The automorphism and mapping class groups of a shift of finite type

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Topological dynamical systems

Today: A topological dynamical system TDS is a homeomorphism of a compact metrizable space, $T : X \to X$.

A homomorphism of TDSs $(X, T) \to (X', T')$ is a continuous map $f : X \to X'$ such that $T'f = fT$.

Here f is an isomorphism (topological conjugacy) if it is bijective. (Think of f as translating names of points, say from English to Spanish, but respecting all topological dynamical properties.)

Full shifts

The full shift on k symbols is a TDS (X_k, σ_k), defined with some "alphabet" set $\mathcal{A}$ of k symbols. Usually $\mathcal{A} = \{0, 1, \ldots k-1\}$.

X_k is the set of biinfinite sequences
$x = \ldots x_{-1} x_{-1} x_0 x_1 x_2 \ldots$
with each x_i in $\mathcal{A}$.

Let $\operatorname{dist}(x, y) = 1/(M+1)$, where $M = \min\{|n| : x_n \neq y_n\}$.
If $k > 1$, then X_k is a Cantor set.

σ_k is defined by the shift map σ:
$(\sigma x)_n = x_{n+1}$.

Subshifts

A TDS (X, T) is a subshift of (X_k, σ_k) if $X \subset X_k$ and $T = \sigma|X$.
(X, T) is a subshift of (X_k, σ_k) iff there is a set $\mathcal{W}$ of words on the alphabet such that X is the set of points x in X_k such that no word from $\mathcal{W}$ is a subword of x.

If $\mathcal{W}$ can be chosen to be finite, then the subshift is a *shift of finite type* (SFT). The SFTs are the building blocks of symbolic dynamics, important for dyamical systems, and with other applications.

Edge shifts

Suppose a directed graph $\mathcal{G}$ has n vertices, named $1, \ldots, n$. Then the adjacency matrix of $\mathcal{G}$ is the $n \times n$ matrix A where $A(i, j)$ = number of edges from i to j.

Given a square matrix A with nonnegative integer entries, let A be the adjacency matrix of a dir. graph $\mathcal{G}$ with edge set $\mathcal{E}$. Let X_A be the set of bisequences $x = \ldots x_{-1}x_0x_1 \ldots$ on alphabet $\mathcal{E}$ such that for all i, the terminal vertex of x_i is the initial vertex of x_{i+1}. The shift map $\sigma : X_A \to X_A$ defines an *edge shift*, an SFT. Every SFT is topologcially conjugate to some edge shift.

Mixing and irreducible shifts of finite type

Let A be a square matrix with nonnegative integer entries.

• A is *irreducible* if for every entry (i,j), there is some $k > 0$ such that $A^k(i,j) > 0$.
• A is *primitive* if there is some $k > 0$ such that $A^k(i,j) > 0$ for every entry (i,j).
• An irreducible matrix is *trivial* if it is a permutation matrix.

If A is a nontrivial primitive matrix, then σ_A is a mixing SFT. These SFTs behave qualitatively very much like nontrivial full shifts.

If A is a nontrivial irreducible matrix, then σ_A is a topologically transitive SFT. If also A is not primitive, then for some p, the domain X_A is the union of p cyclically moving disjoint subsets, and the restriction of the pth power of σ_A to any of them is a mixing SFT.

A general SFT is a union of disjoint irreducible SFTs and connecting orbits between them. So, the mixing SFTs are the central case to understand.

Block codes

Suppose (X, T) is a subshift. Let $\mathcal{W}_k(X) = \{x_1 \dots x_k : x \in X\}$, the set of words of length k occuring in points of X.

A homomorphism of subshifts $f : X \to Y$ is always definable by a block code. This means:

there are integers a, b, with $a \leq b$, and
a function $\Phi : \mathcal{W}_k(X) \to \mathcal{W}_0(Y)$, with $k = b - a + 1$,
such that
$(fx)_i = \Phi(x_{i+a} \dots x_{i+b})$, for all x and i.

THE AUTOMORPHISM GROUP OF σ_A

For a topological dynamical system (X, T), $\text{Aut}(T)$ is the group of automorphisms of T.

For any subshift (X, T), the group $\text{Aut}(T)$ is countable (there are only countably many block codes).

We will consider $\text{Aut}(\sigma_A)$.

STANDING CONVENTION: For the discussion of $\text{Aut}(\sigma_A)$, we assume A is primitive and nontrivial.

