

JOINTLY PERIODIC POINTS IN CELLULAR AUTOMATA: COMPUTER EXPLORATIONS AND CONJECTURES

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ABSTRACT. We develop a rather elaborate computer program to investigate the jointly periodic points of one-dimensional cellular automata. The experimental results and mathematical context lead to questions, conjectures and a contextual theorem.

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1. INTRODUCTION AND CONJECTURES

In this paper we consider the action of a surjective one-dimensional cellular automaton f on jointly periodic points. Detailed definitions are recalled below.

This paper is primarily an experimental mathematics paper, based on data from a program written by the second-named author to explore such actions. The experimental results and mathematical context lead us to questions and a conjecture on the growth rate of the jointly periodic points. The program itself is freely available at the website of the first-named author.

We approach our topic from the perspective of symbolic dynamics, which provides some relevant tools and results. However, almost all of this paper—in particular the questions and conjectures—can be well understood without symbolic dynamics. We do spend time on context, and even prove a theorem (Theorem 3.2), for two

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reasons. First, we believe that experimental mathematics should not be too segregated from the motivating and constraining mathematics. Second, workers on cellular automata have diverse backgrounds, not necessarily including symbolic dynamics. (Similarly, perhaps a technique or example unfamiliar to us could resolve one of our questions.)

To express our questions and conjectures clearly, we must suffer some definitions. We let Σ_N denote the set of doubly infinite sequences $x = \dots x_{-1}x_0x_1\dots$ such that each x_i lies in a finite “alphabet” \mathcal{A} of N symbols; usually $\mathcal{A} = \{0, 1, \dots, N-1\}$. A one-dimensional cellular automaton (c.a.) is a function $f : \Sigma_N \rightarrow \Sigma_N$ for which there are integers $a \leq b$ and a function $F : \mathcal{A}^{b-a+1} \rightarrow \mathcal{A}$ (F is a “local rule” for f) such that for all i , $(f(x))_i = F(x_{i+a} \cdots x_{i+b})$. The shift map σ on a sequence is defined by $(\sigma x)_i = x_{i+1}$. We let S_N denote the shift map on Σ_N .

For any map S , we let $P_k(S)$ denote the points of (not necessarily least) period k of S , i.e. the points fixed by S^k , and let $\text{Per}(S) = \cup_k P_k(S)$. Thus, $\text{Per}(S_N)$ is the set of “spatially periodic” points for a one-dimensional cellular automaton on N symbols. The *jointly periodic* points of a cellular automaton map f on N symbols are the points in $\text{Per}(S)$ which are also periodic under f , that is, the points which are “temporally periodic” as well as spatially periodic. (In the usual computer screen display, this would mean vertically and well as horizontally periodic.) There is by this time a lot of work addressing periodic and jointly periodic points for *linear* one-dimensional c.a.; we refer to [9, 10, 15, 21, 24, 26, 31] and their references. Also see [25] regarding the structure of periodic points for these and more general algebraic maps in the setting of [18, 30].

A subset E of Σ_N is *dense* if for every point x in Σ_N and every $k \in \mathbb{N}$ there exists y in E such that $x_i = y_i$ whenever $|i| \leq k$. We say E is *m-dense* if every word of length m on symbols from the alphabet occurs in a point of E .

We can now state our first conjecture.

Conjecture 1.1. *For every surjective one-dimensional cellular automaton, the jointly periodic points are dense.*

Conjecture 1.1 is a known open question [2, 3, 5], justified by its clear relevance to a dynamical systems approach to cellular automata. (Whether points which are temporally but not necessarily spatially periodic for a surjective c.a. must be dense is likewise unknown [2].) That this question, also open for higher dimensional c.a., has not been answered reflects the difficulty of saying anything of a general nature about c.a., for which meaningful questions are often undecidable [16].

It is known that the jointly periodic points of a one-dimensional cellular automaton map f are dense if f is a closing map [5] or if f is surjective with a point of equicontinuity [3]. We justify our escalation of (1.1) from question to conjecture by augmenting earlier results with some experimental evidence. In particular: for every span 4 surjective one-dimensional cellular automaton on two symbols, the jointly periodic points are at least 13-dense (Proposition 5.1).

