The almost Borel structure of surface diffeomorphisms, Markov shifts and their factors

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Dedicated to Roy Adler, in appreciation

Abstract

Extending work of Hochman, we study the almost-Borel structure, i.e., the non-atomic invariant probability measures, of symbolic systems and surface diffeomorphisms.

We first classify Markov shifts and characterize them as strictly universal with respect to a natural family of classes of Borel systems. We then study their continuous factors showing that a low entropy part is almost-Borel isomorphic to a Markov shift but that the remaining part is much more diverse, even for finite-to-one factors. However, we exhibit a new condition which we call 'Bowen type' which gives complete control of those factors.

This last result applies to and was motivated by the symbolic covers of Sarig. We find complete numeric invariants for Borel isomorphism of C^{1+} surface diffeomorphisms modulo zero entropy measures; for those admitting a totally ergodic measure of positive (not necessarily maximal) entropy, we get a classification up to almost-Borel isomorphism.

Keywords. ergodic theory, entropy, symbolic dynamics, surface diffeomorphism, Borel isomorphism, Markov shifts, factors, universality

Contents

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1 Introduction

Much of the richness of dynamical systems theory comes from understanding systems with respect to different structures (smooth, measurable, etc.). In this paper we are interested in the almost-Borel structure of surface diffeomorphisms. More precisely we study them as automorphisms of standard Borel spaces up to sets negligible for all invariant, nonatomic Borel probability measures, following Hochman [23] (see also [48]).

We analyze Markov shifts (generalizing [23] to the non-irreducible, non-mixing case) and especially their factors, both under continuous and what we call Bowen type factor maps. We finally show that this applies to Sarig's symbolic dynamics [45] of surface diffeomorphisms.

1.1 Surface diffeomorphisms

We consider surface diffeomorphisms which are C^{1+} smooth, i.e., with Hölder continuous derivative. Note that, in order to apply Sarig's symbolic dynamics [44], all surfaces in this paper are assumed to be C^∞ smooth. (We refer to Section 2 for definitions and background.) Our main result, Thm. 8.2, implies:

Theorem 1.1. *Any C^{1+}-diffeomorphism of a compact surface is Borel isomorphic to a countable state Markov shift, up to a subset of zero measure with respect to all ergodic measures[3] with positive entropy.*

[3] By measure we will (outside Appendix A) always mean invariant Borel probability measure.

We will deduce a classification involving the periods of ergodic measure-preserving systems (S, μ) defined as follows. Recall that the rational spectrum is:

$$\sigma_{\text{rat}}(S, \mu) := \{e^{2i\pi r} : r \in \mathbb{Q},\ \exists f \in L^2(\mu) \text{ non a.e. constant such that } f \circ S = e^{2i\pi r} f\}. \tag{1.1}$$

A positive integer p is a *period* if $e^{2i\pi/p} \in \sigma_{\text{rat}}(S, \mu)$. In Sec. 8.4, we will prove the following, using a classification of Markov shifts (Thm. 1.5 below).

Theorem 1.2. *Two C^{1+}-diffeomorphisms of compact surfaces are Borel isomorphic, up to a subset negligible with respect to all ergodic measures with positive entropy, if and only if the following data are equal for both: for each $p \geq 1$,*

1. *the supremum[4] of the positive entropies of ergodic measures which have a maximum period that is equal to p;*
2. *if this supremum is positive, the cardinality of the set of nonatomic ergodic measures that achieves the previous supremum.*

1.2 Almost-Borel classification and Markov shifts

We need the generalization to the non-mixing case of the characterization and classification of Markov shifts obtained by Hochman [23].

First some definitions. An automorphism of a standard Borel space is a *Borel system* (see Sec. 2.3). We denote by $\mathbb{P}'_{\text{erg}}(S)$ its set of ergodic, nonatomic measures.

Definition 1.2. Two Borel systems (X, S) and (Y, T) are *almost-Borel isomorphic* if there exists a Borel isomorphism $\psi : X' \to Y'$ with invariant Borel subsets $X' \subset X$ and $Y' \subset Y$ such that:

- $\psi \circ S = T \circ \psi$ on X';
- $X \setminus X'$ and $Y \setminus Y'$ are *almost null sets*: $\mu(X \setminus X') = \nu(Y \setminus Y') = 0$ for all $\mu \in \mathbb{P}'_{\text{erg}}(S)$ and $\nu \in \mathbb{P}'_{\text{erg}}(T)$.

Thus two systems are *almost-Borel isomorphic* if, in the terminology of [23], their free parts are Borel isomorphic on full sets. In particular, the spaces might not even be Borel isomorphic. We refer to the discussion in [50, p. 394] for a comparison with Borel and measurable isomorphisms.

Let T be a Markov shift (a "subshift of finite type over a countable alphabet", see Sec. 2.5 for this and related definitions and facts). Up to an almost null set, it is a disjoint, at most countable, union of irreducible Markov shifts T_i, $i \in I$, not reduced to periodic

[4] We follow the convention that $\sup \emptyset = -\infty$.

orbits. Each of those has a period p_i and an entropy $h_i := h(T_i)$. We set $m_i = 1$ or 0 according to whether T_i has or not a nonatomic measure of entropy h_i. Throughout this paper, all Markov shifts satisfy:

$$\text{all irreducible components have finite entropy.} \tag{1.3}$$

We define two sequences over $\mathbb{N} := \{1, 2, \dots\}$:

$$\begin{aligned} &\bar{u}_T(p) := \sup\Big(\{h_i : i \in I,\ p_i|p\} \cup \{0\}\Big) \in [0, \infty] \text{ and} \\ &\bar{\eta}_T(p) := \sum \{m_i : i \in I,\ (h_i, p_i) = (\bar{u}_T(p), p)\} \in \{0, 1, \dots, \infty\}. \end{aligned} \tag{1.4}$$

We can now state the extension of Hochman's classification proved in Sec. 4.3:

Theorem 1.5. *Two Markov shifts S, T are almost-Borel isomorphic if and only if $(\bar{u}_T, \bar{\eta}_T) = (\bar{u}_S, \bar{\eta}_S)$. Moreover, sequences u, η coincide with sequences $\bar{u}_T, \bar{\eta}_T$ of some Markov shift T if and only if*

$$\forall p \geq 1 \quad u(p) = \sup_{q|p} u(q) \textit{ and } u(p) = \infty \implies \eta(p) = 0. \tag{1.6}$$

In Sec. 4.2, we find a "maximal Markov subsystem", which we call the universal part, inside an arbitrary Borel system:

Theorem 1.7. *Any Borel system (X, S) contains an invariant Borel subset X_U such that:*

1. *X_U is almost-Borel isomorphic to a Markov shift T with $\bar{\eta}_T \equiv 0$;*
2. *if some subsystem $Y \subset X$ satisfies the previous property, then $Y \setminus X_U$ is almost null.*

These two properties define X_U up to an almost null set.

For instance, the universal part of an irreducible SFT T can be taken as the the shift deprived of a Borel subset carrying exactly the unique measure of entropy $h(T)$.

The condition "$\bar{\eta}_T \equiv 0$" is satisfied, e.g., if no irreducible component T_i of the Markov shift has a measure with entropy $h_i > 0$. It cannot be removed: consider the product of a positive entropy shift of finite type with the identity map on the unit interval. This condition and the above result are very natural from the point of view of universality discussed in Section 1.4.

This leads to a characterization of Markov shifts up to almost Borel isomorphism. We say that a measure-preserving system (S, μ) is *p-Bernoulli* ($p \in \mathbb{N}$) if it is isomorphic to the product of a Bernoulli system and a circular permutation on p points.[5] We call it *periodic-Bernoulli* if we don't want to specify p. At the end of Sec. 4.3, we prove:

[5] Note, p is the maximum period of (S, μ) in the terminology of Theorem 1.2.

Corollary 1.8. *A Borel system (X,S) is almost-Borel isomorphic to a Markov shift if and only if there is a sequence $u:\mathbb{N}\to[0,\infty]$ with $u(p)=\max_{q|p}u(q)$ such that:*

1. *for each $p\in\mathbb{N}$ and $t<u(p)$, there is an almost-Borel embedding of an irreducible Markov shift of period p and entropy $>t$ into X;*
2. *the set $\mathcal{M}$ of ergodic measures $\mu\in\mathbb{P}'_{\mathrm{erg}}(S)$ such that for every period p of μ, $h(S,\mu)\geq u(p)$, is at most countable;*
3. *each $\mu\in\mathcal{M}$ is p-Bernoulli for some $p\in\mathbb{N}$ and $h(S,\mu)=u(p)$.*

The mixing case was analyzed by Hochman (see [23, Thm. 1.7] and the discussion that precedes it).

Remark 1.9. This characterization provides an alternate approach to results like Theorem 1.1 by splitting the dynamics between: a "top entropy part" which must be shown to carry only very specific measures; and the rest which carries all possible measures "below some entropy thresholds". If S is a C^{1+} diffeomorphism of a compact manifold and S has no zero Lyapunov exponents, then this second part can be analyzed using Katok's horseshoes (see [11]).

1.3 Factors of Markov shifts

Thus we are led to find conditions guaranteeing that a dynamical system has shifts of finite type as large (in entropy) subsystems. There is an interest of some vintage in this problem (e.g. [24, 33, 38]). In Section 5, we prove

Theorem 1.10. *Let (X,S) be an irreducible Markov shift with period p and let $\pi:(X,S)\to(Y,T)$ be a continuous, not necessarily surjective, factor map into a selfhomeomorphism of a Polish space. Let*

$$h_*(\pi):=\sup\{h(T,\pi_*\mu):\mu\in\mathbb{P}_{\mathrm{erg}}(S)\}.$$

For any $h<h_*(\pi)$, there is an irreducible shift of finite type $X'\subset X$ such that $h_{\mathrm{top}}(X')>h$, X' has period p, and the restriction of π to X' is injective.

Without additional assumptions, $\pi(X)$ can carry measures with entropy $>h_*(\pi)$ and unrelated to those of X (see Prop. 7.1). Even when X is compact and $h_*(\pi)=h_{\mathrm{top}}(\pi(X))=h_{\mathrm{top}}(X)$, the m.m.e.'s, that is, the *ergodic measures maximizing entropy* for $\pi(X)$, do not have to be images of m.m.e.'s of X. In fact, we show that they can include uncountably many copies of measures which are not periodic-Bernoulli (Cor. 7.6).

Next we assume π to be finite-to-one, continuous and with compact domain. This forces that $h_*(\pi)=h_{\mathrm{top}}(\pi(X))=h_{\mathrm{top}}(X)$ and $\pi(X)$ has a unique m.m.e., which is periodic-Bernoulli; but the *periodic-maximal* measures, i.e., the measures maximizing the entropy among measures *with a given period* can still be more or less arbitrary (see Cor. 7.10), in contrast to those of Markov shifts. To control this, we use the following property.

Definition 1.11. Let $\pi : (X, S) \to (Y, T)$ be a Borel factor map from a Markov shift into a Borel system, with B an invariant Borel subset of X. Then π is *Bowen type* on B (or relative to B) if there is a relation $\sim$ on the alphabet of X such that the following hold:

1. $\pi(x) = \pi(w) \iff x \sim w$, for all x, w in B , and
2. $x \sim w \implies \pi(x) = \pi(w)$, for all x, w in X ,

where $x \sim w$ means $x_n \sim w_n$ for all n. If $B = X$, one simply says that π is Bowen type.

This definition is adapted from a property pointed out by Bowen [8, p.13] for surjective continuous factor maps from shifts of finite type to systems associated with Markov partitions. More precisely, these factors (with the quotient topology) are David Fried's *finitely presented dynamical systems* [18, 19]; these (up to topological conjugacy) are exactly the expansive systems which are continuous factors of shifts of finite type.

For a Markov shift Z, the *Sarig regular set* $Z_{\pm\text{ret}}$ of Z is the subset of sequences in which some symbol appears infinitely often in the past and some symbol (not necessarily the same) appears infinitely often in the future. In Sec. 6 we prove:

Theorem 1.12. *Suppose* (X, S) *is a Markov shift satisfying condition* (1.3) *and* $\pi : (X, S) \to (Y, T)$ *is a Borel factor map such that, for each irreducible component* Z *of* X*,*

1. π *is Bowen type on the Sarig regular set* $Z_{\pm ret}$*, and*
2. *the restriction* $\pi|Z_{\pm ret}$ *is finite-to-one.*

Then, letting $\bar{X}$ *be the union of the Sarig regular sets* $Z_{\pm\text{ret}}$ *of the irreducible components* Z *of* X*,*

- $\pi(\bar{X}) \subset Y$ *is almost-Borel isomorphic to a Markov shift;*
- *the induced map* $\mathbb{P}'_{\text{erg}}(\bar{X}) \to \mathbb{P}'_{\text{erg}}(\pi\bar{X})$ *is surjective.*

Condition (1) above is really about the restrictions $\pi|Z$. The construction behind this statement is related to a construction introduced by Manning [32] for the exact counting of periodic orbits.

In Section 8 we shall apply this theorem to Sarig's symbolic dynamics and deduce Theorem 8.2 from which Theorems 1.1 and 1.2 follow.

1.4 The universality heuristic

A Borel system X is *universal* with respect to a class $\mathcal{C}$ of Borel systems, if any system in $\mathcal{C}$ can be almost-Borel embedded into X. If, additionally, X belongs to $\mathcal{C}$, it is said to be *strictly universal* in $\mathcal{C}$. By an observation of Hochman (see Prop. 2.1), strictly universal systems, when they exist, are unique up to almost-Borel isomorphism. In this case, universal systems can be characterized as unions of an essentially unique "maximal" strictly universal system and a complementary part (see Sec. 3).

Hochman showed that many systems of entropy h are h-universal, i.e., universal with respect to the class of Borel systems whose measures have entropy $< h$ (see Thm. 4.1, Prop. 4.2). The complementary system mentioned above then supports exactly the ergodic measures of entropy h, often a unique measure of maximum entropy which is Bernoulli.

This provides a general heuristic: in a suitable class of systems, for a suitable notion of "universal", analyze each system as the union of a (large) standard part (defined using universality) and a complementary part (hopefully managable). This approach gives our almost Borel results on C^{1+} surface diffeomorphisms and Markov shifts, with Hochman's universality refined to address periods. The details of this universality approach are spelled out in Sections 3 and 4.

The existence of a large universal part can be rather robust. For example, any continuous factor Y of a mixing shift of finite type is $h(Y)$-universal (by Theorem 5.1). A related result holds for continuous factors of Markov shifts (Theorem 1.10). In contrast, as indicated earlier, the possibilities for the complementary system in Y can vary wildly without stronger assumptions (see Sec. 7).

Dedicatory

This paper is dedicated to Roy Adler, coinventor of topological entropy [1], with gratitude for his kindness and in appreciation of his mathematical influence. This paper considers entropy and period for the almost Borel classification of Markov shifts; the seminal result of this type was the Adler-Marcus Theorem [3], which classified irreducible shifts of finite type up to almost topological conjugacy by topological entropy and period.

2 Definitions and background

We fix notations and recall some facts that we will use without further explanation.

2.1 Dynamical Systems

In this paper, a *dynamical system* (or system) S is an automorphism of a space X. We shall consider:

- *topological* dynamical systems (or t.d.s.) given by selfhomeomorphisms of (not necessarily compact) Polish spaces;
- *measure-preserving* systems given by automorphisms of probability spaces. We shall often abbreviate ergodic measure-preserving systems, to *ergodic* systems;
- *Borel* systems given by Borel automorphisms of standard Borel spaces (see below).

Recall that a *factor map*, resp. an *embedding*, is a homomorphism, resp. a monomorphism, of the spaces that intertwines the automorphisms. Unless a factor map is said to be *into*, it is assumed to be surjective. A *subsystem* is a system of the same category given by a restriction to an invariant subspace.

We often use the symbol for the space or for the automorphism to refer to the system and its domain and suppress the structure (topological, Borel,...) from the notation, with interpretation by context.

2.2 Borel spaces

A *standard Borel space* [26, Sec. 12] is a set X together with a σ-algebra $\mathcal{X}$ generated by a Polish topology, i.e., a topology defined by some distance which turns X into a separable, complete, metric space. The elements of $\mathcal{X}$ are called the *Borel sets* of X.

$f : X \to Y$ is a *Borel map* if X and Y are standard Borel spaces and the preimage of any Borel subset is Borel. f is a *Borel isomorphism* if it is a bijection such that f and f^{-1} are Borel. Here, no sets are considered negligible. According to Kuratowski's theorem (see [26, (15.6)]), all uncountable standard Borel spaces are isomorphic.

Recall that if $f : X \to Y$ is a Borel map and A is a Borel subset of X such that $f|A$ is injective, then $f(A)$ is Borel and $f : A \to f(A)$ is a Borel isomorphism, according to the Lusin-Souslin Theorem [26, (15.2)].

We denote by $\mathrm{Prob}(X)$ the set of not necessarily invariant probability measures defined over the Borel sets. We endow it with the σ-algebra generated by the maps $\mu \mapsto \mu(E)$, $E \in \mathcal{X}$. This makes $\mathrm{Prob}(X)$ into a standard Borel space (see [26, (17.24)] and [26, beginning of section 17.E]).

2.3 Almost-Borel systems

Let (X, S) be a Borel system. Then $\mathrm{Prob}(S) \subset \mathrm{Prob}(X)$ is the set of S-invariant Borel probability measures of X (henceforth the measures of S) and $\mathbb{P}_{\text{erg}}(S)$ is the subset of ergodic invariant measures. $\mathrm{Prob}(S)$ and $\mathbb{P}_{\text{erg}}(S)$ are Borel subsets of $\mathrm{Prob}(X)$, hence they also are standard Borel spaces. Note that $\mathrm{Prob}(S)$ may be empty.

An *almost null set* for (X, S) is a Borel set of measure zero for every μ in $\mathbb{P}'_{\text{erg}}(S)$, the set of atomless, ergodic measures of S. We say that two Borel systems define the same *almost-Borel system*, if they coincide up to an almost null set. An *almost-Borel map* means a homomorphism of Borel systems defined on the complement of an almost null set. Almost-Borel embeddings, factors, and isomorphisms are defined in the obvious way.