We will see $\text{Aut}(\sigma_A)$ is complicated.

The Marker Method

An homomorphism $U : X_A \to X_A$ might be defined without reference to a block code. Let's see a simple idea (with far reaching elaborations) which shows there are many automorphisms.

Example. For the full shift on three symbols $0, 1, 2$, $U(x)$ is obtained by replacing 12 with 01, wherever 12 occurs in x. e.g.

$$x = \dots\ 0\ \mathbf{1}\ \mathbf{2}\ 1\ 0\ 0\ \mathbf{1}\ \mathbf{2}\ \mathbf{1}\ \mathbf{2}\ 0\ \dots$$

$$U(x) = \dots\ 0\ \mathbf{0}\ \mathbf{1}\ 1\ 0\ 0\ \mathbf{0}\ \mathbf{1}\ \mathbf{0}\ \mathbf{1}\ 0\ \dots$$

This map U is well defined (but not an automorphism).

Also, e.g. : "replace 00 with 01" is not even well defined: because 00 can properly overlap itself (in 000), and the terminal symbol of 01 does not equal the initial symbol.

These problems are addressed by "markers" (introduced in [H]).

For example, given a permutation π in S_n, let $\mathcal{W}$ be a set of n words on symbols 0,1 of the same length. Define an automorphism U_π of the full 3 shift on symbols 0,1,2: for each W in $\mathcal{W}$, obtain Ux from x by replacing $2W$ with $2\pi(W)$, wherever W occurs in x. E.g. perhaps

$2W = 2000 \to 2010 = 2\pi(W)$.

The symbol 2 serves as a marker, preventing proper overlaps of words of this form $2W$, so the map is well defined. For permutations π and ρ, $U_{\pi\circ\rho} = U_\pi \circ U_\rho$. So, each U_π is an automorphism (having finite order), and the map $\pi \mapsto U_\pi$ embeds the symmetric group S_n into $\mathrm{Aut}(\sigma_3)$.

So every finite group embeds as a subgroup of $\mathrm{Aut}(\sigma_3)$.

That is just a start.

Subgroups of Aut(σ_A)

Elaborations and extensions of the marker method have been used to show Aut(σ_A) contains copies of a variety of groups, e.g.

- Free groups [BLR]
- Direct sum of countably many copies of $\mathbb{Z}$ [BLR]
- Every residually finite countable group which is a union of finite groups [KR]
- The fundamental group of any 2-manifold [KR].

The only obstructions known (to me) to embedding a countable group into Aut(σ_A):
– it must be residually finite
– every finitely generated subgroup must have solvable word problem (easy proof using block codes ...)

Problem: What are other obstructions?

Question [CFKP] Does SL($3, \mathbb{Z}$) embed into Aut(σ_A)?

Using the idea of distortion element from geometric group theory, [CFKP] show a group with logarithmic distortion (such as SL($k, \mathbb{Z}$) with $k \geq 3$) cannot embed in Aut(σ_A) for any zero entropy shift T.

Counting automorphisms

Consider $\mathrm{Bl}(\sigma_A, n)$, the set of block codes $X_A \to X_A$ with $(fx)_0$ determined by $x_1 \dots x_n$. Let $\mathrm{invBl}(\sigma_A, n)$ be the elements of $\mathrm{Bl}(\sigma_A, n)$ defining automorphisms.

What is the size of $\mathrm{invBl}(\sigma_A, n)$, compared to the size of $\mathrm{Bl}(\sigma_A, n)$?

For simplicity, we consider just σ_k, the full shift on k symbols. Then $|\mathrm{Bl}(\sigma_k, n)| = k^{(k^n)}$. So,

$\lim_n (1/n) \log\log|\mathrm{Bl}(\sigma_k, n)| = \lim_n (1/n) \log[\log(k)^{(k^n)}] = \lim_n (1/n) \log[(k^n)\log(k)] = \log k$

Kim and Roush proved $\lim_n (1/n) \log\log|\mathrm{invBl}(\sigma_k, n)| = \log k$.

PROBLEM. For a full shift σ_k, give a better asymptotic formula for $|\mathrm{invBl}(\sigma_k, n)|$.

The doubly exponential growth of the number of automorphisms given by codes of range n causes problems.

• It is difficult to do convincing computational experiments on properties of automorphisms of σ_A.

• It seems completely impractical to prove properties of automorphisms by induction on the range of a defining block code.

How to proceed?