Now we turn to more quantitative questions. Letting for the moment P denote the number of points in $P_k(S_N)$ which are periodic under f as well as S_N (i.e. $P = |\text{Per}(f|P_k(S_N))|$), we set $\nu_k(f, S_N) = P^{1/k}$, and then define

$$\nu(f, S_N) = \limsup_k \nu_k(f, S_N) .$$

Question 1.2. *Is it true for every surjective one dimensional cellular automaton f on N symbols that $\nu(f, S_N) \geq \sqrt{N}$?*

Question 1.3. *Is it true for every surjective one dimensional cellular automaton f on N symbols that $\nu(f, S_N) > 1$?*

We cannot answer Question 1.3 even in the case that f is a “closing” map and we know there is an abundance of jointly periodic points [5].

Conjecture 1.4. *There exists $N > 1$ and a surjective cellular automaton f on N symbols such that $\nu(f, S_N) < N$.*

Conjecture 1.4 is a proclamation of ignorance. From the experimental data in our Tables, it seems perfectly clear that there will be many surjective c.a. f with $\nu(f, S_N) < N$. However, we are unable to give a proof for any example. With the additional assumption that the c.a. is linear, it is known that Conjecture 1.1 is true and the answer to Question 1.2 is yes (Sec. 3).

The relation of Questions (1.2-1.4) to Conjecture 1.1 is the following: if a c.a. map f on N symbols does not have dense periodic points, then $\nu(f, S_N) < N$.

Here is the organization of the sequel. In Section 2, we give detailed definitions and background. In Section 3, we establish some mechanisms by which one can prove lower bounds for $\nu(f, S_N)$ for some f . We also prove (Theorem 3.2) that no property of a surjective c.a. considered abstractly as a quotient map without iteration can establish $\nu(f, S_N) < N$. We also support Question 1.2 with a random maps heuristic. (The potential analogy of c.a. and random maps was remarked earlier by Martin, Odlyzko and Wolfram [24, p.252] in their study of linear c.a.) A list of c.a. used for the computer explorations is given in Section 4.

Our computer program consists of three related subprograms: FDense, FPeriod and FProbPeriod. We use these respectively in Sections 5, 6 and 7. FDense probes approximate density of jointly periodic points of a given shift period. FPeriod provides exact information on jointly periodic points of a given shift period. FProbPeriod provides information on jointly periodic points for a random sample from a given shift period, and thus provides some information at shift periods where the memory demands of FPeriod are too great for it to succeed.

In Sections 5, 6 and 7, we give more information on the algorithms and discuss the many tables of output data in the appendices. In the tables, decimal output data are approximated by truncation; e.g., 1.429 becomes 1.42 rather than 1.43. Detailed information on the program is available, along with the program itself, at the website of the first named author.

2. DEFINITIONS AND BACKGROUND

Let $\mathcal{A} = \{0, 1, \dots, N-1\}$, a finite set of N symbols, with the discrete topology. Let Σ_N be the product space $\mathcal{A}^{\mathbb{Z}}$, with the product topology. We view a point x in Σ_N as a doubly infinite sequence of symbols from \mathcal{A} , $x = \dots x_{-1}x_0x_1\dots$. The space Σ_N is compact and metrizable; one metric compatible with the topology is $\text{dist}(x, y) = 1/(|n| + 1)$ where $|n|$ is the minimum nonnegative integer such that $x_n \neq y_n$. A set E is *dense* in Σ_N in this topology if for every k and every word W in \mathcal{A}^{2k+1} there exists x in E such that $x[-k, k] = W$.

The shift map σ sends a sequence x to the sequence σx defined by $(\sigma x)_i = x_{i+1}$. The shift map defines a homeomorphism S_N on Σ_N . The topological dynamical system (Σ_N, S_N) is called the *full shift on N symbols*, or more briefly the N -shift.