We shall need the following Borel maps (see, e.g., [11]), defined on the complement of an almost null set: (1) a map $M : X \to \mathbb{P}_{\text{erg}}(S)$ such that, for any Borel set $B \subset \mathbb{P}_{\text{erg}}(S)$ and any $\mu \in \mathbb{P}_{\text{erg}}(S)$: $\mu(M^{-1}(B)) = 1$ if and only if $\mu \in B$;[6] (2) the map $h : \mathrm{Prob}(S) \to [0, \infty]$ associating to each measure its Kolmogorov-Sinai entropy (see below).

The following almost-Borel variant of the well-known measurable Schröder-Bernstein theorem [26, (15.7)] is fundamental for us :

Proposition 2.1 (Hochman [23])**.** *Two Borel systems are almost-Borel isomorphic if and only if there are almost-Borel embeddings of one into the other.*

2.4 Entropy

The *topological entropy* of a compact t.d.s. (Y, T) is denoted by $h_{\text{top}}(T)$. The *Kolmogorov-Sinai entropy* of a measure-preserving system (S, μ) is denoted by $h(S, \mu)$. We define the *Borel entropy* of a Borel system (X, S) to be $h(S) := \sup\{h(S, \mu) : \mu \in \mathrm{Prob}(S)\}$. We shall often call any of these the entropy of T, (S, μ) or S.

The *variational principle for entropy* states that if (Y, T) is a compact t.d.s., its Borel entropy $h(T)$ coincides with its *topological entropy* $h_{\text{top}}(T)$. An *ergodic measure of maximum entropy* (or m.m.e.) for (X, S) is a measure $\mu \in \mathbb{P}_{\text{erg}}(S)$ such that $h(S, \mu) = h(S)$. It need not be unique (e.g. the identity map on a nontrivial space) and it need not exist, even for compact t.d.s. with finite smoothness [34].

We will use the Bowen-Dinaburg formulas to compute $h_{\text{top}}(T)$ in terms of *dynamical* (ϵ, n)*-balls* $B(p, \epsilon, n) = \{y \in Y : 0 \leq k < n \implies \text{dist}(T^k p, T^k y) < \epsilon\}$. Recall the following for a compact subset C of Y and $\epsilon > 0$. The integer $r_{\text{span}}(\epsilon, n, C, T)$ is the minimal cardinality of (ϵ, n)-spanning sets for C and $r_{\text{sep}}(\epsilon, n, C, T)$ is the maximal cardinality of an (ϵ, n)-separated subset of C. We have

$$\begin{aligned} h_{\text{sep}}(C, T, \epsilon) &:= \limsup_{n\to\infty} \frac{1}{n} \log r_{\text{sep}}(\epsilon, n, C, T) , \\ h_{\text{span}}(C, T, \epsilon) &:= \limsup_{n\to\infty} \frac{1}{n} \log r_{\text{span}}(\epsilon, n, C, T) , \quad \text{and} \\ h_{\text{top}}(T) &= \lim_{\epsilon\to 0} h_{\text{sep}}(Y, T, \epsilon) = \lim_{\epsilon\to 0} h_{\text{span}}(Y, T, \epsilon) . \end{aligned} \tag{2.2}$$

We refer to [25, 39, 49] for more background.

[6] For compact t.d.s., we can take $M(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{S^k x}$, defined on the Borel set of points for which this weak star limit exists; the complement is a universal null set, by the ergodic theorem and ergodic decomposition.

2.5 Markov shifts

A *countable state Markov shift* (or just Markov shift) is (X,S) where $X \subset V^{\mathbb{Z}}$ for some countable (maybe finite) set V and for some $E \subset V^2$: $X = \{x \in V^{\mathbb{Z}} : \forall n \in \mathbb{Z}\ (x_n, x_{n+1}) \in E\}$ and $S : X \to X$ defined by $S((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$. The directed graph $G = (V,E)$ is a *vertex* presentation of (X,S). The distance $d(x,y) = \exp(-\inf\{|k| : x_k \neq y_k\})$ turns X into a separable, complete metric space and S into a homeomorphism.

A finite or infinite sequence $x = (x_i)_{i\in I}$ is a *path* on the graph G if $I \subset \mathbb{Z}$ is an interval, each $x_i \in V$ and each $(x_i, x_{i+1}) \in E$ whenever $\{i, i+1\} \subset I$. If x has a finite domain I, then we call it a word and define the *cylinder*: $[x]_X$ (or just $[x]$) to be $\{y \in X : \forall i \in I\ x_i = y_i\}$.

If $x \in X$ and $a \leq b$ are two integers, $x|_a^b$ is the word $x_a x_{a+1} \dots x_{b-1}$ of length $b-a$. A *loop* of length n based at a vertex v is a finite word $\ell_0 \dots \ell_{n-1}$ such that $\ell_0 = v$ and $\ell_0 \dots \ell_{n-1}\ell_0$ is a path on G.

The classical *shifts of finite type* (or SFTs) are the topological dynamical systems topologically isomorphic to a compact Markov shift, or equivalently, to a Markov shift that can be presented by a finite graph. We refer to [29] for background.

The Markov shift (X,S) is *irreducible* if it can be presented by a strongly connected graph G, i.e., such that any two vertices u, v can be joined by a path from u to v. In this case, its *period* is the greatest common divisor of the lengths of all loops on G. (X,S) is *mixing* if it is irreducible with period 1.

Any Markov shift (X,S) can be written as the disjoint union of irreducible Markov shifts (X_j, S_j), $j \in J$ with J countable (possibly finite), and a set of measure zero with respect to any invariant measure. This decomposition is unique (up to the indexing) and the Markov subshifts (X_j, S_j), $j \in J$, are called the *irreducible components* of (X,S).

Any infinite entropy shift has uncountably many ergodic measures of infinite entropy. On an irreducible period p Markov shift (X,S) with finite Borel entropy, the measure of maximal entropy (or m.m.e.), if it exists, is unique and *p-Bernoulli*. Moreover, the following is known (it is probably folklore but see [43] for a convenient reference).

Fact 2.3. *For any $h \in (0,\infty)$ and $p \in \mathbb{N}$, one can find two irreducible Markov shifts with entropy h and period p: one with a measure of maximum entropy, one without. Any infinite entropy irreducible shift has uncountably many ergodic measures of infinite entropy.*

Finally, we note that from a directed graph $G = (V,E)$ (now possibly with multiple edges from one vertex to another) one has also the *edge shift* associated to G. This is a Markov shift whose alphabet is the set of edges of G. In terms of the earlier definition, the edge shift of G is defined by a new graph G', whose vertex set is E, in which there is an edge from e_1 to e_2 iff the terminal vertex in G of e_1 equals the initial vertex in G of e_2. We will use the edge shift presentation in Section 7. We refer to [28] for more background on Markov shifts.

2.6 Periods of measures and Borel decomposition

Let (S,μ) be an ergodic system. Recall the notion of periods from eq. (1.1). Note that if p is a period, then any positive divisor of p is also a period and that p is a period iff there is a p-cyclic partition modulo μ, i.e., $\{X_0, X_1, \dots, X_{p-1}\} \subset \mathcal{X}$ such that $\mu(\bigcup_{i=0,\dots,p-1} X_i) = 1$ and $\mu(X_i \cap X_j) = 0$ for all $0 \leq i \neq j \leq p-1$.

Observe that not every measure has a maximum period (consider adding machines). If it exists, then the set of all periods is the set of divisors of the maximum period. Also having maximum period equal to 1 is equivalent to $\sigma_{\mathrm{rat}}(S,\mu) = \{1\}$ and (because (S,μ) is ergodic) it is equivalent to total ergodicity (i.e., the ergodicity of all (S^n,μ), $n \geq 1$).

Fact 2.4. *Given an irreducible Markov shift X with period p and entropy h, the supremum of the entropies of ergodic measures with maximum period p is equal to h. Conversely, for any ergodic invariant measure carried by X, the maximum period, if it exists, is a multiple of p.*

In the above definitions, the partition is relative to μ. It is important for our purposes that we can improve this as follows. By a Borel partition of X, we mean a collection of pairwise disjoint Borel sets whose union is X.

Theorem 2.5 (Borel periodic decomposition)**.** *Let (X,T) be an automorphism of a standard Borel space. For each integer $p \geq 1$, there exists $P(p) := \{P_1, \dots, P_p, P_*\}$, a Borel partition of X such that:*

- $T(P_*) = P_*$ *and* $T(P_i) = P_{i+1}$ *for all* $i = 1, \dots, p$ *(*$P_{p+1} := P_1$*);*
- *for any* $\mu \in \mathbb{P}_{\mathrm{erg}}(T)$, $\mu(P_*) = 0$ *if and only if* p *is a period of* (S,μ).

Though related results exist (see [50, remark on top of page 399]), we could not find this statement in the literature, hence a proof is given in Appendix A.

3 Universal systems

We study Markov shifts as almost-Borel systems. In this section, we perform the part of the analysis that is conveniently done in the language of universality (already used by Hochman [23], following Benjamin Weiss, e.g., [51]).

Definition 3.1. Let $\mathcal{C}$ be a class of almost-Borel systems. An almost-Borel system (X,S) is $\mathcal{C}$*-universal* if it contains (the image of) an almost-Borel embedding of any system in $\mathcal{C}$. If, additionally, $(X,S) \in \mathcal{C}$, then it is said to be *strictly* $\mathcal{C}$-universal.[7]

[7] This is related to but distinct from the notion of a terminal object in category theory.

We build and classify "maximal universal parts" of arbitrary almost-Borel systems. The next section will relate these to Markov shifts by appealing to Hochman's theorem [23].

3.1 Period-universal systems

Following Prop. 2.1, 'the' strictly universal system with respect to a given class, if it exists, is unique up to almost-Borel isomorphism. Hochman identified the strictly universal systems with respect to the classes $\mathcal{B}(t)$, $t \geq 0$, of Borel systems (X, S) such that for all $\mu \in \mathbb{P}'_{\mathrm{erg}}(S)$, $h(S, \mu) < t$.

We consider for each $t \geq 0$ and $p \in \mathbb{N}$, the class $\mathcal{B}(t, p)$ of systems whose measures $\mu \in \mathbb{P}'_{\mathrm{erg}}(S)$ satisfy: p is a period and $h(S, \mu) < t$. For short we write that a system is *t-universal*, resp. *(t, p)-universal* if it is $\mathcal{B}(t)$-universal, resp. $\mathcal{B}(t, p)$-universal. We will repeatedly use (see Prop. 1.4(3) of [23] in the case $p = 1$ –its proof generalizes):

Lemma 3.2. *For $p \in \mathbb{N}$ and $h \in [0, \infty]$, a countable union of strictly (h_n, p)-universal systems, is strictly (h, p)-universal with $h = \sup h_n$.*

The following almost-Borel invariant is important for Markov shifts and related systems.

Definition 3.3. The (*union-entropy-period*) *universality sequence* of an almost-Borel system (X, S) is $u_S : \mathbb{N} \to [0, \infty]$ defined by:

$$u_S(p) := \sup\{t \geq 0 : (X, S) \text{ contains a strictly } (t, p)\text{-universal system}\}.$$

Remarks 3.4. Prop. 4.2 will show that strictly (t, p)-universal systems do exist hence the above invariant is not trivial and can be computed as $u_S(p) = \sup\{t \geq 0 : (X, S) \text{ is } (t, p)\text{-universal}\}$. Also, $u_S(p)$ does not need to be the supremum of the entropies of measures with a period p.

Observe that if q divides p, $\mathcal{B}(t, q) \supset \mathcal{B}(t, p)$ so (t, q)-universality implies (t, p)-universality. Hence:

Fact 3.5. *For all $p \in \mathbb{N}$, $u_S(p) = \max_{q|p} u_S(q)$.*

A condition defines a set *up to an almost null set* if the symmetric difference between any two Borel subsets satisfying it, is an almost null set.

Proposition 3.6. *A Borel system (X, S) contains, for each $p \in \mathbb{N}$, a subsystem (X_{Up}, S_{Up}) characterized up to an almost null set by the two following equivalent properties.*
(1) For all $\mu \in \mathbb{P}'_{\mathrm{erg}}(S)$:

$$\mu X_{Up} = 1 \iff p \text{ is a period of } \mu \text{ and } h(S, \mu) < u_S(p)\ . \tag{3.7}$$

(2) (X_{Up}, S_{Up}) is a strictly p-universal subsystem and contains any other strictly p-universal subsystem of X up to an almost null set.

Moreover, (X_{Up}, S_{Up}) is strictly $(u_S(p), p)$-universal.

Proof. Conditions (1) and (2) separately imply uniqueness up to an almost null set so it suffices to build a solution (X_{Up}, S_{Up}) to (1) and check that it satisfies also (2) and the last claim.

Theorem 2.5 gives Borel subsystems C_p, $p \geq 1$, such that for any $\mu \in \mathbb{P}'_{\text{erg}}(S)$, $\mu(C_p) = 1$ if and only if p is a period of μ. Recall that the functions $M(\cdot)$ and $h(S, \cdot)$ from Sec. 2.3 are Borel. Hence for any $t \in (0, \infty]$ there is an invariant Borel subset V^t of X such that, for all $\mu \in \mathbb{P}'_{\text{erg}}(S)$, $\mu(V^t) = 1$ if and only if $h(S, \mu) < t$. Set $X_{Up} = C_p \cap V^t$ with $t = u_S(p)$. Clearly X_{Up} is a solution to (1).

We turn to condition (2). First, (X_{Up}, S_{Up}) is strictly $(u_S(p), p)$-universal by Lemma 3.2. Second, if $X' \subset X$ is strictly p-universal, then it must be (t, p)-universal with $t \leq u_S(p)$. Thus $X' \subset X_{Up}$ up to an almost null set by (3.7). (2) and the last claim are satisfied. □

3.2 Union-entropy-period universal parts

The following class of Borel systems will help us analyze Markov shifts without assuming irreducibility.

Definition 3.8. For a sequence $u : \mathbb{N} \to [0, \infty]$, $\mathcal{C}(u)$ denotes the union-entropy-period class of Borel systems (X, S) such that any $\mu \in \mathbb{P}'_{\text{erg}}(S)$ has some period p such that $h(S, \mu) < u(p)$. A *strictly u.e.p.-universal system* is a strictly $\mathcal{C}(u)$-universal system for some $u : \mathbb{N} \to [0, \infty]$.

Considering the subsystems $X_p := C_p \cap V^{u(p)}$ as in the proof of Proposition 3.6 easily yields:

Fact 3.9. *For any $u : \mathbb{N} \to [0, \infty]$, $(X, S) \in \mathcal{C}(u)$ if and only if $X = \bigcup_{p \in \mathbb{N}} X_p$ with $X_p \in B(u(p), p)$ for all $p \in \mathbb{N}$. If X is strictly $\mathcal{C}(u)$-universal, then each X_p is strictly $(u(p), p)$-universal.*

An arbitrary Borel system (X, S) contains a 'maximal' strictly u.e.p.-universal subsystem:

Theorem 3.10. *For any Borel system (X, S) satisfying:*

$$\forall \mu \in \mathbb{P}'_{\text{erg}}(S)\ h(S, \mu) < \infty, \tag{3.11}$$

there is a subsystem (X_U, S_U) characterized up to an almost null set by each of the following three equivalent properties.

(1) $X_U = \bigcup_{p \in \mathbb{N}} X_{Up}$ up to an almost null set.

(2) For all $\mu \in \mathbb{P}'_{\text{erg}}(S)$,

$$\mu X_U = 1 \iff \mu \text{ has a period } p \text{ s.t. } h(S,\mu) < u_S(p)\ . \tag{3.12}$$

(3) (X_U, S_U) is a strictly u.e.p.-universal subsystem that contains any strictly u.e.p.-universal subsystem up to an almost null set.

Moreover, (X_U, S_U) is strictly $\mathcal{C}(u_S)$-universal and its universality sequence coincides with u_S.

Definition 3.13. The subsystem (X_U, S_U) above is called the (union-entropy-period) *universal part* of (X, S).

The following are easy consequences of universality.

Corollary 3.14. *Suppose (X,S) and (Y,T) are Borel systems. Then*

1. *There is an almost-Borel embedding $(X_U, S_U) \to (Y_U, T_U)$ if and only if $u_S \le u_T$.*
2. *(X_U, S_U) and (Y_U, T_U) are almost-Borel isomorphic if and only if $u_S = u_T$.*
3. *Suppose for all $\mu \in \mathbb{P}'_{\text{erg}}(X)$ there is a period p of μ such that $h(S,\mu) < u_T(p)$. Then the systems $(X,S) \cup (Y,T)$, $(X,S) \sqcup (Y,T)$, and (Y,T) are almost-Borel isomorphic.*

The proof of Theorem 3.10 relies on the following lemma, whose proof we defer to the end of the section. Say that a Borel system (X,S) is *stable* if there is an almost-Borel embedding of $(X \times \{0,1,\dots\}, S \times \mathrm{id})$ into (X,S). Note that any strictly universal system X with respect to some class $\mathcal{C}$ among $\mathcal{B}(t)$, $\mathcal{B}(t,p)$, or $\mathcal{C}(u)$, is stable: indeed, $X \times \{0,1,2,\dots\}$ trivially belongs to $\mathcal{C}$, hence can be embedded into X by universality. Moreover, countable unions of stable systems are stable.

Lemma 3.15. *A countable union $\bigcup_{n\ge0} X_n$ of stable subsystems is almost-Borel isomorphic to the corresponding disjoint union $\bigsqcup_{n\ge0} X_n$.*

Proof of Theorem 3.10. Each of the conditions (1), (2) and (3) implies uniqueness up to an almost null set. It suffices to show that X_U as in condition (1) with $S_U := S|X_U$ satisfies the two other claims. For Claim (2) this follows from Condition (1) of Prop. 3.6.

To prove the universality stated in Claim (3), let $(Y,T) \in \mathcal{C}(u_S)$. By Fact 3.9, $Y = \bigcup_{p\in\mathbb{N}} Y_p$ with $Y_p \in \mathcal{B}(u_S(p), p)$ and $Z_p := Y_p \setminus \bigcup_{q<p} Y_q$, $p \in \mathbb{N}$, is a partition. By Prop. 3.6, each X_{Up} is strictly $(u_S(p), p)$-universal so there is an almost-Borel embedding of $Z_p \subset Y_p$ into X_{Up} for all $p \ge 1$. Now, Lemma 3.15 lets us assume that $X_U = \bigcup_{p\in\mathbb{N}} X_{Up}$ is a partition, proving $\mathcal{C}(u_S)$-universality. It is strict since $(X_U, S_U) \in \mathcal{C}(u_S)$ by Claim (2).