How can we learn something about a complicated group, such as $\mathrm{Aut}(\sigma_A)$?

We look to guidance provided by The Bible:

BY THEIR ACTIONS, YE SHALL KNOW THEM .

There are two (and so far, essentially only two) actions of $\mathrm{Aut}(\sigma_A)$ which we have been able to learn from:

- Action on periodic points.

- Action on the dimensional module.

Action on periodic points

P_n := set of σ_A-periodic points of least period n.
For each n, P_n is finite and σ_A invariant.
Aut($\sigma_A|P_n$ is finite. For U in Aut(σ_A), let $U_n := U|P_n$.
Then $U \mapsto U_n$ defines a homomorphism from Aut(σ_A) into the finite group Aut($\sigma_A|P_n$).

Because the periodic points of σ_A are dense, the maps $U \mapsto U_n$ separate points. So, Aut(σ_A) is residually finite.

Contrast: for various subshifts T, Aut(T) is not residually finite:
• $\exists$ a minimal subshift T with Aut(T) containing a copy of $\mathbb{Q}$.
• Many reducible shifts (some SFTs, and many more – Salo and Schraudner ...) contain a copy of S_∞, the union of the symmetric groups S_n.
(S_n=permutations of $\{1, 2, \ldots, n\}$.)

Action on the dimension module

Suppose A is $k \times k$. Let G_A be the direct limit group
$\mathbb{Z}^k \to \mathbb{Z}^k \to \mathbb{Z}^k \to \mathbb{Z}^k \cdots$
where each arrow is the homomorphism given by $x \mapsto xA$.

A induces an automorphism $\hat{A} : G_A \mapsto G_A$. The pair $(G_A, \hat{A})$ is a presentation of $\mathcal{M}_A$, the *dimension module* of σ_A.

An automorphism of $\mathcal{M}_A$ is a group automorphism $G_A \mapsto G_A$ which commutes with $\hat{A}$ and respects a minor order condition we won't describe.

Example. $A = [2]$. $G_A \cong \mathbb{Z}[1/2]$. For $n \in \mathbb{Z}$, let $\phi_n : G_A \to G_A$ be $\phi_n : x \mapsto 2^n x$. The map $n \mapsto \phi_n$ defines a group isomorphism $\mathbb{Z} \to \mathrm{Aut}(\mathcal{M}_A)$.

Other examples give e.g.

$G_A \cong \mathbb{Z}[1/6] \oplus \mathbb{Z}^3$,
$\mathrm{Aut}(\mathcal{A}) \cong \mathbb{Z}^2 \oplus \mathrm{SL}(3, \mathbb{Z})$.

The groups $\mathrm{Aut}(\mathcal{M}_A)$ can be understood very concretely, though we won't have time for this.

Usually but not always, $\mathrm{Aut}(\mathcal{M}_A)$ is finitely generated.

The dimension representation

FACT: an element U of $\mathrm{Aut}(\sigma_A)$ induces an automorphism $\widehat{U}$ of $\mathcal{M}_A$. The rule $U \mapsto \widehat{U}$ defines a group homomorphism

$$\rho_A : \mathrm{Aut}(\sigma_A) \to \mathrm{Aut}(\mathcal{M}_A).$$

This homomorphism, describing the action of $\mathrm{Aut}(\sigma_A)$ on the module $\mathrm{Aut}(\mathcal{M}_A)$, is called the *dimension representation*.

Let $\mathrm{Aut}_0(\sigma_A)$ denote the kernel of the ρ_A (the group of *inert* automorphisms). This is the large, complicated part of $\mathrm{Aut}(\sigma_A)$. The possible actions of $\mathrm{Aut}_0(\sigma_A)$ on finite subsystems (or any proper subsystem) are, remarkably, completely understood.

The sign and gyration homomorphisms

Given n, let $x_1, \ldots, x_k$ be a set of representatives of the σ_A orbits of size n. An automorphism U of σ_A acts on P_n by a rule

$$U : x_i \mapsto \sigma^{m(i)}(x_j)$$

where $j = \pi(i)$, with π a permutation of $\{1, \ldots, k\}$.
We have homomorphisms

$\mathrm{sign}_n : U \mapsto \mathrm{sign}(\pi_U) \in \mathbb{Z}/2\mathbb{Z}$

$\mathrm{gy}_n : U \mapsto \sum_i m(i) \in \mathbb{Z}/n\mathbb{Z}$

The SGCC homomorphism

(SGCC stands for sign gyration compabitility condition.)

