gamma) \to 0$ when $n \to \infty$. Clearly, Homeo(X) is complete and $[T]$ is closed with respect to τ_{uw}. It follows from Theorem 4.5 that, in general, $\overline{[[T]]}^{\tau_{uw}}$ is a subset of $[T]$. In this subsection, we are going to show that the density of $[[T]]$ in $[T]$ with respect to τ_{uw} is equivalent to other topological properties of (X, T).

We first remark that more thorough analysis of the preceding theorem shows that the sequence (γ_s) found in the proof of Theorem 4.5 gives a kind of "weak approximation" of a homeomorphism $\gamma \in [T]$ on the arbitrary "large" clopen sets. The exact statement is given in Theorem 4.6 where we use the notations from the proof of Theorem 4.5.

Theorem 4.6. *Let γ and γ_s be as in Theorem 4.5. Then there exist clopen sets $Z_s \supset Y_s$ and $\hat{Z}_s \supset \gamma(Y_s)$ such that $\cup_s Z_s = \cup_s \hat{Z}_s = X$ and*

$$\lim_{s\to\infty}[\sup_{x\in Z_s} d(\gamma(x), \gamma_s(x)) + \sup_{x\in \hat{Z}_s} d(\gamma^{-1}(x), \gamma_s^{-1}(x))] = 0. \tag{4.10}$$

Proof. Let (ξ_t) be a refining sequence of K-R partitions as in Theorem 4.5. Let us denote for $J \in \alpha_t$ and $J' \in \alpha'_t$ (we use the notations from Theorem 4.5)

$$F_1^t(h_J - 1, J) = \bigcup_{p\in J} D^t_{h(p,t)-1,p}, \quad F_2^t(0, J') = \bigcup_{p'\in J'} D^t_{0,p'}.$$

Because (ξ_t) generates the clopen topology on X, we get that

$$\sup_{J\in\alpha_t}(\mathrm{diam}(F_1^t(h_J - 1, J))) + \sup_{J'\in\alpha'_t}(\mathrm{diam}(F_2^t(0, J'))) \to \infty \ \ (t \to \infty). \tag{4.11}$$

In the proof of Theorem 4.5 we have found a sequence $(\tau(s))$, $s \in \mathbb{N}$, such that $(\xi_{\tau(s)})$ is a subsequence of refining partitions generating topology. We will slightly change $\xi_{\tau(s)}$ to produce a new sequence $(\xi'_{\tau(s)})$. Determine $\xi'_{\tau(s)}$ as the partition of X with atoms $(\{D^\tau_{q,p} : (q, p) \in \cup_{|j|\le s} R_j(0)\},\ \{F_1^\tau(r, J) : (r, J) \in \cup_{0<j\le s} R_j(1)\},\ \{F_2^\tau(r', J') : (r', J') \in \cup_{-s\le j<0} R_j(2)\},\ X - Z_s)$. Here $\tau = \tau(s)$. Note that the diameter of every atom in ξ'_τ goes to 0 as $s \to \infty$ by (4.11). It follows that the sequence (ξ'_τ) also generates the clopen topology and Z_s is a ξ'_τ-set.

Fix some s and find $t = t(s)$ and $\tau = \tau(s)$ as in the proof of Theorem 4.5. In fact, τ can be taken so large that all atoms $D^t_{q,p}$ of ξ_t and sets $\gamma(D^t_{q,p})$, $\gamma^{-1}(D^t_{q,p})$ become ξ'_τ-sets. Now we prove that for every $D^t_{q,p}$

$$\gamma_s(D^t_{q,p} \cap Z_s) = \gamma(D^t_{q,p}) \cap \hat{Z}_s. \tag{4.12}$$

Let $C^\tau_{l,k}$ be an atom of ξ'_τ taken in the ξ'_τ-set $D^t_{q,p} \cap Z_s$. Then there exists a point $x \in C^\tau_{l,k}$ such that $\gamma_s(x) = \gamma(x)$. It follows that $\gamma_s(C^\tau_{l,k}) \cap \gamma(D^t_{q,p}) \neq \emptyset$ and $\gamma_s(C^\tau_{l,k}) \subset \gamma(D^t_{q,p})$. By the same reason, $\gamma_s(C^\tau_{l,k}) \subset \hat{Z}_s$. Therefore $\gamma_s(D^t_{q,p} \cap Z_s) \subset \gamma(D^t_{q,p} \cap Z_s)$. Using the same arguments for γ^{-1}, γ_s^{-1} and $\gamma(D^t_{q,p}) \cap \hat{Z}_s$, we obtain the opposite inclusion.

Note now that for $x \in D^t_{q,p} \cap Z_s$ we have that $d(\gamma(x), \gamma_s(x)) \leq \mathrm{diam}(\gamma(D^t_{q,p}))$ (the case of γ^{-1} and γ_s^{-1} is considered similarly). To finish the proof, we apply (4.11) and the fact that (ξ_t) generates the topology on X. $\square$

Remind that the set of coboundaries, B_T, is formed by all functions $g = f \circ T - f$ where $f : X \to \mathbb{Z}$ is a continuous function. Denote

$$Inf(X,T) = \{g \in C(X,\mathbb{Z}) : \int_X g(x)d\mu(x) = 0 \ \text{ for every } \ \mu \in M_1(T)\}$$

where $M_1(T)$ is the set of all T-invariant probability measures. Functions from $Inf(X,T)$ are called infinitesimal. Clearly, $Inf(X,T) \supset B_T$.