For the second half of (3), let (Y, T) be a strictly $\mathcal{C}(v)$-universal subsystem of (X, S) for some $v : \mathbb{N} \to [0, \infty]$. Fact 3.9 implies $Y = \bigcup_{p \in \mathbb{N}} Y_p$ and $v \leq u_S$. By Prop. 3.6, $Y_p \subset X_{Up} \cup N_p$ for some almost null N_p: $Y \subset X_U \cup \bigcup_{p \in \mathbb{N}} N_p$ and Claim (3) follows.

Finally, let u_U be the universality sequence of (X_U, S_U). As $X_U \subset X$, $u_U \leq u_S$. The converse inequality follows from the strict universality of each X_{Up}. □

Proof of Lemma 3.15. It suffices to build an almost Borel embedding $\Psi : \bigcup_{n \geq 0} X_n \times \{n\} \hookrightarrow \bigcup_{n \geq 1} X_n$ (the reverse embedding is obvious and the lemma then follows from Prop. 2.1). We claim that there exist subsystems $Z_0, Z_1, \dots$ and maps $\phi_0, \phi_1, \dots$ such that:

1. each set $Z_n \subset X_0 \cup \cdots \cup X_n$ is almost Borel isomorphic to X_n;
2. ϕ_n is an almost-Borel embedding of $X_n \times \{0, 1, \dots\}$ into Z_n;
3. the sets $\phi_\ell(X_\ell \times \{\ell\})$, $0 \leq \ell < n$, are pairwise disjoint.
4. $Z_n \cap \phi_\ell(X_\ell \times \{\ell, n+1, n+2, \dots\}) = \emptyset$ for $0 \leq \ell < n$.

Then, $\Psi : \bigcup_{n \geq 0} X_n \times \{n\} \hookrightarrow \bigcup_{n \geq 0} X_n$ defined by $\Psi(x, n) = \phi_n(x, n)$ proves the lemma.

We proceed by induction. To begin with, let $\phi_0 : X_0 \times \{0, 1, \dots\} \hookrightarrow Z_0 := X_0$ be given by the stability assumption. Properties $(1)_0, (2)_0, (3)_0$, and $(4)_0$ (i.e., (1),…,(4) for n taking the value 0) are satisfied.

For $n \geq 1$, we assume $(1)_m, (2)_m, (3)_m, (4)_m$ for $0 \leq m < n$ and, letting $\tilde{X}_k := X_k \setminus (X_0 \cup \cdots \cup X_{k-1})$, we set:

$$Z_n := \tilde{X}_n \cup \bigcup_{k=0}^{n-1} \phi_k((\tilde{X}_k \cap X_n) \times \{n\}). \tag{3.16}$$

First note that, using $(1)_k$ for $k < n$, $Z_n \subset \tilde{X}_n \cup \bigcup_{k<n} X_k \subset \bigcup_{k \leq n} X_k$. Second we check that the union in (3.16) is disjoint. Note, $\tilde{X}_n \cap \phi_k(X_k \times \{0, 1, \dots\}) \subset \tilde{X}_n \cap (X_0 \cup \cdots \cup X_k) = \emptyset$ for $0 \leq k < n$. So it is enough to note that for all $0 \leq \ell < k < n$, $(4)_k$ yields:

$$\phi_\ell((\tilde{X}_\ell \cap X_n) \times \{n\}) \cap \phi_k((\tilde{X}_k \cap X_n) \times \{n\}) \subset \phi_\ell(X_\ell \times \{k + (n-k)\}) \cap Z_k = \emptyset.$$

The disjointness in (3.16) implies that Z_n is isomorphic to X_n so $(1)_n$ holds. Moreover, the stability assumption gives ϕ_n as in condition $(2)_n$.

We prove $(4)_n$ for $0 \leq \ell < n$. We use (3.16) to expand Z_n. As before $\tilde{X}_n \cap Z_\ell = \emptyset$ so we need only to show that, for $0 \leq k < n$:

$$\phi_k(X_n \times \{n\}) \cap \phi_\ell(X_\ell \times \{\ell, n+1, n+2, \dots\}) = \emptyset. \tag{3.17}$$

If $\ell = k$, (3.17) follows from the injectivity of ϕ_k. If $\ell < k$, it follows from $(4)_k$ as $\phi_k(X_k \times \{0,1,\dots\}) \subset Z_k$ and $\{\ell, k+1, k+2, \dots\} \supset \{\ell, n+1, n+2, \dots\}$. If $k < \ell$, it follows from $(4)_\ell$ using $\phi_\ell(X_\ell \times \{0,1,\dots\}) \subset Z_\ell$ and $n \geq \ell + 1$.

(3.17) and therefore condition $(4)_n$ are established. Eq. (3.17) also implies condition $(3)_n$, completing the inductive step. □

4 Finite entropy Markov shifts

In this section, we prove Theorems 1.5 and 1.7 as well as Corollary 1.8 by relating the universal parts studied in Sec. 3 to Markov shifts using the work of Hochman [23].

4.1 Markov shifts and universality

For $h \geq 0$ the *h-slice of* (X,S) is a Borel subsystem which, for $\mu \in \mathbb{P}'_{\text{erg}}(X)$, has μ measure 1 if and only if $h(S,\mu) < h$. "The" h-slice subsystem is unique up to an almost null set. Note that the 0-slice is an almost null set and that a system (X,S) with no measure of maximum entropy, is equal to its $h(S)$-slice up to an almost null set. We recall the main result of [23], combining his statements 1.4, 1.5 and 1.6:

Theorem 4.1 (Hochman [23])**.** *Let X be a mixing SFT or more generally a mixing Markov shift and let $0 < h \leq h(X)$ be finite. Then X is h-universal and its h-*slice *is strictly h-universal.*

Proposition 4.2. *For $p \in \mathbb{N}$ and $h \in [0,\infty]$, the following systems are strictly (h,p)-universal (and therefore almost Borel isomorphic).*

1. *h-slices of irreducible period p, entropy h Markov shifts.*
2. *Irreducible Markov shifts with period p and entropy h with no measure of maximal entropy. (Recall Fact 2.3: such shifts exist exactly when $0 < h < \infty$.)*
3. *Countable unions of period p irreducible Markov shifts with entropies strictly less than h and with supremum equal to h.*

Proof. All of this is in Hochman's work for the case $p = 1$ (see Theorems 1.5 and 1.6, Proposition 1.4 in [23]). The remark about almost-Borel isomorphism follows from Prop. 2.1. For $p > 1$, observe that a Borel system (X,S) is (h,p)-universal if it contains a cyclically moving subset with a period p such that the restriction of S^p to it is $h(S^p)$-universal. □

Recall the notions of p-maximal and p-Bernoulli measures (see before Cor. 1.8).

Lemma 4.3. *An irreducible Markov shift (X,S) with entropy h and period p satisfying* (3.11) *has $h<\infty$ and is the disjoint union of a strictly $(h(S),p)$-universal system and a system supporting at most one measure from $\mathbb{P}'_{\mathrm{erg}}(S)$, which if it exists is the unique measure of maximal entropy of S, a p-Bernoulli measure.*

Proof. (This follows the proof of [23] for $p=1$.) The $h(S)$-slice of (X,S) is strictly $(h(S),p)$-universal (Prop. 4.2). There is at most one measure of maximum entropy [21], which if it exists is a countable state Markov chain, and therefore p-Bernoulli (by [36] for $p=1$ and then for general p by the argument of [2]) and is supported on the complement of the $h(S)$-slice. □

4.2 Characterizing Markov shifts

Recall that (X_U,S_U) is the universal part of (X,S) (Thm. 3.10) and that $u_S:\mathbb{N}\to[0,\infty]$ is the universality sequence (Def. 3.3).

Theorem 4.4. *Let (X,S) be a Borel system satisfying the finite entropy condition* (3.11). *Then the following are equivalent:*

1. *(X,S) is almost-Borel isomorphic to a Markov shift.*
2. *$\mathbb{P}'_{\mathrm{erg}}(X\setminus X_U)$ is at most countable and each $\mu\in\mathbb{P}'_{\mathrm{erg}}(X\setminus X_U)$ is p-Bernoulli with entropy equal to $u_S(p)<\infty$ for some $p\in\mathbb{N}$.*

It will be convenient to define $\mathrm{Prob}(p)$ as the collection of p-Bernoulli measures carried by $X\setminus X_U$ and let

$$\eta_S(p):=\#\mathrm{Prob}(p). \tag{4.5}$$

Proof. First, let (X,S) be a Markov shift. It is a countable union $\bigcup_{i\in I}X_i$ where each X_i is an irreducible Markov shift with period p_i and entropy h_i (ignoring almost null sets).

Applying Lem. 4.3, we get $h_i<\infty$ and $X_i=X_i'\sqcup X_i''$ where X_i' is strictly (h_i,p_i)-universal and X_i'' is either empty or carries a p_i-Bernoulli measure of entropy h_i (and no other measure). Therefore the universal part of X contains $\bigcup_{i\in I}X_i'$. Hence $X\setminus X_U$ carries at most the previous countably many periodic-Bernoulli measures. The period p and entropy h of any periodic-Bernoulli measure not carried by X_U must satisfy $h=h_i\geq u_S(p)$ whenever $p_i=p$ (see Thm. 3.10). But $u_S(p)\geq h_i$ whenever $p_i=p$. Hence $h=u_S(p)$. This proves (1) $\Longrightarrow$ (2).

Conversely, let (X,S) be a Borel system as in (2). By Thm. 3.10, $X_U=\bigcup_{p\in\mathbb{N}}X_{Up}$. According to Lem. 3.15, this is almost-Borel isomorphic to a disjoint union $\bigsqcup_{p\in\mathbb{N}}V_p$ of some strictly $(u_S(p),p)$-universal systems V_p. By Prop. 4.2, each V_p, and therefore X_U itself, is isomorphic to a Markov shift.

Each $\mu \in \mathbb{P}'_{\text{erg}}(X \setminus X_U)$ is a periodic-Bernoulli measure. By Fact 2.3, there is an irreducible, positive recurrent Markov shift W_μ with the same period p and entropy $u_S(p)$ as μ. As above, one can write $W_\mu = W'_\mu \sqcup W''_\mu$ where W'_μ is strictly universal and W''_μ carries μ and no other measure. Now, the disjoint union of any finite or countably infinite collection of strictly $(u_s(p), p)$-universal systems is itself strictly $(u_S(p), p)$-universal. Hence, V_p is almost-Borel isomorphic to $V_p \sqcup \bigsqcup_{\mu \in \text{Prob}(p)} W'_\mu$. Therefore X is almost-Borel isomorphic to the union $\bigsqcup_{p \in \mathbb{N}} V_p \sqcup \bigsqcup_{\mu \in \text{Prob}(p)} W_\mu$, a Markov shift. □

This implies (note that Lem. 3.15 does not apply):

Corollary 4.6. *If X is the (not necessarily disjoint) union of countably many systems X_n, each of which is almost-Borel isomorphic to a Markov shift satisfying* (3.11)*, then X is almost-Borel isomorphic to a Markov shift, itself satisfying* (3.11).

We now relate Markov shifts with strictly u.e.p.-universal systems.

Lemma 4.7. *For a Markov shift, the conditions* (1.3) *and* (3.11) *are equivalent. For a Borel system (X, S), the sequences $\bar{u}_S, \bar{\eta}_S$ and u_S, η_S (from* (1.4)*,* (4.5)*, and Def. 3.3) coincide. Moreover, the following are equivalent:*

1. *(X, S) is strictly u.e.p.-universal;*
2. *(X, S) is almost-Borel isomorphic to a Markov shift with $\bar{\eta}_S \equiv 0$.*

Proof. We write $X = \bigcup_{i \in I} X_i$ with p_i, h_i as in (1.4). Any $\mu \in \mathbb{P}'_{\text{erg}}(S)$ is carried by some X_i by ergodicity. The equivalence of (1.3) and (3.11) follows. Prop. 4.2 implies $u_S \geq \bar{u}_S$ and $u_S(p) > \bar{u}_S(p)$ would give a measure with maximum period p and entropy $> \bar{u}_S(p)$. $\eta_S \equiv \bar{\eta}_S$ follows from Theorem 4.4.

Point (3) of Thm. 3.10 shows that a Borel system is strictly u.e.p.-universal if and only if it coincides with its universal part. Thm. 4.4 shows that this is equivalent to condition (2) above. □

Given Lemma 4.7, Theorem 1.7 is equivalent to Theorem 3.10.

4.3 Classification of Markov shifts

Proof of Thm. 1.5. The sequences u_S, η_S coincides with $\bar{u}_S, \bar{\eta}_S$ according to Lem. 4.7. Clearly the former are invariants of almost-Borel isomorphism. To see that these are complete, let (X, S) and (Y, T) be two Markov shifts satisfying (1.3) and $(u_S, \eta_S) \equiv (u_T, \eta_T)$. By Cor. 3.14, S_U and T_U are almost-Borel isomorphic. By Thm. 4.4, $X \setminus X_U$ carries only periodic-Bernoulli measures. Let $p \in \mathbb{N}$. Using the periodic decomposition Thm. 2.5, one finds a Borel subset $X^{(p)} \subset X \setminus X_U$ carrying exactly the p-Bernoulli measures of $X \setminus X_U$. Those measures have entropy $u_S(p)$ by Thm. 4.4. Hence the almost-Borel isomorphism class of $X^{(p)}$ is defined by $(p, u_S(p), \eta_S(p))$. To conclude, remark that $X \setminus U = \bigsqcup_{p\in\mathbb{N}} X^{(p)}$ up to an almost null set.

We turn to Claim (1.6). The necessity of its first half follows from Fact 3.5, while its second half is a consequence of the finite entropy condition (3.11). Conversely, given (u, η) satisfying (1.6), let us build a Markov shift (X, S) realizing these invariants.

First, let $X' := \bigcup_{p\in\mathbb{N},u(p)>0} V_p$ with V_p a strictly $(u(p), p)$-universal Markov shift (Prop. 4.2). By Fact 3.5, $u_S(p) = \sup_{q|p} u_S(q)$, which is $u(p)$. Second, let $X'' := \bigcup_{p\in\mathbb{N},\eta(p)>0} W_p \times 1_{\eta(p)}$ where W_p is an irreducible Markov shift of entropy $u(p)$ and period p with exactly one measure of maximum entropy and $1_{\eta(p)}$ is the identity on a set of cardinality $\eta(p)$. This is possible as $\eta(p) > 0$ only if $u(p) < \infty$ (Lem. 4.3). The Markov shift $X' \cup X''$ satisfies $u_S = u$ and $\eta_S = \eta$. □

Proof of Cor. 1.8. For (X, S) almost-Borel isomorphic to a Markov shift T, let $u := u_S$ its universal sequence. Prop. 4.2 implies Claim (1). The set $\mathcal{M}$ defined in Claim (2) is contained in $\mathbb{P}'_{\text{erg}}(X \setminus X_U)$ and Thm. 4.4 implies (2) and (3).

Conversely, let (X, S) be a Borel system satisfying conditions (1)-(3) for some $u : \mathbb{N} \to [0, \infty]$. (1) implies $u_S \geq u$ and therefore $\mathcal{M} \subset \mathbb{P}'_{\text{erg}}(X \setminus X_U)$. If $u(p) > u_S(p)$, $\mathcal{M}$ would be uncountable. Finally, (2)-(3) with $u = u_S$ imply condition (2) of Thm. 4.4 so X is almost-Borel isomorphic to a Markov shift. □

5 Continuous factors of Markov shifts: universality

We prove Theorem 1.10. We first deal with the following compact case and then reduce the general case to this one through an entropy formula.

Theorem 5.1. *Let (X, S) be an irreducible SFT with period p and let $\pi : (X, S) \to (Y, T)$ be a continuous factor map. Then, for any $0 \leq h < h(T)$, there is a period p, irreducible SFT $X' \subset X$ such that $h(X') > h$ and the restriction of π to X' is injective. In particular, (Y, T) is $(h(T), p)$-universal.*

Remark 5.2. The universality claim of Theorem 5.1 fails badly for Borel factor maps, even if finite to one. For example, from a mixing shift of finite type with entropy $h > 0$, with the Borel Periodic Decomposition one can show that there is a Borel at most 2-to-1 map which collapses all ergodic measures with maximum period 2 to ones with maximum period 1, and is the identity on supports of other ergodic measures. The image is not h-universal.

To prove Theorem 5.1, we will use the formulas (2.2) for the topological entropy of a t.d.s. in terms of separated and spanning sets. We slightly extend the usual notion saying that two points are (ϵ, a, b)-*separated* if, for some $a \leq k < b$, their kth iterates are at a distance at least ϵ.

Section 2.5 recalls some standard definitions and notations for Markov shifts including $[w]_X$, $[w]$, and $x|_a^b$.

If v, w are two finite words over the alphabet of X, then $|v|, |w|$ are their lengths and $[v.w] := \sigma^{|v|}[v] \cap [w]$ is the cylinder $\{x \in X : x|_{-|v|}^0 = v \text{ and } x|_0^{|w|} = w\}$. We define $v^\infty.w^\infty$ as the unique point in all $[v^n.w^n]$ for $n \geq 1$ and $v^\infty := v^\infty.v^\infty$. To simplify notations, we let sometimes a word stand for its length, e.g., $n \geq \ell^A L_1$ actually means $n \geq A|\ell| + |L_1|$.