$\mathrm{sggc}_n : \mathrm{Aut}(\sigma_A) \to \mathbb{Z}/n\mathbb{Z}$
$\mathrm{sggc}_n = \mathrm{gy}_n + \Big(\sum_k \mathrm{sign}_{n/(2^k)} \Big)(n/2)$

The last sum is over integers $k \geq 1$ such that 2^k divides n. E.g.

$\mathrm{sggc}_n = \mathrm{gy}_n$ if n is odd,
$\mathrm{sggc}_{24} = \mathrm{gy}_{24} + (\mathrm{sign}_{12} + \mathrm{sign}_6 + \mathrm{sign}_3)\, 12$

For any $U \in \mathrm{Aut}(\sigma_A)$, $\mathrm{sggc}_n(U)$ is either
$\mathrm{gy}_n(U) \quad \text{or} \quad \mathrm{gy}_n(U) + n/2$.

The Factorization Theorem (Kim-Roush-Wagoner)

FACTORIZATION THEOREM For all n, there is a homomorphism $\gamma_n : \text{Aut}(\mathcal{M}_A) \to \mathbb{Z}/n\mathbb{Z}$ such that $\text{sggc}_n = \gamma_n \circ \rho_A$.

So, if U is in the kernel of the dim. representation ρ_A, then $\text{sggc}_n(U) = 0$ for all n.

This is a major obstruction to extending an automorphism of a subsystem of (X_A, σ_A) to an automorphism in the kernel of the dim. repn.
By constructions of several people – but especially, KRW – the sgcc = 0 constraint is the ONLY obstruction to extending an automorphism of a subsystem to an inert automorphism of σ_A.

There can be more obstructions to an automorphism extending to a composition of elements of finite order in $\mathrm{Aut}(\sigma_A)$. We know for some A that $\mathrm{Aut}_0(\sigma_A)$ is not generated by elements of finite order. (But possibly the finite order elements always generate a subgroup of finite index in $\mathrm{Aut}_0(\sigma_A)$.)

Mastery of actions on subsystems of X_A is generally not enough for global questions. For example,

PROBLEM. Suppose for all n that an automorphism U of σ_A permutes the orbits of size n by an even permutation. Must U be in the commutator of $\mathrm{Aut}(\sigma_A)$?

Isomorphism of Aut(σ_A) and Aut(σ_B)?

For all we know, there is such an isomorphism only if σ_B is topologically conjugate (isomorphic) to σ_A or $(\sigma_A)^{-1}$.

The only tool known to give examples of Aut(σ_A) and Aut(σ_B) not isomorphic is very crude:
Ryan's Theorem: the center of Aut(σ_A) is the powers of σ_A.

E.g., the 2-shift has no square root, but the 4-shift has a square root, so their automorphism groups are not isomorphic

PROBLEM Are the automorphism groups of the 2-shift and 3-shift isomorphic?

PROBLEM Is every group isomorphism $\mathrm{Aut}_0(\sigma_A) \to \mathrm{Aut}_0(\sigma_B)$ induced by a homeomorphism $X_A \to X_B$ which is a conjugacy to σ_B or $(\sigma_B)^{-1}$?

For the last problem, note a huge difference between it and the corresponding question for full groups of Cantor systems. There is a rich supply of full group elements which are the identity on large open sets. But here, points with dense orbits are dense; if a conjugacy is the identity on such a point, then it is the identity everywhere.

Flow equivalence

Let $T : X \to X$ be a homeomorphism of a compact metric space. (We are interested in $T = \sigma_A : X_A \to X_A$.)

The mapping torus of T is the quotient $Y(T)$ of $X \times \mathbb{R}$ by the identifications $(x, s+n) \sim (\sigma_A)^n(x), s)$, for all $x \in X_A$, $s \in \mathbb{R}$, $n \in \mathbb{Z}$.

$Y(T)$ is the image of $X \times [0, 1]$, under the identifications $(x, 1) \sim (T(x), 0)$.

There is a continuous $\mathbb{R}$ action (flow) on $X \times \mathbb{R}$, for which the time t map is $(x, s) \mapsto (x, s+t)$. This flow pushes down to a flow on the mapping torus (the “suspension flow”).

DEFN Two homeomorphisms S, T are *flow equivalent* if there is a homeomorphism $F : Y(S) \to Y(T)$ mapping flow orbits to flow orbits, preserving the direction of the flow.
(Such an F is called a flow equivalence.) There is a long history to the study of flow equivalence.