Definition 4.7. We say that a C. m. system (X,T) is *saturated* if any two clopen sets A and B from X such that $\mu(A) = \mu(B),\ \mu \in M_1(T)$, are $[[T]]$-equivalent.

Theorem 4.8. *Let (X,T) be a C. m. system. The following statements are equivalent.*

(i) (X,T) is saturated;
(ii) $Inf(X,T) = B_T$;
(iii) $\overline{[[T]]}^{\tau_w} \supset [T]$;
(iv) $\overline{[[T]]}^{\tau_w} = \overline{[T]}^{\tau_w}$;
(v) $\overline{[[T]]}^{\tau_{uw}} = [T]$.

Proof. The equivalence of (iii) and (iv) is evident. Suppose (iii) is true. For two clopen sets A and B from X such that $\mu(A) = \mu(B),\ \mu \in M_1(T)$, find $\gamma \in [T]$ such that $\gamma(A) = B$ [Proposition 2.6, GW]. Then take a sequence (γ_s) that τ_w-converges to γ where $\gamma_s \in [[T]]$ for all s. By [Remark 1.6, GPS2], there exists some N such that $\gamma_s(A) = \gamma(A)$ for all $s > N$. Therefore, (i) holds. Conversely, suppose (X,T) is saturated and take some γ in $[T]$. Let (ξ_t) be a sequence of partitions of X into clopen sets generating the clopen topology. Then one can construct $\gamma_t \in [[T]]$ such that $\gamma_t(D) = \gamma(D)$ and $\gamma_t^{-1}(\gamma(D)) = D$ for every $D \in \xi_t$. Then (γ_t) converges to γ in τ_w that proves (iii).

Assume now that (ii) is true. If A and B are two clopen subsets from X such that $\mu(A) = \mu(B),\ \mu \in M_1(T)$, then $\chi_A - \chi_B \in Inf(X,T)$ and therefore is a T-coboundary. It follows from [Lemma 3.3, GPS2] that A and B are $[[T]]$-equivalent. Conversely, assume that (X,T) is saturated and take $f \in Inf(X,T)$. It follows from [GPS2, GW] that there exists $\gamma \in [T]$ such that $Inf(X,T) = B_\gamma$. Then $f(x) = g(\gamma^{-1}(x)) - g(x)$ where $g(x) = \sum_{n\in I} c_n \chi_{E_n}(x)$, $|I| < \infty$. We get that

$$\begin{aligned} g(\gamma^{-1}(x)) - g(x) &= \sum_{n\in I} c_n(\chi_{\gamma(E_n)}(x) - \chi_{E_n}(x)) \\ &= \sum_{n\in I} c_n(\chi_{\sigma_n(E_n)}(x) - \chi_{E_n}(x)) \end{aligned}$$

where $\sigma_n \in [[T]]$ and $\sigma_n(E_n) = \gamma(E_n)$. Since every $\chi_{\sigma_n(E_n)}(x) - \chi_{E_n}(x)$ is represented as a finite sum of T-coboundaries, then $f(x) = g(\gamma^{-1}(x)) - g(x)$ is a T-coboundary.

Clearly, (v) implies (iii) since τ_{uw} is stronger than τ_w. Next assume that (X,T) is saturated and show that (v) holds. Take $\gamma \in [T]$ and find a sequence (γ_s) from $[[T]]$ such that (γ_s) τ_u-converges to γ. By Theorem 4.6, we get that there exists a subsequence (again denoted by (γ_s)) satisfying (4.10) and (4.12). Having (γ_s), we are going to construct $(\tilde{\gamma}_s)$ such that $\tilde{\gamma}_s$ is still τ_u-converging to γ and furthermore is τ_w-converging to γ. Fix some s. It follows from (4.12) and the proof of Theorem 4.6 that for every $\mu \in M_1(T)$ and every atom $D^t_{q,p}$ of ξ_t

$$\mu(D^t_{q,p} \cap Z_s) = \mu(\gamma(D^t_{q,p}) \cap \hat{Z}_s)$$

and therefore $\mu(D^t_{q,p} - Z_s) = \mu(\gamma(D^t_{q,p}) - \hat{Z}_s)$. Take $\gamma'(t) \in [[T]]$ such that $\gamma'(t)(D^t_{q,p} - Z_s)) = \gamma(D^t_{q,p}) - \hat{Z}_s)$ where $D^t_{q,p}$ does not belong to Z_s. Define

$$\tilde{\gamma}_s(x) = \begin{cases} \gamma_s(x) & \text{if } x \in Z_s \\ \gamma'(t)(x) & \text{if } x \in D^t_{q,p} - Z_s. \end{cases}$$

Thus, (4.10) and the fact that $\mathrm{diam}(\gamma(D^t_{q,p})) \to 0$ $(t \to \infty)$ imply τ_{uw}-convergence of $(\tilde{\gamma}_s)$ to γ. □

We note that the Chacon flow is not saturated. It follows from Theorem 2.6 and the computations at the end of subsection **3.3**. On the other hand, every C. m. system constructed as in Example 2.7 is saturated.

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