Proof of Theorem 5.1. Observe that the claim about universality follows immediately from the embedding claim according to Proposition 4.2. Notice that $h(T) > 0$. Let G be a strongly connected, finite graph presenting X. Fix $0 < \zeta < 1$ small enough and then h' such that $h < (1-\zeta)h(T) < h' < h(T)$. Let $\eta_1 > 0$ small enough such that the separation entropy at scale $4\eta_1$ satisfies $h_{\text{sep}}(T, \pi(X), 4\eta_1) > h' > h$ (recall $\pi(X) = Y$). Observe that

$$h_{\text{sep}}(T, \pi(X), 4\eta_1) = \sup_{v \in G} h_{\text{sep}}(T, \pi([v]), 4\eta_1)\ . \tag{5.3}$$

G is finite, hence this supremum is achieved at some vertex v, which we will denote by 0:

$$h_{\text{sep}}(T, \pi([0]), 4\eta_1) > h' > h\ . \tag{5.4}$$

Claim 1. *Let ℓ and $\tilde{\ell}$ be loops in G based at vertex 0 such that $P := \pi(\ell^\infty) \neq \tilde{P} := \pi(\tilde{\ell}^\infty)$. Then there are a positive multiple M of p and a number $0 < \eta < \eta_1$ such that for all integers $A, C \geq M$, if $x, y \in \pi([\ell^A.\tilde{\ell}^C])$ and $-\ell^A + M \leq k \leq \tilde{\ell}^C - M$, then*

$$k = 0 \iff \max_{0 \leq j \leq \ell^A - M} d(T^{-j}x, T^{k-j}y) < \eta. \tag{5.5}$$

Moreover, for any $x, y \in X$,

$$x|_{-M}^M = y|_{-M}^M \implies d(\pi(x), \pi(y)) < \eta/4. \tag{5.6}$$

Proof of Claim 1. Let $Z = \pi(\ell^\infty.\tilde{\ell}^\infty)$. As Z is a heteroclinic point, its orbit is discrete. Define $r_0 = \min(d(Z, \mathcal{O}(Z) \setminus \{Z\}), \eta_1) > 0$. The uniform continuity of π gives $M \in p\mathbb{N}$ such that, for all $u, v \in X$, $u|_{-M}^M = v|_{-M}^M$ implies $d(\pi(u), \pi(v)) < r_0/16$. We will prove Claim 1 for this M and $\eta = r_0/4$.

Let $\hat{x}, \hat{y} \in [\ell^A.\tilde{\ell}^C]$, $x = \pi(\hat{x}), y = \pi(\hat{y})$ and $-\ell^A + M \leq k \leq \tilde{\ell}^C - M$. Note, $\hat{x}|_{-\ell^A}^M = \hat{y}|_{-\ell^A}^M$ so, if $k = 0$:

$$0 \leq j \leq \ell^A - M \implies d(T^{-j}x, T^{k-j}y) < r_0/16 = \eta/4\ .$$

Also, $\hat{y}|_{k-M}^{k+M} = (\ell^\infty.\tilde{\ell}^\infty)|_{k-M}^{k+M}$. $d(T^k y, T^k Z) < r_0/16$. The same holds with x instead of y. If $k \neq 0$,

$$\begin{aligned}
\max_{0 \leq j \leq \ell^A - M} d(T^{-j}x, T^{k-j}y) &\geq d(x, T^k y) \\
&\geq d(Z, T^k Z) - d(Z, x) - d(T^k y, T^k Z) \\
&> r_0 - r_0/16 - r_0/16 = (7/8)r_0 > \eta\ .
\end{aligned}$$

This proves Claim 1. □

We fix M, ℓ, η according to Claim 1. Recall $\zeta > 0$.

Claim 2. There is $M_0 \in \mathbb{N}$ such that for all large $M \in p\mathbb{N}$, there is a family Γ_N of N-loops based at vertex 0 such that $\#\Gamma_N \geq e^{h'N}$ and the following holds.

If $\{\bar{x}^\gamma \in X : \gamma \in \Gamma_N\}$ is such that $\bar{x}^\gamma|_0^N = \gamma$, then for all $\gamma \neq \gamma'$ in Γ_N, two separation properties are satisfied:

(S1) $\pi(\bar{x}^\gamma)$ and $\pi(\bar{x}^{\gamma'})$ are $(\eta, M + M_0, N - (M + M_0))$-separated;

(S2) $\pi(\bar{x}^\gamma)$ is $(\eta, M + M_0, N - (M + M_0))$-separated from $\pi(\hat{z})$ whenever $\hat{z} \in X$ satisfies, for some $k \in \mathbb{Z}$ and $m := \lceil \zeta N \rceil$: (i) $\hat{z}|_k^{k+m} = \ell^\infty|_0^m$ and (ii) $[k, k+m] \subset [M + M_0, N - (M + M_0)]$.

Proof of Claim 2. We choose $M_0 \in p\mathbb{N}$ such that, for any vertex v in the graph G, from which there is a path to 0 of length a multiple of p, we choose paths of length M_0: one, denoted $p^{\to v}$, from vertex 0 to v and another, denoted $p^{v\to}$, from v to 0.

Because $\eta < \eta_1$ and the inequality in (5.4) is strict, there is an $\epsilon > 0$ such that for any sufficiently large n there is a $(4\eta, n)$-separated subset S_n of $\pi([0])$ such that $\#S_n \geq e^{(1+\epsilon)h'n}$. For each $x \in S_n$, pick $\hat{x} \in \pi^{-1}(x) \cap [0]$ and define the following concatenation:

$$\gamma(\hat{x}) \ := \ p^{\to \hat{x}_{-M}} \cdot \hat{x}|_{-M}^{n+M} \cdot p^{\hat{x}_{n+M}\to} \ .$$

Given n, define $N = n + 2M_0 + 2M$; for x in S_n, $\gamma(\hat{x})$ is a loop of length N based at 0. Define

$$\begin{aligned} \widehat{\Gamma_N} \ &= \ \{\gamma(\hat{x}) : x \in S_n\} \ , \\ \Gamma_N \ &= \ \{\gamma \in \widehat{\Gamma_N} : \gamma \text{ satisfies } (S2)\} \ . \end{aligned}$$

We will show that for all sufficiently large n, Claim 2 holds for this Γ_N.

For distinct $w, x \in S_n$, there is an integer $0 \leq k < n$ such that $d(\pi(\sigma^k \hat{w}), \pi(\sigma^k \hat{x})) > 4\eta$. Hence, given any $\bar{w}, \bar{x}$ in X such that $\bar{w}|_0^N = \gamma(\hat{w})$ and $\bar{x}|_0^N = \gamma(\hat{x})$, we have from (5.6) in Claim 1 some k in the interval $[M+M_0, n+M+M_0] = [M+M_0, N-(M+M_0)]$ such that:

$$\begin{aligned} &d\big(T^{k+M+M_0}\pi(\bar{w}), T^{k+M+M_0}\pi(\bar{x})\big) \\ &\qquad > d\big(T^k\pi(\hat{w}), T^k\pi(\hat{x})\big) - 2\eta/4 \ > \ \eta \ . \end{aligned}$$

As $k + M + M_0 \in [M + M_0, N - M - M_0]$, this shows that Γ_N will satisfy the separation property $(S1)$.

Let S_n' be the set of points $x \in S_n$ such that $\gamma(\hat{x})$ fails the separation property $(S2)$. Pick H such that $h_{\text{top}}(Y) < H < h'/(1-\zeta)$. By (2.2) we can find a number $C < \infty$ such that

$$\forall m \geq 0 \quad r_{\text{span}}(\eta/2, m, \pi(X), T) \leq Ce^{Hm} . \tag{5.7}$$

As $Y = \pi(X)$ is compact and π uniformly continuous,

$$\exists C' < \infty \; \forall m \geq 0 \quad r_{\text{span}}(\eta/2, m, \pi[\ell^{[m/\ell]}]_X, T) \leq C' . \tag{5.8}$$

Now suppose $m := \lceil \zeta N \rceil$ with $[k, k+m] \subset [M + M_0, N - (M + M_0)]$ as in (S2). It follows from (5.7) and (5.8) that the set of all $\pi(\hat{z})$ such that $\hat{z}|_k^{k+m} = \ell^\infty|_0^m$ is contained in at most $Ce^{kH} \times C' \times Ce^{(N-k-\zeta N)H} = C'C^2e^{(1-\zeta)HN}$ dynamical $(\eta/2, N)$-balls. No such set can contain two $(\eta, M+M_0, N-(M+M_0))$-separated points. Thus, considering the union over k we have $\#S_n' \leq NC'C^2e^{(1-\zeta)HN}$ and therefore for large $N = n+2(M+M_0)$ and for $C'' = e^{-2(M+M_0)(1+\epsilon)h'}$,

$$\begin{aligned} |\Gamma_N| \;&=\; |\widehat{\Gamma_N}| - |S_n'| = |S_n| - |S_n'| \\ &\geq\; C''e^{(1+\epsilon)h'N} - NC'C^2e^{(1-\zeta)HN} \;\geq \frac{C''}{2}e^{(1+\epsilon)h'N} >\; e^{h'N} \end{aligned} \tag{5.9}$$

where the last inequality holds for large N because $(1-\zeta)H < h'$. This finishes the proof of Claim 2. $\square$

As X has period p, we may fix loops L_1, L_2 based at vertex 0 such that $|L_2| = |L_1| + p \in p\mathbb{N}$. As in Claim 1, we fix loops ℓ and $\tilde{\ell}$ and obtain an integer M. We also fix integers M_0, N (and possibly also increase M) and a set of words Γ_N as in Claim 2. We use markers of the form $m_i := \ell^A\tilde{\ell}^C L_i$, $i = 1, 2$, for some integers $A, C \geq M$. To recognize markers, we increase the integers C and then A so that:

$$|\tilde{\ell}^C| > \zeta N + 2M + M_0 \text{ and } |\ell^A| > \tilde{\ell}^C + \max_{i=1,2} L_i + \zeta N + 2M + M_0. \tag{5.10}$$

We consider the subshift of finite type $X_K \subset X$ defined as the set of paths obtained from concatenations of words of the form $m_a w_1 w_2 \ldots w_K$ where K is fixed, but large, $a = 1, 2$ and $w_1, w_2, \ldots, w_K \in \Gamma_N$.

Observe that X_K is irreducible. The two lengths $|m_a| + K|w_i|$, for $a = 1, 2$ (and any i) are multiple of p and differ by p. Hence the period of X_K is exactly p. Moreover, by (5.9), the topological entropy of X_K satisfies:

$$h_{\text{top}}(X_K) \;\geq\; \frac{K \log \#\Gamma_N}{KN + |m_2|} \;>\; \frac{1}{1 + \frac{|L_2| + |\ell^A\tilde{\ell}^C|}{KN}} h' ,$$

with the right side greater than h for large K (given N). It only remains to show that $\pi : X_K \to Y$ is injective. Let $\bar{x}, \bar{y} \in X_K$ with $\pi(\bar{x}) = \pi(\bar{y})$.

We first prove $\mathfrak{M}(\bar{x}) = \mathfrak{M}(\bar{y})$ where $\mathfrak{M}(\bar{x})$ is the set of positions where a marker m_i appears. Assume that $0 \in \mathfrak{M}(\bar{x})$ so: $\bar{x}|_0^{\ell^A} = \ell^A$. We claim that the corresponding subword of $\bar{y}$ must also be part of marker (mostly). Indeed, the separation property (S2) from Claim 2 implies that, if, for any integer r, $\bar{y}|_r^{r+N}$ coincides with some $w_i \in \Gamma_N$, then $[r + M + M_0, r + N - M - M_0]$ cannot overlap $\bar{x}|_0^{\ell^A} = [0, \ell^A]$ on a set of length $\geq \zeta N$. Thus, $\bar{y}|_{\zeta N+M+M_0}^{\ell^A-\zeta N-M-M_0}$ must be part of a marker $m_a = \ell^A \tilde{\ell}^C L_a$ ($a = 1$ or 2).

It follows that $\mathfrak{M}(\bar{y})$ contains some k with $-\tilde{\ell}^C - L_i - \zeta N - M - M_0 \leq k \leq \zeta N + M + M_0$. Thanks to (5.10), $-\ell^A + M \leq k \leq \tilde{\ell}^C - M$ and Claim 1 applied to $\sigma^{\ell^A}\bar{x}, \sigma^{\ell^A - k}\bar{y} \in [\ell^A.\tilde{\ell}^C]$ yields $k = 0$. It follows that $\mathfrak{M}(\bar{x}) = \mathfrak{M}(\bar{y})$ by symmetry.

Let $n_1 < n_2$ be two consecutive elements of $\mathfrak{M}(\bar{x}) = \mathfrak{M}(\bar{y})$. By construction,

$$\exists a, b \in \{1, 2\} \exists w, w' \in (\Gamma_N)^K \quad \bar{x}|_{n_1}^{n_2} = m_a w_1 \dots w_K \text{ and } \bar{y}|_{n_1}^{n_2} = m_b w'_1 \dots w'_K.$$

Note $|m_a| = |m_b|$, so $a = b$. Assume by contradiction that there is some $s = 1, \dots, K$, such that, denoting $r := n_1 + |m_a| + (s-1)N < n_2$, $\gamma := \bar{x}|_r^{r+N}$, $\gamma' := \bar{y}|_r^{r+N}$ are distinct elements of Γ_N. Now, for all $0 \leq k < N$,

$$0 = d(\pi(\sigma^{r+k}\bar{x}), \pi(\sigma^{r+k}\bar{y})) > d(\pi(\sigma^k \bar{x}^\gamma), \pi(\sigma^k \bar{x}^{\gamma'})) - \eta/2$$

but this should be positive for some $k \in [M + M_0, N - M - M_0]$ by the separation property (S1) in Claim 2. As $\inf \mathfrak{M}(\bar{x}) = -\infty$ and $\sup \mathfrak{M}(\bar{x}) = \infty$, we obtain $\bar{x} = \bar{y}$, concluding the proof. $\square$

Theorem 1.10 is now an obvious consequence of the next Proposition (whose proof follows).

Proposition 5.11. *Let $\pi : (X, S) \to (Y, T)$ be a continuous factor map from an irreducible, period p Markov shift into a self-homeomorphism of a Polish space. For any $\mu \in \mathbb{P}_{\text{erg}}(S)$ and $0 < h < h(T, \pi_*\mu)$, there exists $\nu \in \mathbb{P}_{\text{erg}}(S)$ with compact support and $h(T, \pi_*\nu) > h$. In particular,*

$$\begin{aligned} \sup\{h(T, \pi(\Sigma)) : \Sigma \subset X,\ \Sigma \textit{ irreducible period } p \textit{ SFT}\} \\ = \sup\{h(T, \pi_*\mu) : \mu \in \mathbb{P}_{\text{erg}}(S) \textit{ and } \operatorname{supp}\mu \textit{ compact}\} \\ = \sup\{h(T, \pi_*\mu) : \mu \in \mathbb{P}_{\text{erg}}(S)\}. \end{aligned} \tag{5.12}$$

To prove the above proposition, we need some definitions and notations. For a Borel partition P, ∂P denotes the union of the boundaries of the elements of P. For $x \in X$, $P(x)$ is the unique element of P containing x. P^n is the set of words $v = v_0 \dots v_{n-1}$ on P of length n. Any such word defines a cylinder $[v] := v_0 \cap T^{-1}v_1 \cap \dots \cap T^{-n+1}v_{n-1}$. v is the P, n-name of any point in $[v]$. P^n will also denote the set of cylinders defined by words on P of length n. Depending on the setting $P^n(x)$ will mean either the P, n-name or cylinder of x.

Proof of Prop. 5.11. It is enough to prove the first claim. Indeed, the first equality in (5.12) is easily checked: any invariant compact subset of an irreducible, period p Markov shift is contained in a SFT and any SFT is included in one which is irreducible and with period p. The second equality in (5.12) is an obvious consequence of the first claim, to which we turn.

Let $\delta := (h(T, \pi_*\mu) - h)/h > 0$. As Y is Polish, there exists a finite Borel partition P such that

$$h(T, \pi_*\mu) < h(T, \pi_*\mu, P) + \delta h/10 \text{ and } \pi_*\mu(\partial P) = 0. \tag{5.13}$$

Fix $t_0 > 0$ such that, for all large n, the number of subsets of $\{1, \ldots, n\}$ with cardinality at most $t_0 n$ is less than $e^{(\delta h/20)n}$. As π is continuous, there exist an integer M and a Borel set $X_1 \subset X$ such that $\mu(X_1) > 1 - \min(\delta h/(40 \log \#P), t_0/2)$ and

$$\forall x \in X_1 \ \forall w \in X \ x|_{-M}^{M} = w|_{-M}^{M} \implies P(\pi(x)) = P(\pi(w)).$$

Let 0 be a vertex of G with $\mu([0]) > 0$. Define X_0 to be the set of points in X such that $x_n = 0$ for infinitely many positive n and also for infinitely many negative n. By ergodicity, $\mu(X_0) = 1$.

Claim 5.14. *There exists a period p SFT $\bar{X} \subset X_0$ and a continuous factor map $p : X_0 \to \bar{X}$ such that, if $X_2 := \{x \in X_0 : p(x)|_0 \neq x|_0\}$, then:*

$$\mu(X_2) < \frac{\min\left(\delta h/(40 \log \#P), t_0/2\right)}{2M+1} . \tag{5.15}$$

Proof of Claim 5.14. The first return words at 0 are the words w such that $w0$ is a word of X, $w_0 = 0$, and $0 \notin \{w_1, \ldots, w_{|w|-1}\}$. The loop graph at 0 is the graph $\hat{G}$ with:

- vertices: 0 and (w, k) for $0 < k < |w|$ and w is a first return word at 0;
- edges: $0 \to (w, 1)$, $(w, k) \to (w, k+1)$, and $(w, |w| - 1) \to 0$, for w a first return word at 0 and $0 < k < |w| - 1$.

The *loop shift* (see, e.g., [9]) for G at 0 is the Markov shift $\hat{X}$ presented by $\hat{G}$. Note, $\hat{X}$ like X has period p. Let $\psi : X_0 \to \hat{X}$ be the obvious topological conjugacy.

Given an enumeration $w^1, w^2, \ldots$, without repetition, of the first return words at 0, let $\hat{X}_N$ be the SFT defined by the finite subgraph $\hat{G}_N$ of $\hat{G}$ obtained by restricting the previous construction to the words w^n for $n \leq N$. We fix N large enough so that $\hat{G}_N$ has the same period p (g.c.d. of loop lengths) as $\hat{G}$; for all $n \geq N$, np is a sum of lengths of first return loops to 0 in $\hat{G}_N$; and $[0] \cup \bigcup \{[(w^n, k)]_X : n \leq N, 0 < k < |w^n|\}$ has $\psi_*\mu$-measure close enough to 1 so that (5.15) will hold. Then we define the SFT $\bar{X} = \psi^{-1}\hat{X}_N \subset X_0$.

We can define a map $q : \hat{X} \to \hat{X}_N$ by replacing each w^n, $n > N$, by some concatenation $\tilde{w}^n$ of w^i's for $i \leq N$ with total length $|w^n|$ (making choices depending only on $|w^n|$). We define $p : X_0 \to \bar{X}$ by $p = \psi^{-1} \circ q \circ \psi$. $\square$

We denote by $\bar{\pi}$ the restriction of π to $\bar{X} \subset X$ and set $\nu := p_*\mu$.