Let $\mathcal{F}(T)$ be the group of self flow equivalences of T. Let $\mathcal{F}_o(T)$ be the subgroup of homeos F isotopic to the identity in $\mathcal{H}(T)$.

DEFN The mapping class group of T, MCG(T), is the group $\mathcal{F}(T)/\mathcal{F}_0(T)$.

The mapping class group plays the role for flow equivalence that the automorphism group plays for topological conjugacy.

The mapping class group of a shift of finite type

From here an edge shift X_A, σ_A is assumed to be irreducible and nontrivial. (Every irreducible SFT is flow equivalent to a mixing SFT.)

We will consider MCG(σ_A), contrasted to Aut(σ_A) (always, for σ_A irreducible and nontrivial).

A fundamental tool is the following classification theorem.

THEOREM (Franks, after Bowen-Franks, Parry-Sullivan):

Nontrivial irreducible SFTs σ_A, σ_B are flow equivalent
iff
(1) the groups cok$(I - A)$ and cok$(I - B)$ are isomorphic and
(2) det$(I - A) = $ det$(I - B)$.

Properties of MCG(σ_A)

These are taken from joint work B-Chuysurichay unless indicated.

• Recall, every automorphism of a subshift can be defined by a block code. There is an analogous notion [BCE]: for a subshift, every element of its mapping class group has a representative defined by a "flow code". (This looks like a block code, with words in place of symbols.)

• MCG(σ_A) is a countable group.
(There are only countably many flow codes.)

• MCG(σ_A) is not residually finite.
(Contains a copy of S_∞.)

• The center of MCG(σ_A) is trivial.

• An automorphism of σ_A induces a flow equivalence. The corresponding homomorphism Aut(σ_A) $\to$ MCG(σ_A) has kernel equal to $< \sigma_A >$, the powers of the shift.

• If σ_B flow equivalent to σ_A, then MCG(σ_B) $\cong$ MCG(σ_A); so, a flow equivalence induces an embedding of Aut(σ_B)$/ < \sigma_B >$ into MCG(σ_A).

• Many elements of MCG(σ_A) cannot arise from automorphisms in this way (as automorphisms of return maps to cross sections).

Circles and extensions

Periodic points of σ_A give rise to circles in the mapping torus. MCG(σ_A) acts by permutations on $\mathcal{C}$, the countable set of circles in the mapping torus.

• The action of MCG(σ_A) on $\mathcal{C}$ by permutations is faithful.
• The action of MCG(σ_A) on $\mathcal{C}$ by permutations is n-transitive, for all n.

• [BCE] If $F : Y_1 \to Y_2$ is a flow equivalence from one subsystem of $Y(\sigma_A)$ to another, then F extends to a flow equivalence $Y(\sigma_A) \to Y(\sigma_A)$

The Bowen-Franks representation

• [B] Analogous to the dimension representation for $\text{Aut}(\sigma_A)$ is the "Bowen-Franks representation" , a group homomorphism $\beta_A : \text{MCG}(\sigma_A) \to \text{Aut}(\text{cok}(I - A))$.

In contrast to the dimension representation:

β_A is surjective, for all A.
The range group is finitely generated, for all A.

Let $\text{MCG}_0(\sigma_A)$ be the kernel of β_A: this is the big, mysterious part of $\text{MCG}_0(\sigma_A)$

Questions about the mapping class group

- Does the map $\pi : \mathcal{H}(\sigma_A) \to \mathrm{MCG}(\sigma_A)$ split?
(i.e. is there a subgroup which π maps bijectively to $\mathrm{MCG}(\sigma_A)$?)

- Is $\mathrm{MCG}_0(\sigma_A)$ simple? (I suspect, yes.)
- Is $\mathrm{MCG}_0(\sigma_A)$ equal to its commutator? generated by involutions? finitely generated?

- For σ_A and σ_B not flow equivalent, we know nothing at all about whether their mapping class groups are always/sometimes/never isomorphic as groups.
PROBLEM Is every group isomorphism $\mathrm{MCG}_0(\sigma_A) \to \mathrm{MCG}_0(\sigma_B)$ induced by a homeomorphism of mapping tori, $Y(\sigma_A) \to Y(\sigma_B)$.

- Is $\mathrm{MCG}(\sigma_A)$ sofic?

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