Observe that, for $x \in X$:

$$P(\bar{\pi}p(x)) \neq P(\pi(x)) \implies x \notin X_1 \text{ or } p(x)|_{-M}^{M} \neq x|_{-M}^{M}$$
$$p(x)|_{-M}^{M} \neq x|_{-M}^{M} \implies x \in S^{-M}X_2 \cup \cdots \cup S^{M}X_2.$$

Hence, by the Birkhoff ergodic theorem, there exists $X_3 \subset X$ such that $\mu(X_3) > 9/10$ and for all large n, all $x \in X_3$,

$$\frac{1}{n}\#\{0 \leq k < n : P(\bar{\pi}p(T^k x)) \neq P(\pi(T^k x))\} < \rho := \min\left(\frac{\delta h}{20 \log \#P}, t_0\right). \quad (5.16)$$

For any two words $v, w \in P^n$, define the relation:

$$v \sim w \iff \#\{0 \leq k < n : v_k \neq w_k\} < \rho n.$$

Note that for $v \in P^n$ for n large enough, by choice of t_0 we have

$$\#\{w : w \sim v\} \leq e^{(\delta h/20)n} \times \#P^{\rho n} \leq e^{(\delta h/10)n}\,. \quad (5.17)$$

The theorem of Shannon-McMillan-Breiman applied to $(T, \bar{\pi}_*\nu)$ gives sets E_n of P, n-words such that, for all large n, writing $[E_n] := \bigcup_{v \in E_n}[v]$,

$$\bar{\pi}_*\nu\left([E_n]\right) > 9/10 \text{ and } \#E_n \leq \exp(h(T, \bar{\pi}_*\nu) + \delta h/10)n. \quad (5.18)$$

Let $F_n := p^{-1}\bar{\pi}^{-1}([E_n]) \cap X_3$. It is a Borel set, so its continuous image, $\pi(F_n)$, is Borel too (up to a subset included in a set with zero $\pi_*\mu$-measure, see universal measurability of analytic sets in [26, Theorem 21.10]). Using $\bar{\pi}_*\nu := \mu \circ p^{-1} \circ \bar{\pi}^{-1}$,

$$\pi_*\mu(\pi(F_n)) = \mu(\pi^{-1}\pi(F_n)) \geq \mu(F_n) \geq \bar{\pi}_*\nu([E_n]) - \mu(X \setminus X_3) > 8/10.$$

Let n be large and $x \in F_n \subset X_3$. By construction of F_n, $v := P^n(\bar{\pi}p(x))$ belongs to E_n. Eq. (5.16) gives $P^n(\pi(x)) \sim v$.

$$P^n(\pi(x)) \subset \bigcup\big\{[v] : v \sim P^n(\bar{\pi}p(x))\big\}.$$

Thus, $G_n := \bigcup_{v \in E_n}\{w : w \sim v\}$ satisfies $[G_n] \supset \pi([F_n])$ so, using eqs. (5.18) and (5.17),

$$\pi_*\mu([G_n]) \geq \pi_*\mu(\pi(F_n)) > 8/10 \text{ and}$$
$$\#G_n \leq \#E_n \times \exp(\delta h n/10) \leq \exp\left((h(T, \bar{\pi}_*\nu) + \tfrac{2}{10}\delta h)n\right).$$

Applying the Shannon-McMillan-Breiman Theorem this time to $\pi_*\mu$ and P and recalling (5.13), we get:

$$h + \delta h = h(T, \pi_*\mu) \leq h(T, \pi_*\mu, P) + \delta h/10 \leq h(T, \bar{\pi}_*\nu) + \tfrac{2}{5}\delta h.$$

Hence, $h(T, \bar{\pi}_*\nu) > h$, proving the first claim of the Proposition. □

6 Bowen factors of Markov shifts

In this section we prove Theorem 1.12, which states conditions satisfied by Sarig's symbolic dynamics under which a factor of a Markov shift is almost-Borel isomorphic to a Markov shift.

Recall Definition 1.11 for Bowen type factor maps. A prototypical Bowen type map is a one-block code from an SFT onto a sofic shift; in this case, the relation $\sim$ on symbols is transitive. When $\pi : X \to Y$ is a continuous Bowen type factor map from an SFT and Y is not zero dimensional, the relation $\sim$ on symbols cannot be transitive. For our almost-Borel purposes, the condition (2) in Definition 1.11 is only a notational convenience.

Definition 6.1. Let X be a Markov shift with alphabet $\mathcal{A}$. For $a, b \in \mathcal{A}$, $X_{a,b}$ is the set of x in X such that $x_n = a$ for infinitely many negative n and $x_n = b$ for infinitely many positive n. X_a is the subset of X consisting of points x such that $x_n = a$ for infinitely many positive n and infinitely many negative n. The *return set* of X is $X_{\text{ret}} := \cup_a X_a$. The *Sarig regular set* of X is $X_{\pm\text{ret}} := \cup_{a,b} X_{a,b}$.

One virtue of the Sarig regular set of a Markov shift X is that it contains every compact subshift of X.

We will use the following consequence of Theorem 1.10 (proved in Sec. 5) to establish the universality claim of Theorem 1.12.

Proposition 6.2. *Let $\pi : (X, S) \to (Y, T)$ be a Borel factor map, from an irreducible Markov shift of period p. Assume that it is countable to one and Bowen type on the Sarig regular set $X_{\pm\text{ret}}$. Then $(\pi(X_{\pm\text{ret}}), T)$ is $(h(S), p)$-universal.*

The Bowen type assumption is key here - compare with Rem. 5.2.

Proof. It suffices to show that $\pi(X_{\pm\text{ret}})$ is $(h(S) - \epsilon, p)$ universal for every $\epsilon > 0$ (Prop. 4.2). Given ϵ, let Σ be an irreducible SFT of period p contained in $X_{\pm\text{ret}}$ such that $h(\Sigma) > h(S) - \epsilon$. Let $\overline{\Sigma}$ be $\pi\Sigma$, endowed with the quotient topology; as in [19] $\overline{\Sigma}$ is a compact metrizable dynamical system – use, e.g., Prop. B.2 with Σ compact metrizable and the quotient relation a closed set in $\Sigma \times \Sigma$ (π is Bowen type on $\Sigma \subset X_{\pm\text{ret}}$). It follows from Thm. 5.1 that $\overline{\Sigma}$ is $(h(S) - \epsilon, p)$-universal.

A countable-to-one map from a standard Borel space into another one has a Borel section [26, (18.10) and (18.14)]. It follows that $\pi\Sigma$ is a Borel set, and a set is Borel in $\overline{\Sigma}$ or $\pi\Sigma$ if and only if its preimage in Σ is Borel. Consequently the identity map $\overline{\Sigma} \to \pi\Sigma$ is a Borel isomorphism. Therefore $\pi\Sigma$, like $\bar{\Sigma}$, is $(h(S) - \epsilon, p)$-universal. □

The key step for the proof of Theorem 1.12 is the following. It is related by a classical construction of Manning [32]. We will let $\mathcal{A}(S)$ or $\mathcal{A}(X)$ denote the alphabet (symbol set) of a shift space (X, S). In the setting of Theorem 1.12, we have:

Proposition 6.3. *Let (X, S) be a Markov shift and let $\pi : (X, S) \to (Y, T)$ be as in Thm. 1.12: X satisfies the finite entropy condition* (1.3)*, π is a Borel factor map such that for each irreducible component Z of X,*

1. *π is Bowen type on the Sarig regular set $Z_{\pm ret}$, and*
2. *the restriction $\pi|Z_{\pm ret}$ is finite-to-one.*

Let $\bar{X}$ be the union of the Sarig regular sets $Z_{\pm \text{ret}}$ of the irreducible components Z of X.

Then the induced map $\mathbb{P}'_{\text{erg}}(\bar{X}) \to \mathbb{P}'_{\text{erg}}(\pi\bar{X})$ is surjective. Moreover, there is a countable collection of Borel factor maps $\pi' : (X', S') \to (Y', T') \subset (Y, T)$ for which the following hold.

1. *(X', S') is an irreducible Markov shift.*
2. *π' is both Bowen type and finite to one on the Sarig regular set $X'_{\pm \text{ret}}$.*
3. *If $\nu \in \mathbb{P}'_{\text{erg}}(T|\pi(\bar{X}))$, then there exists some π' in the collection and some $\mu' \in \mathbb{P}'_{\text{erg}}(S')$ such that $\pi' : (S', \mu') \to (T, \nu)$ is a measure-preserving isomorphism.*

Remark 6.4. In the above proposition, if π is continuous, or Hölder-continuous with respect to the exponential distance (see 2.5), then so are the extensions $\pi' : S' \to Y'$. Such a Hölder-continuous factor map, one-to-one on a set of full measure for a given measure has been obtained independently by Sarig[8].

Remark 6.5. Even though measures are supported on the return sets, our proof of Proposition 6.3 appeals to π being Bowen type on the (larger) Sarig regular sets.

Proof of Proposition 6.3. Let $\nu \in \mathbb{P}'_{\text{erg}}(T|\pi\bar{X})$. The set $\pi\bar{X}$ is the union of the countable collection of invariant sets $\pi Z_{\pm \text{ret}}$. Since $\pi|\bar{X}$ is at most countable to one, these sets are Borel. As ν is ergodic, there exists Z such that $\nu(\pi Z_{\pm \text{ret}}) = 1$. Because π is finite to one on $Z_{\pm \text{ret}}$ there exists $\mu \in \mathbb{P}'_{\text{erg}}(S|Z_{\pm \text{ret}})$ with $\pi\mu = \nu$ (Prop. B.1 and ergodic decompositon).

Thus $\mathbb{P}'_{\text{erg}}(\bar{X}) \to \mathbb{P}'_{\text{erg}}(\pi\bar{X})$ is surjective, as claimed. The rest of the proof is devoted to the construction of the factor maps $\pi' : (X', S') \to (Y', T')$.

Because μ is ergodic, there is a positive integer m and a set E in $Z_{\pm \text{ret}}$ of μ-measure one such that for every y in πE:

- y has exactly m preimages in E, and
- with ν_y denoting the measure assigning mass $1/m$ to each preimage point of y in E, for every Borel set B in X
$$\mu B = \int_Y \nu_y\big((\pi^{-1}y) \cap B\big)\, d\nu(y)\ .$$

[8]Private communication.

If $m = 1$, $\pi' := \pi$ already satisfies condition (3). Now suppose $m > 1$. Let E_k be the set of x in E such that, if $x^1, \dots, x^m$ are the distinct preimages in E of πx, then the m words $x^i[-k, k]$ are distinct. For large enough k, $\mu E_k > 0$. After passing to a higher block presentation of (Z, S), we may assume $k = 0$.

Let $\sim$ be some relation on $\mathcal{A}(Z)$ with respect to which π is Bowen type on $Z_{\pm\mathrm{ret}}$. Let (F_m, S_m) denote the m-fold fibered product system of $(Z, S|_Z)$ over $\sim$. Here

$$F_m := \{x = (x^1, \dots, x^m) \in Z^m : x^i \sim x^j, 1 \le i \le j \le m\}$$

(recall $x^i \sim x^j$ means $x^i_n \sim x^j_n$ for all n) and S_m is the restriction to F_m of the product map $S \times \cdots \times S$. Thanks to the Bowen property, (F_m, S_m) is a Markov shift, whose alphabet $\mathcal{A}(S_m)$ is a subset of the set of m-tuples of symbols from $\mathcal{A}(Z)$ which are mutually related. For $1 \le r \le m$, let $p_r : F_m \to X$ be the coordinate projection map $x \mapsto x^r$. Define $\widetilde{\pi} : F_m \to Y$ as the composition $\widetilde{\pi} = \pi \circ p_r$, for any p_r. Here $\widetilde{\pi}$ is well defined since $x \sim y \implies \pi(x) = \pi(y)$, for all $x, y \in Z$.

We define an S_m-invariant measure $\widetilde{\mu}$ on F_m as follows. For each y in πE, define a measure $\widetilde{\nu}_y$ on $\tilde{\pi}^{-1}(y)$ as follows: $\widetilde{\nu}_y$ assigns mass $1/m!$ to each m-tuple $(x^1, \dots, x^m)$ such that the m entries are distinct preimages of y (there are $m!$ such tuples for μ-a.e. y). Then for any Borel set B in F_m define

$$\widetilde{\mu}B = \int_Y \widetilde{\nu}_y\big((\pi^{-1}y) \cap B\big)\, d\nu(y) \; .$$

Then $p_r\widetilde{\mu} = \mu$ and $\widetilde{\pi}\widetilde{\mu} = \pi\mu = \nu$. Because μ is ergodic, we may take μ'' an ergodic measure from the ergodic decomposition of $\widetilde{\mu}$ such that $p_r\mu'' = \mu$, for $1 \le r \le m$, and $\widetilde{\pi}\mu'' = \nu$.

Claim 6.6. *For μ''-a.e. $x = (x^1, \dots, x^m) \in F_m$, for all $n \in \mathbb{Z}$:*
(i) the m symbols $x^1_n, \dots, x^m_n$ are pairwise distinct;
(ii) for $1 \le i, j \le m$: $x^i_n x^j_{n+1}$ is an S-word if and only if $j = i$.

Proof of Claim 6.6. Because $p_1\mu'' = \mu$ and $\mu E_0 > 0$, the set

$$E''_0 := \{(x^1, \dots, x^m) \in F_m : x^i_0 \ne x^j_0, 1 \le i < j \le m\}$$

satisfies $\mu'' E''_0 = \mu E_0 > 0$. Let $a = (a^1, \dots, a^m)$ be an m-tuple of distinct symbols such that $[a] := \{x \in F_m : x_0 = a\} \subset E''_0$ satisfies $\mu''[a] > 0$.

We note that (i) follows from (ii) and prove this last assertion of the claim. For a contradiction, assume that there are symbols $b = (b^1, \dots, b^m)$ and $c = (c^1, \dots, c^m)$ in $\mathcal{A}(S_m)$ such that $\mu''[bc] > 0$ and (say) b^2c^1 is an S-word (i.e. the transition $b^2 \to c^1$ is allowed in S).

The following hold for all $x = (x^1, \dots, x^m)$ from a set of full μ'' measure, (1) because, for each r, $p_r(\mu'') = \mu$ which is ergodic and (2) by ergodicity of μ'':

1. There is a symbol which in every x^i occurs with positive frequency in positive and in negative coordinates.

2. There are sequences of integers (i_n), (j_n) (depending on x) with $i_1 < j_1 < i_2 < j_2 < \cdots$ such that for all n, $x_{j_n} x_{j_n+1} = bc$ and $x_{i_n} = a$.

Pick one such x. For each $n \geq 1$, define a point $z^{(n)}$ in S by setting

$$\begin{aligned} (z^{(n)})_t &= (x^1)_t \quad \text{if } t \geq j_n + 1 \\ &= (x^2)_t \quad \text{if } t \leq j_n \,. \end{aligned}$$

Then for all n, $z^{(n)} \sim x^1$, so $\pi(z^{(n)}) = \pi(x^1)$. If $\ell > n$, then $(z^{(n)})_{i_\ell} = a^1$ and $(z^{(\ell)})_{i_\ell} = a^2$, so $z^{(n)} \neq z^{(\ell)}$. By condition (1), the points $z^{(n)}$ are all in Z_{ret}. This contradicts π being finite to one on $Z_{\pm\text{ret}}$, and proves (ii). $\square$

Let $(\widetilde{X_m}, \widetilde{S_m})$ be the Markov shift contained in the Markov shift (F_m, S_m) and which is defined by the following conditions:

1. $\mathcal{A}(\widetilde{S_m})$ is the set of $a = (a_1, \ldots, a_m)$ in $\mathcal{A}(S_m)$ such that the symbols $a_1, \ldots, a_m$ from $\mathcal{A}(S)$ are distinct.

2. There is a transition from $a = (a_1, \ldots, a_m)$ to $b = (b_1, \ldots, b_m)$ if and only if the following holds: for $1 \leq i, j \leq m$ there is an S transition $a_i \to b_j$ if and only if $i = j$.

The claim 6.6 implies that μ'' assigns measure one to the Markov shift $\widetilde{X_m}$. By ergodicity of μ'', there is a unique irreducible component (X'', S'') of $\widetilde{X_m}$ such that $\mu'' X'' = 1$.

Now define (X', S') to be the shift space (on a countable alphabet) which is the image of (X'', S'') under the one-block map ψ defined by the rule $\psi : (a_1, \ldots, a_m) \mapsto \{a_1, \ldots, a_m\}$. The map ψ is right resolving: i.e., if $A_0 A_1$ is a word of length two occuring in a point of X', and $\psi : \widetilde{a_0} \mapsto A_0$, then there exists a unique symbol $\widetilde{a_1}$ following $\widetilde{a_0}$ in $\widetilde{X_m}$ such that $\psi : \widetilde{a_1} \mapsto A_1$. Therefore X' is a Markov shift and it is also irreducible. The map ψ is likewise left resolving. Thus, for every x in X', for every $\widetilde{a}$ in $\mathcal{A}(\widetilde{X_m})$ such that $\psi : \widetilde{a} \mapsto x_0$, there exists a unique preimage $\widetilde{x}$ of x such that $\widetilde{x_0} = \widetilde{a}$. Every point of X' has exactly $m!$ preimage points in $\widetilde{X_m}$.

The map ψ only collapses points which have the same image under $\widetilde{\pi}$. Therefore there is a Borel map $\pi' : (X', S') \to (Y', T')$ defined by $\widetilde{\pi} = \pi' \psi$, where $Y' = \tilde{\pi}(X'') = \pi'(X')$ and T' is the restriction of T to Y'. Let $\sim$ also denote the natural relation on the alphabet of X': $\{a_1, \ldots, a_m\} \sim \{b_1, \ldots, b_m\}$ iff $a_i \sim b_j$ for all i, j. If w', x' are in $X'_{\pm\text{ret}}$, there are w'', x'' in $X''_{\pm\text{ret}}$ such that $\psi x'' = x'$, $\psi w'' = w'$. (This is the one point where the proof would fail if we used Z_{ret} rather than $Z_{\pm\text{ret}}$.) Then $p_1 x'' = x \in Z_{\pm\text{ret}}$ and $p_1 w'' = w \in$

$Z_{\pm\text{ret}}$; and, $\pi'(x') = \pi'(w')$ if and only if $\pi(x) = \pi(w)$. Because π is Bowen type on $Z_{\pm\text{ret}}$, it follows that π' is Bowen type on $X'_{\pm\text{ret}}$.

A set in X'' of full measure for μ'' is $E'' = \{(x_1, \ldots, x_m) \in X'' : x_i \in E, 1 \leq i \leq m\}$. Points in E'' with the same $\widetilde{\pi}$ image are mapped by ψ to the same point in X'. Setting $\mu' = \pi\mu''$, the map

$$\pi' : (S', \mu') \to (T, \nu)$$

is an isomorphism of measure-preserving systems.

The Markov shift (X'', S'') constructed above given ν was an irreducible component of the Markov shift obtained by restricting $\widetilde{X_m}$ to a higher block presentation. The higher block presentation was a notational convenience, but in any case there are only countably many higher block presentations of a given $\widetilde{X_m}$. Any Markov shift has only countably many irreducible components. Consequently, we build only countably many irreducible Markov shift extensions. $\square$

Proof of Theorem 1.12. Prop. 6.3 implies the surjectivity of the induced map $\mathbb{P}'_{\text{erg}}(S|\bar{X}) \to \mathbb{P}'_{\text{erg}}(T|\pi\bar{X})$. The characterization of Markov shifts in terms of universal subsystems from Thm. 4.4 will yield the almost-Borel isomorphism of $\pi(\bar{X})$ to a Markov shift as follows.

Let ν be an ergodic and invariant probability measure of $(\pi(\bar{X}), T)$. Let $\pi' : (X', S') \to (Y', T)$ be the extension given by Prop. 6.3 with $\mu' \in \mathbb{P}'_{\text{erg}}(S')$ such that $\pi'\mu' = \nu$. Letting q denote the period of the irreducible Markov shift (X', S'), we note:

1. The set of periods of (T, ν) coincides with that of (S', μ') and therefore contains q;

2. The image of $(X'_{\pm\text{ret}}, S')$ contains a strictly $(h(S'), q)$-universal system (by Proposition 6.2, because π' is finite to one, Bowen type on $X'_{\pm\text{ret}}$).

Using that entropy is a Borel function of the measure and the Borel Periodic Decomposition (Thm. 2.5), we obtain an invariant Borel subset $Z \subset \pi'(X')$ such that, for all measures m on $\pi'(X'_{\pm\text{ret}})$, $m(Z) = 1$ if and only if q is a period of m and $h(T, m) < h(S')$. It follows from (2) above that Z is strictly $(h(S'), q)$-universal. Note that Z depends only on the extension π', hence there are at most countably many such sets Z, also: $u_T(q) \geq h(Z) = h(S')$.

Thus, either μ' is the measure of maximal entropy for (X', S'), or $h(T, m) = h(S', \mu') < h(S')$ so $m(Z) = 1$. Altogether, then, $(\pi(\bar{X}), T)$ is almost-Borel isomorphic to a countable union of:

1. strictly $(u_T(p), p)$-universal systems (using Lemma 3.2);

2. systems supporting a single measure μ of $\mathbb{P}'_{\text{erg}}(T)$, such that there exists p with $h(T, \mu) = u_T(p)$ and (T, μ) is p-Bernoulli.

Theorem 3.10 implies that $\pi(\bar{X}) \setminus \pi(\bar{X})_U$ (in the notation of that theorem) carries only measures from (2) above. By Theorem 4.4, it follows that $\pi(\bar{X})$ is almost-Borel isomorphic to a Markov shift. $\square$

7 Continuous factors of Markov shifts: pathology

The results of this section will give limits to any strengthening of our two main theorems (1.12 and 1.10) about continuous factors of Markov shifts. Recalling the discussion after Theorem 1.10 we build examples with large sets of

- measures with entropy greater than the entropy $h_*(\pi)$ from Theorem 1.10 in Proposition 7.1.
- m.m.e.'s for a factor which is not finite to one, in Corollary 7.6.
- period-maximal measures for a finite-to-one but not Bowen type factor in Corollary 7.10.

We also remark that a factor of an irreducible Markov shift by a continuous map need not be a factor by a Bowen type map, even if it is a compact expansive system. Indeed, among subshifts (up to topological conjugacy, the compact zero-dimensional expansive systems), the continuous factors of irreducible Markov shifts are exactly the coded systems [16]. But among these, the factors by one-block codes are the factors by Bowen type maps, and form a proper subset of the coded systems [16].

7.1 Arbitrary dynamics in high entropy

It is well known that the entropy of irreducible Markov shifts can increase under one-block codes (which are continuous and Bowen type factor maps); see e.g. [15, 16, 17, 38]. The following construction, resembling [38, Examples 3.3,3.4], further shows that a one-block code image of the nonrecurrent part of a Markov shift can have virtually no almost-Borel relation to that Markov shift. The quantity $h_*(\pi)$ in the statement of Proposition 7.1 comes from Theorem 1.10.

Proposition 7.1. *Suppose Y is a subshift of $\{0,1\}^{\mathbb{Z}}$ and $\epsilon > 0$. Then there is a locally compact irreducible Markov shift X and a one-block code π from X into $\{0,1,2\}^{\mathbb{Z}}$ such that X is the disjoint union of Borel subsystems X', X'', X''' for which the following hold.*

1. *$\pi(X')$ is almost-Borel isomorphic to X with $\pi|X'$ one-to-1 ;*
2. *$\pi(X'')$ is almost-Borel isomorphic to Y with $\pi|X''$ countable-to-1.*
3. *$\pi(X''')$ is a fixed point and X''' is a finite orbit.*

4. $h_*(\pi) = h(X) < \epsilon$.

5. $\pi(X)$ *is compact and almost-Borel isomorphic to the disjoint union of* Y *and* X.

Proof. We build in stages a labeled graph G defining π. The Markov shift X will be the edge shift defined by G. Each edge will be labeled by a symbol from $\{0, 1, 2\}$. The one-block code will be the rule replacing an edge with its label.

First, there is a labeled subgraph G^+ which has for every Y-word W(including the empty word $\emptyset$) a vertex v_W, and for $i \in \{0, 1\}$ with Wi a Y-word, has an edge labeled i from v_W to v_{Wi}. Then for each z in Y, there is a unique path from $v_\emptyset$ labeled by the onesided sequence $z[0, \infty) = z_0 z_1 \dots$. Similarly build a graph G^- such that for each y in Y there is a unique left infinite path into $v_\emptyset$ labeled by $y(-\infty, -1] = ...y_{-2}y_{-1}$.

Let X'' be the edge shift presented by $G^- \cup G^+$. Note, $v_\emptyset$ is the only common vertex of G^-, G^+. The image $\pi X''$ is the set of all shifts of sequences that are concatenations $y(-\infty - 1]z[0, \infty)$ with y, z in Y. For $n \in \mathbb{N}$, define

$$B_n \;=\; \{y \in \pi(X'') \setminus Y : \; y[-n, n] \text{ is not a } Y\text{-word}\} \;,$$

a possibly empty wandering subset of Y. Because $\pi(X'') \setminus Y = \cup_n B_n$, an almost null set, the inclusion $Y \subset \pi(X'')$ gives an almost-Borel isomorphism. Any $x \in X''$ is determined by $\pi(x)$ and x_0, and therefore $\pi|X''$ is countable-to-one. Claim (2) ensues.

The definition of X will depend on positive integer parameters to be specified later: $(n_k)_{k=1}^\infty$, $(m_k)_{k=1}^\infty$ and M. For each integer $k \geq 1$ we add edges labeled by 2 as follows. Let $\mathcal{V}_k^-$ and $\mathcal{V}_k^+$ be the sets of vertices in G^- and G^+ corresponding to words of length k. For each v_- in $\mathcal{V}_{n_k}^-$ and each v_+ in $\mathcal{V}_{n_k}^+$, add in an otherwise isolated extra path from v_+ to v_- of length m_k. We also add an extra loop based at $v_\emptyset$ with length M (the loop is used to make the image of π compact).

Now fix (n_k) an arbitrary strictly increasing sequence of positive integers. Then for large M and (m_k) any sequence of large enough positive integers, we have $h(X) < \epsilon$. For a formal proof of this (obvious) fact, one can use for example the Gurevič entropy formula [21], which states that $h(X)$ is the growth rate of the number of loops based at $v_\emptyset$ when their length goes to infinity. We choose $\{m_1 < m_2 < \dots\} \cap M\mathbb{N} = \emptyset$.

Define $X''' = \pi^{-1}(2^\infty)$; X''' is the finite orbit corresponding to the special M loop at $v_\emptyset$. Then (3) holds. Next we show π is injective on X', the complement of $X'' \cup X'''$. If $y \in \pi(X')$, then there is at least one maximal block of 2s in y which is bordered by a 0 or 1. The length of the block (∞, m_k for some k, or a multiple of M) determines a vertex in G (more precisely, among the ones with ingoing or outgoing edge labeled 2) from which the preimage of y is uniquely determined. Because all nonatomic measures on X are supported on X', Claim (1) follows, and also (4).

The almost-Borel isomorphism claim of (5) then follows from (1) and (2) because $\pi(X) = \pi(X') \sqcup \pi(X'') \sqcup \pi(X''')$.

It remains to check the compactness. Suppose $z \in \overline{\pi(X)}$. If 2 does not occur in z, then z must be in $\pi(X'')$, which is compact. Now suppose $z = \lim \pi(x^n)$ for a sequence (x^n), 2 occurs in z and $z \neq 2^\infty$. If a finite maximal block of 2s occurs in z, then by considering the unique G-path above that block, one sees $z \in \pi(X')$. So suppose there is no such block. Suppose $z_i \neq 2$ and $z[i+1,\infty) = 2^\infty$. Let v_n be the terminal vertex of $(x^n)_i$. If a subsequence (v_n) goes to $+\infty$, then $z(-\infty, i]$ must be the left half of a point in Y; otherwise, a subsequence of (v_n) is constant and $z \in \pi(X')$. The argument for the case $z(-\infty, i] = 2^\infty$ is essentially the same. □

Remark 7.2. It is an exercise to show that X in Proposition 7.1 can in addition be chosen to be SPR (positive recurrent, and exponentially recurrent with respect to its measure of maximal entropy – see [9] for equivalent conditions and reference to [22] for more). In some ways, the SPR Markov shifts behave like shifts of finite type – but not here.

7.2 Wild Maximal Entropy

The next result realizes a wide class of systems T as equal entropy subsystems of continuous factors of SFTs. This will be used to prove Corollary 7.6.

First, we need to recall some definitions. A system is zero dimensional if its topology is generated by clopen sets. Every such system is topologically isomorphic to an inverse limit $X = X_1 \leftarrow X_2 \leftarrow \cdots$ where for all $n \in \mathbb{N}$, X_n is a subshift and the bonding map $X_n \leftarrow X_{n+1}$ is surjective. A continuous factor of a system is finite/zero dimensional, etc. if as a space it is finite/zero dimensional/etc.

The property *entropy-expansive* was defined by Bowen [7]. A zero dimensional t.d.s. is entropy-expansive if and only if the above inverse limit satisfies $h(X) = h(X_n)$ for some n. The property *asymptotically h-expansive* was a generalization defined by Misiurewicz [35] (under the name "topological conditional entropy", which is now probably best avoided [13, Remark 6.3.18]). Any asymptotically h-expansive system has finite entropy and has a measure of maximal entropy [35]. The asymptotic h-expansiveness property plays an important role in the entropy theory of symbolic extensions [13]. A zero dimensional compact t.d.s. is asymptotically h-expansive if and only if it is topologically isomorphic to a subsystem of a product $\prod_{k=1}^{\infty} X_k$ of some subshifts X_k such that $\sum_k h(X_k) < \infty$ (see [12] or [13, Theorem 7.5.9]).

Theorem 7.3. *Suppose T is a compact zero dimensional topological dynamical system which is asympotically h-expansive and is not entropy expansive. Then there is a continuous factor map from a mixing SFT onto a system Y such that $h(T) = h(Y)$ and Y contains a subsystem topologically conjugate to T.*

Proof. Without loss of generality, we assume $T \subset X = \prod_{k=1}^{\infty} X_k$ where each X_k is a mixing SFT with a fixed point, alphabet $\mathcal{A}_k$, and $\sum_k h(X_k) < \infty$. Then X is a factor of a mixing SFT [10, Theorem 7.1]. So it is enough to find a continuous factor map $\gamma : X \to Y$ such that $\gamma|T \equiv \mathrm{id}$, $T \subset Y \subset \prod_{k\geq 1}(\mathcal{A}_k \sqcup \{0\})^{\mathbb{Z}}$, and $h(Y) = h(T)$.

We introduce some notations. Suppose R is a subshift and M is a positive integer. Then $\mathcal{W}(M,R)$ is the set of words of length M occuring in points of R. We let $\hat{X}_N = X_1 \times \cdots \times X_N$ and T_N be the projection of T in $\hat{X}_N$. We write $x \in X$ as $(x_1, x_2, \dots)$ with $x_k \in X_k$. We denote by $(x_1,\dots,x_N)|J$ the restriction of these sequences to an integer interval J. Given $N, M \geq 1$, $x \in X$, we define

$$I(x,N,M) := \{j \in \mathbb{Z} : (x_1,\dots,x_N)|[j,j+M) \in \mathcal{W}(M,T_N)\}$$

and let $J(x,N,M,L)$ be the union of integer intervals of length L that are contained in $I(x,N,M)$.

We shall select two non-decreasing sequences of positive integers M_N, L_N, $N \geq 1$, and define $\gamma_N : \hat{X}_N \to (\mathcal{A}_N \sqcup \{0\})^{\mathbb{Z}}$ by:

$$\gamma_N(x) = (y_j)_{j\in\mathbb{Z}} \text{ with } y_j = \begin{cases} x_N|j \text{ if } j \in J(x,M_N,L_N) \\ 0 \text{ otherwise.} \end{cases}$$

We also define $\hat{\gamma}_N : X \to \prod_{1\leq k\leq N}(\mathcal{A}_k \sqcup \{0\})^{\mathbb{Z}}$ by:

$$x \mapsto (\gamma_1(x_1), \gamma_2(x_1,x_2), \dots, \gamma_N(x_1,\dots,x_N)),$$

and, finally, $\gamma : X \to \prod_{N\geq 1}(\mathcal{A}_N \sqcup \{0\})^{\mathbb{Z}}$ by:

$$\gamma(x) := (\gamma_1(x_1), \gamma_2(x_1,x_2), \dots) \text{ and let } Y := \gamma(X).$$

Y is a compact t.d.s. and a factor of X and $\gamma|T \equiv \mathrm{id}$.

Because T is not entropy expansive, we have for all N (perhaps after telescoping) that $h(T_{N+1}) > h(T_N)$. Hence, we can fix a sequence of numbers h_N, $N \geq 1$ such that $h(T_N) < h_N < h(T_{N+1})$ for all $N \geq 1$.

It now suffices to show that there are sequences of integers M_N, L_N such that:

Claim. *For all $N \geq 1$, there is $C_N < \infty$ such that, for all $\ell \geq 0$:*

$$\#\{\hat{\gamma}_N(x)|[0,\ell) : x \in X\} \leq C_N e^{h_N \ell}. \tag{7.4}$$

We extend the above claim to $N = 0$, by putting $\hat{\gamma}_0(x) := 0^\infty$, so $C_0 = 1$ and $h_0 = 0$ satisfy it for arbitrary M_0, L_0. We let $N \geq 1$, fix $0 < \epsilon < (h_N - h(T_N))/3$ and assume the claim for $N-1$ for some choice of M_{N-1}, L_{N-1}.

Pick $M := M_N \geq M_{N-1}$ such that, for some $K_1(M) < \infty$, for all $j \geq 0$:

$$\#\mathcal{W}(j,T_N) \leq (\#\mathcal{W}(M,T_N))^{j/M+1} \leq K_1(M) e^{(h(T_N)+\epsilon)j}. \tag{7.5}$$

By construction, the maximal integer intervals in $J(x,N,M,L)$ have length at least L. Therefore, letting $\mathcal{J}_\ell(N,M,L) := \{J(x,N,M,L) \cap [0,\ell) : x \in X\}$, we have, for $L := L_N$ large enough:

1. for all $\ell \geq 0$, $\#\mathcal{J}_\ell(N, M, L) \leq K_2(L)e^{\epsilon\ell}$;
2. $C_{N-1}K_1(M) \leq e^{\epsilon L}$.

Note that the elements of $\hat{\gamma}_N(x)|[0, \ell-1]$, $x \in X$, can be determined by specifying:

1. $J := J(x, N, M, L) \cap [0, \ell)$;
2. for each maximum integer interval I' in J, $\hat{\gamma}_N(x)|I'$;
3. for each maximum integer interval I'' in $[0, \ell) \setminus J$, $\hat{\gamma}_N(x)|I'' = \hat{\gamma}_{N-1}(x)|I'' \times 0^{I''}$.

For (1), the number of possibilities is bounded by:

$$\#\mathcal{W}(\ell, Z_L) \leq K_2(L)e^{\epsilon\ell}.$$

Fix one of these. Then, there are at most $\ell/L + 2$ intervals I' as in (2), so writing ℓ' for the sum of their lengths, the number of possibilities for (2) is at most:

$$K_1(M)^{\ell/L+2}e^{\ell'(h(T_N)+\epsilon)}.$$

For (3), we similarly get the bound:

$$(C_{N-1})^{\ell/L+2}e^{\ell'' h_{N-1}}.$$

Thus, the number of possibilities for $\hat{\gamma}_N(x)|[0, \ell-1)$ is bounded by:

$$K_2(L)(K_1(M)C_{N-1})^2 e^{(h(T_N)+3\epsilon)\ell}$$

As $h(T_N) + 3\epsilon \leq h_N$, (7.4) follows for an obvious choice of C_N. The induction and therefore the proof is complete.

□

Corollary 7.6. *For any ergodic, finite entropy, measure-preserving system Z, there is a continuous factor of a mixing SFT which admits among its ergodic measures of maximal entropy uncountably many copies of the product of Z with a Bernoulli system.*

Proof. Let $B = \prod_{n\geq 1} B_n$, where the B_n are positive entropy mixing SFTs with fixed points such that $h(B) < \infty$. B has a unique measure μ of maximum entropy, the product of the unique maximum entropy measures μ_n of the B_n. Each (B_n, μ_n) is a mixing Markov chain and therefore Bernoulli (by [20]). It then follows from [37, Theorem 1] that (B, μ) is also isomorphic to a Bernoulli shift

By the Jewett-Krieger theorem, there is a strictly ergodic subshift S which is measurably isomorphic to Z. Let $W = S \times \prod_{n=1}^{\infty} W_n$ with each W_n the identity map on a two point space. Then $B \times W$ is asymptotically h-expansive and not h-expansive so Theorem 7.3 applies with $T = B \times W$. □

Note that the Bernoulli factor is only used to ensure the topological condition of asymptotic h-expansivity without entropy-expansiveness. Moreover, if Z in Cor. 7.6 has positive entropy and the weak Pinsker property[9] then (of course) the conclusion holds for Z itself, with no need to take a product with a Bernoulli system.

The next proposition shows that the assumption that T not be entropy expansive was necessary for it to be embedded as a proper full entropy subsystem of a continuous factor of a mixing SFT.

Proposition 7.7. *Suppose X is a mixing SFT, Y is a zero dimensional continuous factor of X and T is an entropy expansive subsystem of Y such that $h(T) = h(Y)$. Then $T = Y$.*

Proof. Let Y be given as an inverse limit of subshifts Y_n by surjective bonding maps $p_n : Y_{n+1} \to Y_n$. Let $\pi_n : Y \to Y_n$ be the projection and let T_n be the subshift $\pi_n T$. With p_n also denoting the restriction of p_n to T, we have T as the inverse limit $T_n \leftarrow T_{n+1}$ by surjective bonding maps. Suppose $Y \neq T$.

Pick N such that $h(T_N) = h(T)$. We assume by contradiction, $T_N \neq Y_N$. Let $\gamma : X \to Y$ be the continuous factor map. Then $\pi_N \circ \gamma := \gamma_N$ is a factor map onto Y_N which is therefore mixing sofic. Hence $h(T_N) < h(Y_N) \leq h(Y)$, a contradiction. □

7.3 Wild period-maximal measures subsection

We now consider the case that $\pi\colon X \to Y$ is a bounded to one continuous factor map from an irreducible SFT X onto a zero dimensional system Y. In this case, Y has a unique measure of maximal entropy, which must be period-Bernoulli. If Y is expansive, then Y is irreducible sofic and almost-Borel isomorphic to a Markov shift. If Y is not expansive then the Borel structure of Y at a period can be very different from that of a Markov shift.

Below Y_1 and T_1 denote the restrictions of Y and T to ergodic measures with maximum period 1 (see the Borel periodic decomposition Thm. 2.5).

Proposition 7.8. *Suppose T is a subshift. Then there is a period 2 irreducible SFT X and a continuous factor map π from X onto a zero dimensional metrizable system Y such that the following hold.*

1. $|\pi^{-1}(y)| \leq 2$ *, for all* $y \in Y$.
2. $\pi^{-1}T = \{x \in X : |\pi^{-1}(\pi(x))| = 2\}$.
3. $Y \setminus Y_1$ *is almost-Borel isomorphic to* X.

[9] This property holds for all positive entropy ergodic systems according to the Weak Pinsker Conjecture [46, 47] (which remains open).

4. Y_1 is almost-Borel isomorphic to T_1.

Moreover, X can be chosen with $h(X)$ arbitrarily close to $h(T)$.

Proof. We choose (X, σ) of the form $X = X' \times (\mathbb{Z}/2\mathbb{Z})$, with $\sigma : (x, g) \mapsto (\sigma x, g + 1)$, where (X', σ) is any mixing SFT into which T continuously embeds with entropy arbitrarily close to $h(T)$. Let E' be the quotient relation of the map $T \times \mathbb{Z}/2\mathbb{Z} \to T$ defined by $(x, g) \mapsto x$. Let E be the union of E' and the diagonal of X. Define Y as the quotient space X/E (with quotient topology) and identify the image in Y of $T \times \{0, 1\}$ with T. Then Y is compact metrizable, since E is a closed equivalence relation (Proposition B.2). Let us check that Y is zero-dimensional. For an X' word $W_{-n} \dots W_n$, let $U_w = \{x \in X' : x[-n, n] = W\}$. If W is not a T-word, then πU_W is clopen in Y; if W is a T-word, then $\pi(W \times \mathbb{Z}/2\mathbb{Z})$ is clopen in Y. Therefore each point in Y has a neighborhood basis of clopen sets.

The system $X' \backslash T$ contains mixing SFTs with entropy arbitrarily close to $h(X)$. Hence $Y \setminus Y_1$ is the union of a strictly $(h(X), 2)$-universal Borel system and a period-2 Bernoulli measure of entropy $h(X)$. Therefore $Y \setminus Y_1$ is almost-Borel isomorphic to X. The rest is clear. $\square$

We'll give two easy corollaries of Proposition 7.8 which already show Y_1 can be very different from what can arise in a Markov shift.

Corollary 7.9. *Suppose (W, ν) is a totally ergodic, finite entropy, measure-preserving system. Then there is a period 2 irreducible SFT X and a continuous, at most 2-to-1 factor map $\pi \colon X \to Y$ such that Y_1 is almost-Borel isomorphic to (W, ν).*

Proof. This follows from Prop. 7.8 and the Jewett-Krieger Theorem. $\square$

Let R be the map on $\mathbb{T}^2$ defined by $(t, y) \mapsto (t, y+t)$. Let $P_0 = \{(x, y) \in \mathbb{T}^2 \colon 0 \le x \le y \le 1\}$ and $P_1 = \mathbb{T}^2 \setminus P_0$. Let Z be the subshift on symbols $0, 1$ which is the closure of R-itineraries through the partition $\{P_0, P_1\}$. Z is a disjoint union of Sturmian shifts (one for each irrational rotation) and countably many periodic orbits. Now Z_1 is the restriction of Z to the complement of the periodic orbits of period greater than 1 (including exactly one copy of each Sturmian shift and a fixed point).

Corollary 7.10. *Suppose (W, ν) is a weakly mixing, finite entropy, ergodic transformation. There is a period 2 irreducible SFT X and a continuous at most 2-to-1 factor map $\pi \colon X \to Y$, such that Y_1 is almost-Borel isomorphic to $Z_1 \times (W, \nu)$. In particular, the measures of Y_1 are uncountably many and have entropy $h(W)$.*

Proof. By the Jewett-Krieger Theorem, let W' be a strictly ergodic shift, which with its invariant measure is isomorphic to (W, ν). Set T in Prop. 7.8 to be $Z \times W'$. A product of irrational rotation (or fixed point) and weakly mixing remains totally ergodic so Y_1 and T_1 are isomorphic to $Z_1 \times W'$. $\square$

Obviously, the possible almost-Borel structure of Y_1 in Prop. 7.8 can be much more varied than shown in the two corollaries.

8 C^{1+} surface diffeomorphisms

8.1 Sarig's Symbolic Dynamics

For each compact surface C^{1+}-diffeomorphism $f : M \to M$ and number $\chi > 0$, Sarig [45] defined $\hat{\pi}, \hat{\Sigma}, \hat{\Sigma}^{\#}, \mathcal{R}, \sim$ such that $\hat{\Sigma}$ is a Markov shift with countable alphabet $\mathcal{R}$; $\hat{\pi}$ is a Borel factor map from $\hat{\Sigma}$ into M; and there is a relation on the elements of $\mathcal{R}$ of being "affiliated" (which we will write as $\sim$). We note that $\hat{\Sigma}^{\#}$ (the "regular set") is the Sarig regular set $\hat{\Sigma}_{\pm\text{ret}}$ of Definition 6.1.

Summary 8.1. *The items above satisfy the following.*

1. *If $\mu \in \mathbb{P}_{\text{erg}}(f)$ and has both its positive and negative Lyapunov exponents outside $(-\chi, \chi)$, then $\mu\hat{\pi}(\hat{\Sigma}^{\#}) = 1$.*
2. *If $\mu \in \mathbb{P}_{\text{erg}}(f)$ and $h(f, \mu) \geq \chi$, then $\mu\hat{\pi}(\hat{\Sigma}^{\#}) = 1$.*
3. *Each point $z \in \hat{\pi}(\hat{\Sigma}^{\#})$ has only finitely many preimages in $\hat{\Sigma}^{\#}$.*
4. *$\hat{\pi}$ is Bowen type on $\hat{\Sigma}^{\#}$ for the relation $\sim$ (see Defn. 1.11).*
5. *For all $R \in \mathcal{R}$, $\{R' \in \mathcal{R} : R' \sim R\}$ is finite.*
6. *$\hat{\pi}$ is Hölder-continuous.*
7. *$\hat{\Sigma}$ is locally compact.*

This symbolic dynamics is an embarassment of riches. To apply Theorem 1.12, we only need that $\hat{\pi}$ is finite-to-one Bowen type on $\hat{\Sigma}^{\#}$, which follows from (3,4). Properties (5,6,7) are given for context.

Properties (1,2) are of course essential to relating the symbolic dynamics to the diffeomorphism. We note that the main theorems of [45] quote property (2). This is weaker than (1): as is well-known (see [24]), for a surface diffeomorphism, an ergodic measure with nonzero entropy must have no zero Lyapunov exponent. However the proofs deal with the set $\mathrm{NUH}_\chi(f)$ which is defined [45, p. 348] in terms of the exponents, not the entropy, which is never used in the rest of the paper.[10]

We will see below that the properties in the summary are explicitly or essentially contained in [45].

[10] The author has confirmed to us that the *remark on χ-largeness* [45, p.344] contains a misstatement: there, "both Lyapunov exponents" should replace "at least one Lyapunov exponent".

8.2 The theorem for surface diffeomorphisms

We recall that all surfaces are assumed to be C^∞ smooth.

Theorem 8.2. *Every C^{1+} surface diffeomorphism (X, f) is the union of two Borel subsystems Y and Z such that:*

- *Y is almost-Borel isomorphic to a Markov shift;*
- *Z carries only zero entropy measures.*

Moreover, a nonatomic ergodic measure is carried by Z if and only if it satisfies all of the following conditions:

(i) its entropy is zero;

(ii) at least one of the Lyapunov exponents is zero;

(iii) it has no period which is the maximal period of an ergodic, invariant probability with positive entropy.

Remark 8.3. The conditions (i)-(iii) are not independent. As discussed above, (ii) implies (i). Also (iii) is equivalent to:

(iii') the measure has no period which is the maximal period of a nonatomic, ergodic, invariant probability which has no zero Lyapunov exponent.

Remark 8.4. Note that the "universal" part of Y above could alternately be argued from Corollary 1.8 and Katok's horseshoes (see [11], where this is done in any dimension, assuming no zero Lyapunov exponents). But to control measures with entropy maximal at a period, we depend on Sarig's symbolic dynamics.

Proof of Theorem 8.2. For $\chi = 1/n$, we apply Sarig's work to get a Markov shift $\hat{\Sigma}_n$ and factor map $\hat{\pi}_n : \hat{\Sigma}_n \to X$ satisfying 8.1(1-4). Let $\overline{\Sigma}_n$ be the union of the Sarig regular sets of all irreducible components of Σ_n. By properties 8.1(3,4) and Thm. 1.12, $\hat{Y}_n := \hat{\pi}_n(\overline{\Sigma}_n)$ is almost-Borel isomorphic to a Markov shift. Let $Y_0 = \cup_n \hat{Y}_n$; by Cor. 4.6, Y_0 is almost-Borel isomorphic to a Markov shift.

If $\mu \in \mathbb{P}'_{\text{erg}}(f)$ satisfies neither (i), nor (ii), then, by properties 8.1(1,2), there exists $\hat{\mu} \in \mathbb{P}'_{\text{erg}}(\hat{\Sigma}_n)$ with $\hat{\pi}_*(\hat{\mu}) = \mu$. In particular, $\hat{\mu}(Z) = 1$ for some irreducible component of $\hat{\Sigma}_n$, so $\hat{\mu}(Z_{\pm\text{ret}}) = 1$, and therefore $\mu(Y_0) = 1$. We enlarge Y_0 into Y carrying all measures not satisfying all of (i)-(iii) as follows.

First, let $\lambda^u(x) := \limsup_{n\to\infty} \frac{1}{n} \log \|D_x f^n\|$. It is a Borel function such that, for all $\mu \in \mathbb{P}_{\text{erg}}(f)$, for μ-a.e. $x \in X$, $\lambda^u(x)$ is the largest exponent of μ. By this observation (and the same applied to the smallest exponent), we get an invariant Borel subset X'' which has full measure for $\mu \in \mathbb{P}_{\text{erg}}(f)$ if and only if μ has a zero Lyapunov exponent.

Now let P be the set of integers $p \geq 1$ such that there is some ergodic, invariant probability measure μ with nonzero entropy with maximal period p. For each p in P, Σ contains an irreducible Markov shift Σ_p with some period dividing p and positive entropy, and therefore $u_{Y_0}(p) > 0$. For each $p \in P$, the Borel periodic decomposition (Theorem 2.5) provides an invariant Borel subset X'_p of X such that for $\mu \in \mathbb{P}_{\text{erg}}(X)$, $\mu(X'_p) = 1$ if and only if p is a period of μ. Define $Y_p := X'_p \cap X''$ and $Y := Y_0 \cup \bigcup_{p \in P} Y_p$. Because all measures on Y_p have zero entropy and $u_{Y_0}(p) > 0$ for p in P, by Corollary 3.14(3) the systems Y and Y_0 are almost-Borel isomorphic.

Thus $X = Y \sqcup Z$, with $Z := X \setminus Y$, is an invariant, Borel decomposition such that Y satisfies (1) and (2) and carries any $\mu \in \mathbb{P}'_{\text{erg}}(f)$ failing to satisfy one of (i),(ii),(iii). Conversely, $\mu(Z) > 0$ implies (i), (ii), and $\mu(Y_p) = 0$ for all $p \in P$, hence (iii). $\square$

As an invariant, ergodic probability measure with trivial rational spectrum has maximal period equal to 1, this yields:

Corollary 8.5. *Consider a positive entropy, C^{1+} diffeomorphism of a compact surface.*

It is almost-Borel isomorphic to a Markov shift if it has a totally ergodic measure with positive entropy.

It is almost-Borel isomorphic to a mixing Markov shift if it has a totally ergodic measure which is the unique measure of maximum entropy.

Remark 8.6. The situation of the corollary occurs in some natural settings. In particular, Berger [5] has shown that for a positive Lebesgue measure subset of parameters, Hénon maps have a unique measure of maximal entropy that is mixing. Their invariant measures are carried by a forward invariant compact disk and therefore one can apply the above corollary: these Hénon maps are almost-Borel isomorphic to a mixing Markov shift. In particular, they are h-universal, where h is their Borel entropy (equal to their topological entropy after restricting to the invariant disk).

8.3 Proof of the properties of Sarig's construction

We now discuss how the Summary 8.1 properties come from Sarig's paper. For (1,2,3,6,7), see [45, Theorems 1.3, 12.5, 12.8]. Property (5) is a statement within the proof of Lemma 12.7. To explain (4), we need some facts and notations from Sarig's paper [45].

The set $\mathcal{V}$ of Pesin charts and the Markov shift $\Sigma(\mathcal{G})$

Sarig builds a countable collection $\mathcal{V}$ of triplets (Ψ_x, p^s, p^u) where $p^s, p^u > 0$ and Ψ_x is a Pesin chart defined using the Oseledets theorem applied at point x. Charts are diffeomorphisms onto their image with Lipschitz constant at most 2 and the domain of Ψ_x contains

$(-p^s, p^s) \times (-p^u, p^u)$. We often write p for $\min(p^u, p^s)$ and, following Sarig, write the triplet as $\Psi_x^{p^s,p^u}$ and continue to call it a chart (despite the extra information p^u, p^s).

Sarig defines a graph $\mathcal{G}$ over $\mathcal{V}$. In particular, $\Psi_x^{p^s,p^u} \to \Psi_y^{q^s,q^u}$ in $\mathcal{G}$ implies that, at least on the rectangle $(-10p, 10p)$, $f_{x,y} := \Psi_y^{-1} \circ f \circ \Psi_x$ is uniformly hyperbolic and $\Psi_y^{-1} \circ \Psi_x$ is very close to the identity. More precisely, for $(u, v) \in (-p^s, p^s) \times (-p^u, p^u)$

$$f_{x,y}(u, v) = \begin{pmatrix} A_{x,y} & 0 \\ 0 & B_{x,y} \end{pmatrix} . \begin{pmatrix} u & v \end{pmatrix} + h(u, v)$$

with $C_f^{-1} < |A_{x,y}| < e^{-\chi}, e^{\chi} < |B_{x,y}| < C_f$ and $\|h(0)\| \le \epsilon q$ and $\|h'(0)\| \le 2\epsilon p^{\beta/3} < \epsilon$ (see [45, Prop. 3.4, p.14]).

It follows that, for any sequence $\underline{v} = (\Psi_{x_n}^{p_n^s,p_n^u})_{n\in\mathbb{Z}} \in \Sigma(\mathcal{G})$, i.e., defining a path on the graph $\mathcal{G}$ (see Sec. 2.5), there is a unique sequence $\underline{t} \in (\mathbb{R}^2)^{\mathbb{Z}}$ such that

$$f_{x_n,x_{n+1}}(t_n) = t_{n+1} \in B(0, p_{n+1})$$

for all $n \in \mathbb{Z}$. The projection $\pi : \Sigma(\mathcal{G}) \to M$ defined by Sarig [45, Proposition 4.15, Theorem 4.16] satisfies: $\pi(\underline{v}) = \Psi_{x_0}^{p^u_0,p^s_0}(t_0)$ and $t_n \in B(0, p_n/100)$ for all $n \in \mathbb{Z}$.

According to [45, Theorem 5.2], if $\pi(\underline{v}) = \pi(\underline{w})$ for $\underline{v}, \underline{w} \in \Sigma(\mathcal{G})^\#$, then, for each integer $n \in \mathbb{Z}$, the charts $v_n = \Psi_{x_n}^{p_n^u,p_n^s}$ and $w_n = \Psi_{y_n}^{q_n^u,q_n^s}$ are very close: on $B(0, \epsilon)$ (ϵ is much larger than p, q, see [45, Def.2.8 and Lem.2.9])

$$\Psi_{y_n}^{-1} \circ \Psi_{x_n}(t) = \pm t + \delta(u) \text{ where } \|\delta(0)\| < q_n/10,\ \|\delta'\| \le \epsilon^{1/3}. \tag{8.7}$$

Cover $\mathcal{Z}$ by large rectangles

Sarig then defines a cover:

$$\mathcal{Z} := \{Z(v) : v \in \mathcal{V}\} \text{ with } Z(v) := \{\pi(\underline{v}) : \underline{v} \in \Sigma(\mathcal{G})^\#,\ v_0 = v\}$$

Proposition 4.11 of [45] implies that $\Psi_x^{-1}(Z(v)) \subset B(0, q/100)$, well inside the domain of the chart.

Partition $\mathcal{R}$ by small rectangles

Sarig refines the cover $\mathcal{Z}$ into a "Markov partition" $\mathcal{R}$, following an elaborate version of the Bowen-Sinaĭ construction used in the uniformly hyperbolic case. $\hat{\Sigma}$ is then the Markov shift defined by the countable oriented graph with vertices $R \in \mathcal{R}$ and arrows $(R, R') \in \mathcal{R}^2$ if and only if $f(R) \cap R' \neq \emptyset$. The map $\hat{\pi} : \hat{\Sigma} \to M$ satisfies:

$$\{\hat{\pi}((R_n)_{n\in\mathbb{Z}})\} = \bigcap_{n\in\mathbb{Z}} f^{-n}(\overline{R_n}) = \bigcap_{n\in\mathbb{Z}} f^{-n}(\overline{Z_n})$$

for some $Z_n \in \mathcal{Z}$, $Z_n \supset R_n$.

Affiliated small rectangles

Sarig defines two small rectangles $R, R' \in \mathcal{R}$ to be **affiliated** (see before Lemma 12.7 in [45]) when there are two large rectangles $Z, Z' \in \mathcal{Z}$ such that:

$$R \subset Z,\ R' \subset Z' \text{ and } Z \cap Z' \neq \emptyset.$$

Proof of 8.1(4). Claim 2 in the proof of Theorem 12.8 in [45] asserts precisely that, for $R, R' \in \hat{\Sigma}$, if $\hat{\pi}(R) = \hat{\pi}(R') \in \hat{\pi}(\hat{\Sigma}^{\#})$ then R_n and R'_n are affiliated for each $n \in \mathbb{Z}$. Thus, it suffices to prove: for all $R, R' \in \hat{\Sigma}$, if R_n and R'_n are affiliated for each $n \in \mathbb{Z}$, $\hat{\pi}(R) = \hat{\pi}(R')$. Let $x = \hat{\pi}(R)$, $y = \hat{\pi}(R')$. For each $n \in \mathbb{Z}$, writing $Z_n = Z(\Psi_{x_n}^{p_n^s, p_n^u})$,

$$f^n x \in \overline{R_n} \subset \overline{Z_n} \text{ and } t_n := \Psi_{x_n}^{-1}(f^n x) \in \Psi_{x_n}^{-1}(Z_n) \subset B(0, p_n/100).$$

Likewise,

$$u_n := \Psi_{y_n}^{-1}(f^n y) \subset B(0, q_n/100).$$

Now, using $q_n \leq e^{\epsilon^{1/3}} p_n$ and eq. (8.7), we get, for all $n \in \mathbb{Z}$,

$$u'_n := \Psi_{x_n}^{-1} \circ \Psi_{y_n}(u_n) \in B(0, p_n/10 + (1 + e^{\epsilon^{1/3}})p_n/100) \subset B(0, p_n)$$

so $u'_{n+1} = F_n(u'_n)$ where $F_n := \Psi_{x_{n+1}}^{-1} \circ f \circ \Psi_{x_n}$. The uniform hyperbolicity of these maps on their domains $B(0, p_n)$ implies that $u'_n = t_n$ for all $n \in \mathbb{Z}$. In particular, $x = y$. □

8.4 Classification from measures of given maximum period

Proof of Theorem 1.2. Isomorphic diffeomorphisms have equal data (1) and (2), since those only depend on positive entropy measures. We turn to the converse. By Theorem 1.1, it suffices to classify the isomorphic Markov shifts up to almost-Borel isomorphism. By Theorem 1.5, it suffices to show that the data (1) and (2) are equal to $\bar{u}_S(\cdot)$ and $\bar{\eta}_S(\cdot)$ for any isomorphic Markov shift S. We fix $p \geq 1$ and use Fact 2.4.

First the Fact implies that $\bar{u}_S(p)$ is indeed equal to the supremum in (1). Second, let $\mathcal{M}(p)$ be the measures counted in (2) and $\mathcal{S}(p)$ be the irreducible subshifts counted by $\bar{\eta}_S(p)$. Associate to any $\mu \in \mathcal{M}(p)$ the irreducible shift Σ_i carrying its image in S.

The Fact implies $p_i | p$, hence $h_i \leq \bar{u}_S(p)$ so μ is a m.m.e. of Σ_i. Thus $p_i = p$ and $\Sigma_i \in \mathcal{S}(p)$. Since the m.m.e. of Σ_i is unique, $\mu \mapsto \Sigma_i$ is injective. Conversely, for any $\Sigma_i \in \mathcal{S}(p)$, (the image on the surface of) its m.m.e. belongs to $\mathcal{M}(p)$. Hence, $\mu \mapsto \Sigma_i$ is a bijection and $\#\mathcal{M}(p) = \bar{\eta}_S(p)$. □

9 Open problems

We select and discuss a few open problems. Observe that the universality results in this paper and [23] address only systems with topological embeddings of positive entropy SFTs (often as the consequence of hyperbolicity). However, the following result of Quas and Soo suggests that this strong kind of hyperbolicity is not necessary for Borel universality.

Recall that a toral automorphism arising from matrix A is *quasi-hyperbolic* if A has an irrational eigenvalue on the unit circle [30]. It is *irreducible* if the characteristic polynomial of A is irreducible. Lindenstrauss and Schmidt [31] showed that irreducible quasi-hyperbolic toral automorphisms cannot contain nonperiodic homoclinic points, and therefore cannot contain (or be a continuous factor of) any positive entropy SFT.

Nevertheless, Quas and Soo [41] have proven an analogue of the Krieger generator theorem (which is the starting point of Hochman's result) for this class. This generalization raises the following, which is a probe into the problem of understanding more sharply dynamical conditions which guarantee "universal" behavior.

Problem 9.1. *Suppose (X, T) is a mixing quasihyperbolic toral automorphism*[11]*. Must (X, T) be $h(T)$-universal (as in Theorem 4.1)?*

A different question related to the absence of hyperbolicity is:

Problem 9.2. *Complete the almost-Borel classification of C^{1+} surface diffeomorphisms (i.e., extend Theorem 1.1 to address all nonatomic, ergodic measures).*

In another direction, our proofs require C^{1+}-smoothness (for the application of Sarig's [45] symbolic dynamics and ultimately Pesin theory [40, 6]). Rees' examples [42] (see also [4] and references therein) show that our results do not extend to homeomorphisms.

Problem 9.3. *Are C^1 surface diffeomorphisms Borel isomorphic to Markov shifts away from zero entropy measures? In positive topological entropy, can they have ergodic period-maximal measures that are not period-Bernoulli, or have uncountably many ergodic period-maximal measures?*

Finally, in light of Theorem 1.1, we ask the following.

Problem 9.4. *Which Markov shifts of finite positive entropy can be almost-Borel isomorphic to a C^{1+} surface diffeomorphism?*

We are not able to rule out the possibility that every Markov shift of finite positive entropy is almost-Borel isomorphic to a surface diffeomorphism.

[11] More generally, the question can be asked about the class of maps considered by [41]: compact t.d.s. that satisfy almost weak specification, asymptotic entropy expansiveness, and the small boundary property.

A Borel periodic decomposition

This Appendix provides a proof of Theorem 2.5. We freely use the notations of the Theorems and definitions and facts from Sec. 2.6. We assume $p \geq 2$, the case $p = 1$ being trivial.

The space of finite measurable partitions of X into $p+1$ atoms is:

$$\mathcal{P} = \{(P_1, \ldots, P_p, P_{p+1}) : P_i \text{ is Borel; } P_i \cap P_j = \emptyset \text{ if } i \neq j;\ \cup_i P_i = X\}.$$

If $C := (C_1, \ldots, C_p)$ is a p-cyclic partition for some measure $\mu \in \mathcal{M}$, set $\hat{C} := (\hat{C}_1, \ldots, \hat{C}_p, X \setminus \cup_i \hat{C}_i)$ where

$$C_i' = C_i \setminus \cup_{j \neq i} C_j \text{ and } \hat{C}_i = C_i' \cap (\cap_{n \in \mathbb{Z}} T^n(\cup_{j=1}^p C_j')),$$

so $\hat{C} \in \mathcal{P}$. Moreover, $\mu(\hat{C}_i \Delta C_i) = 0$ and $T(\hat{C}_i) = \hat{C}_{i+1}$ (again $\hat{C}_{p+1} = C_1$) for all $i = 1, \ldots, p$ and $(\hat{C}_1, \ldots, \hat{C}_p)$ is still a p-cyclic partition for μ.

Finally each $\mu \in \mathbb{P}(X)$ defines a pseudometric ρ_μ on $\mathcal{P}$: $\rho_\mu(P, Q) = \frac{1}{2} \sum_{j=1}^{p+1} \mu(P_j \triangle Q_j)$. We will appeal to the following theorem of Kieffer and Rahe.

Theorem A.1. *[27, Thm. 5] Let $\mathcal{D}$ be a Borel subset of $\mathbb{P}_{\text{erg}}(T)$ and let $\{\mathcal{P}_\mu : \mu \in \mathcal{D}\}$ be a collection of nonempty subsets of $\mathcal{P}$ such that*

1. *each $\mathcal{P}_\mu$ is ρ_μ-closed, and*
2. *for each P in $\mathcal{P}$, the map $\rho_P : \mathcal{D} \to [0, 1]$ defined by $\mu \mapsto \inf\{\rho_\mu(P, Q) : Q \in \mathcal{P}_\mu\}$ is Borel measurable.*

Then $\cap_\mu \mathcal{P}_\mu \neq \emptyset$.

Proof of Theorem 2.5. Let $\mathcal{D} = \{\mu \in \mathbb{P}_{\text{erg}}(T) : e^{2i\pi/p} \in \sigma_{\text{rat}}(T, \mu)\}$.

Given $\mu \in \mathcal{D}$, let $\mathcal{P}_\mu$ be the set of $\hat{C} \in \mathcal{P}$ for all p-cyclic partitions C for μ. It remains to show $\cap_\mu \mathcal{P}_\mu \neq \emptyset$. Note, each $\mathcal{P}_\mu$ is ρ_μ-closed, so condition (1) of Theorem A.1 is satisfied.

Given $\mu \in \mathcal{D}$, there are distinct ν_i in $\mathbb{P}_{\text{erg}}(T^p)$, $1 \leq i \leq p$, such that $\mu = \frac{1}{p} \sum_i \nu_i$ and $T\nu_i = \nu_{i+1}$, $1 \leq i \leq p$ (ν_{p+1} means ν_p). Given μ, let $C_1, \ldots, C_p$ be disjoint sets such that $\nu_i(C_i) = 1$, $1 \leq i \leq p$. Observe that the ergodicity of μ implies that elements of $\mathcal{P}_\mu$ coincide modulo μ up to a cyclic permutation of their first p elements. Thus, modulo μ, $\mathcal{P}_\mu$ contains exactly p elements, the cyclic permutations $(C_{1+d}, \ldots, C_{p+d}, C_*)$, $d = 0, \ldots, p-1$.

To check that $\mathcal{D}$ is a Borel subset of the Borel set $\mathbb{P}_{\text{erg}}(T)$, we appeal to some background facts. An injective Borel measurable map into a Borel space has a Borel image, and a Borel measurable inverse [26, (15.2)]. The fixed point set of a Borel automorphism is Borel. For E a separable metric space, the Borel field of $\mathbb{P}(E)$ (and hence of any Borel subset of $\mathbb{P}(E)$ is the smallest field for which the maps $\mu \mapsto \mu(A)$, A ranging over the Borel sets of E, are measurable [26, Theorem 17.24]. Consequently, the sets F_i, G_1, G_2, G_3 below are Borel:

$$\begin{aligned} F_i &= \{\mu \in \mathbb{P}(T^i) : T^i\mu = \mu\} & G_1 &= \mathbb{P}_{\mathrm{erg}}(T^p) \setminus \cup_{i=1}^{p-1} F_i \\ G_2 &= \{\frac{1}{p}\mu : \mu \in G_1\} & G_3 &= \left\{ \sum_{i=1}^{p} T^i\mu : \mu \in G_2 \right\} \quad . \end{aligned}$$

We claim that $\mathcal{D} = G_3$. If $\nu \in \mathcal{D}$ and γ is the assumed factor map onto $\{e^{2\pi i/k} : k = 0, 1, \ldots, p-1\}$, let μ be p times the restriction of ν to $\gamma^{-1}(1)$. Then $\mu \in G_1$ (because μ is ergodic for T) and $\nu = \sum_{i=1}^{p} \mu$. Therefore $\mathcal{D}$ is contained in G_3. For the other direction, suppose $\mu \in G_1$. Given $1 \leq i \leq p-1$, write the measure $T^i\mu$ as $\nu_c + \nu_s$, where $\nu_c = f\mu$ (f the Radon-Nikodym derivative) and ν_s is singular with respect to μ. The function f is T^p-invariant, because the measures μ and $T^i\mu$ are T^p-invariant, so by ergodicity of μ for T^p, f is constant μ a.e. Because $T^i\mu \neq \mu$, there is then a set C_i of μ-measure 1 and $T^i\mu$-measure zero. Let $C = \cap_{i=1}^{p-1} C_i$ and $D_i = T^iC$, $0 \leq i \leq p-1$. It follows that $\mu(D_i \cap D_j) = 0$ for $0 \leq i < j \leq p-1$. Now $\sum_{i=0}^{p-1} \frac{1}{p} T^i\mu$ is a T invariant probability eigenfunction defined a.e. by $x \to e^{2\pi i/p}$ if $x \in D_i$. Therefore G_3 is contained in $\mathcal{D}$.

It remains to verify condition (2) of Theorem A.1. We will construct a Borel selection β for the Borel map $\phi : G_2 \to \mathcal{D}$ defined by $\nu \mapsto \sum_{i=1}^{p} T^i\nu$ (i.e., $\beta : \mathcal{D} \to G_2$ is Borel and $\phi \circ \beta$ is the identity on $\mathcal{D}$).

Define a Borel measurable order $\prec$ on G_2 (for example, via a Borel injective map $G_2 \to \mathbb{R}$). Let $B = \{m \in G_2 : m \prec T^j m, 1 \leq j < p\}$, a Borel set in G_2. Then the restriction $B \xrightarrow{\phi} \mathcal{D}$ is a Borel bijection and $\beta = (\phi|B)^{-1}$ is our selection.

Now suppose $P = (P_1, \ldots, P_{p+1}) \in \mathcal{P}$. For $\mu \in \mathcal{D}$, set $\mu' = \beta(\mu)$. Given $Q = (Q_1, \ldots, Q_{p+1})$ in $\mathcal{P}_\mu$, there is some $d \in \{1, \ldots, p\}$ such that for $1 \leq j \leq p$ we have

$$\begin{aligned} (T^{j+d}\mu')(Q_j) &= \mu(Q_j) \,, \\ (T^{j+d}\mu')(X \setminus Q_j) &= 0 \,, \end{aligned}$$

and $\mu(Q_{p+1}) = 0$. Therefore

$$\begin{aligned} \rho_\mu(P, Q) &= \frac{1}{2} \sum_{j=1}^{p+1} \mu(P_j \triangle Q_j) = \frac{1}{2} \sum_{j=1}^{p+1} \mu(P_j) + \mu(Q_j) - \mu(P_j \cap Q_j) \\ &= 1 - \frac{1}{2} \sum_{j=1}^{p} \mu(P_j \cap Q_j) = 1 - \frac{1}{2} \sum_{j=1}^{p} (T^{j+d}\mu')(P_j) := \phi_d(\mu) \,. \end{aligned}$$

We conclude that $\inf\{\rho_\mu(P, Q) : Q \in \mathcal{P}_\mu\} = \min\{\phi_d(\mu) : 1 \leq d \leq p\}$, which is a Borel function of μ. $\square$

B Miscellany

We include in this section some basic results for lack of a direct reference.

Proposition B.1. *Let $\pi : (X,S) \to (Y,T)$ be a Borel factor map. Let $\nu \in \mathrm{Prob}(T)$ satisfy: for ν-a.e. $y \in Y$, $0 < \#\pi^{-1}(y) < \infty$. Then there exists $\mu \in \mathrm{Prob}(S)$ such that $\pi_*\mu = \nu$.*

Proof. Observe that we can replace Y by $\bigcap_{n\in\mathbb{Z}} T^{-n}Y'$ where Y' is a Borel set of full ν-measure implied by the assumption.

We claim that there are a Borel map $N : Y \to \mathbb{N}$, $N(y) := \#\pi^{-1}(y)$, and a Borel isomorphism $\psi : X \to \hat{Y} := \{(y,k) \in Y \times \mathbb{N} : 1 \le k \le N(y)\}$ such that $\pi \circ \psi(y,k) = y$ on $\hat{Y}$. This follows from the uniformization theorem for Borel maps with countable fibers [26, (18.10) and (18.14)].

Now, $\psi \circ S \circ \psi^{-1}(y,k) = (T(y), \sigma_y(k))$ where

$$\sigma_y : \{1,\dots,N(y)\} \to \{1,\dots,N(Ty)\}.$$

S and T being automorphisms, $N \circ T = N$ and σ_y is a permutation of $\{1,2,\dots,N(y)\}$. Hence, S must preserve

$$\mu := \sum_{n\ge 1} (\psi^{-1})_* \left((\nu|N^{-1}(n)) \times \tfrac{1}{n}(\delta_1 + \cdots + \delta_n) \right).$$

□

Proposition B.2. *Suppose $f : X \to Y$ is a continuous surjection, Y has the quotient topology, X is compact metric and $E := \{(x,w) : f(x) = f(w)\}$ is closed in $X \times X$. Then Y is compact metrizable.*

Proof. Let p_1, p_2 be the projections from $X \times X$ to X. If K is a closed subset of the compact Hausdorff space X, then $f^{-1}(f(K)) = \pi_2(\pi_1^{-1}K))$ is closed in X. Now f is a closed map with compact fibers and X is metrizable, so Y is metrizable [14, Theorem 5.2]. □

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