A map $f : \Sigma_N \rightarrow \Sigma_N$ is continuous and shift-commuting ($f\sigma = \sigma f$) if and only if f is a *block code*, i.e. there exist integers a, b and a function $F : \mathcal{A}^{b-a+1} \rightarrow \mathcal{A}$ such

that $(f(x))_i = F(x[a, b])$ for integers i , for all $x \in \Sigma_N$. Such a map f is called a one-dimensional cellular automaton. There is a well known dichotomy for such maps f : either (i) f is surjective and for some integer M every point has at most M preimages, or (ii) image points typically have uncountably many preimages, and f is not surjective [14, 19, 22]. In Case (i), almost all points have the same number of preimages; this number is the *degree* of f .

We restrict our attention to surjective maps in this paper because we are interested in periodic points of f , which must be contained in $\bigcap_{k>0} f^k \Sigma_N$, the eventual image of f . We separate our ignorance about periodic points from additional difficulties involving the passage to the eventual image [23].

Polynomials can be used to define cellular automata; for example, if we refer to the c.a. f defined on the N -shift by the polynomial $2x_{-1} + x_0(x_2)^3$, we mean that f is defined by the block code $(fx)_i = 2x_{i-1} + x_i(x_{i+2})^3$, where the arithmetic is interpreted modulo N . The *span* of such a code is 1 plus the maximum difference of coordinates with nonzero coefficients; in this example, it is $1 + 2 - (-1) = 4$. The code is left permutative if for every x , permuting inputs to the leftmost variable, with inputs to other variables fixed, permutes the outputs. Likewise there is the notion of right permutative. The previous example is left permutative and it is not right permutative. When the number N of symbols is prime, every c.a. map f has such a polynomial representation [14]. (For general N , there is a representation by a product of polynomial representations over finite fields [24].)

A block code on S_N depending on coordinates $[0, j-1]$ can be described by a “lookup code”, a word W of length N^j on alphabet $\{0, \dots, j-1\}$ defined as follows. List the N^j possible blocks of length j in lexicographic order; then the i th symbol of W is the output symbol under f for the i th input block. For example, for the code $x_0 + x_1x_2$ on S_2 , the input words in lexicographic order are 000,001,010,011,100,101,110,111 and the corresponding word W is 0001 1110.

For the N shift, the number of coding rules of span at most j is N^{N^j} . If $\text{inj}(j, N)$ denotes the number of these which define injective (and thus surjective [14]) codes, then we still [17] see a superexponential growth rate in j , $\lim_j \log \log(\text{inj}(j, N)) = \log N$, even though surjective span j maps become very sparse in the set of all span j maps, as j increases.

A block code $f : \Sigma_N \rightarrow \Sigma_N$ is *right-closing* if it never collapses distinct left-asymptotic points. This means that if $f(x) = f(x')$ and for some I it holds that $x_i = x'_i$ for all i in $(-\infty, I]$, then $x = x'$. Any right permutative map is right closing. The definition of *left closing* is given by replacing $(-\infty, I]$ with $[I, \infty)$. The map f is *closing* if it is either left or right closing. An endomorphism of a full shift S_N is constant-to-one if and only if it is both right and left closing (i.e., it is biclosing). A closing map is surjective. Closing maps are important in the coding theory of symbolic dynamics [1, 19, 22]. They also have a very natural description from the viewpoint of hyperbolic dynamics [8]: right closing maps are injective on unstable sets, left closing maps are injective on stable sets.

We now discuss some previous work involving periodic points and cellular automata. We let $P_n(S)$ denote the points of period n of S , and $P_n^o(S)$ the points of least period n . These finite sets are mapped into themselves by any c.a. map f ; thus any periodic point of S is at least preperiodic for f . For a preperiodic (possibly periodic) point x , the *preperiod* of x is the least nonnegative integer j such that $f^j(x)$ is periodic, and the *period* of x is its eventual period, the smallest positive

integer k such that $f^{m+k}(x) = f^m(x)$ for all large m . A point is jointly periodic if it is periodic under both f and S_N .

In the case f is linear ($f(x) + f(y) = f(x + y)$), Martin, Odlyzko and Wolfram [24] (see also the further work in [9, 10, 15, 21, 26, 31] and their references) gave an algebraic analysis of f -periods and preperiods for points of a given shift period, and also provided some numerical data. One key feature for linear f is an easy observation: among the jointly periodic points of shift period k , there will be a point (generally many points) whose least f -period will be an integer multiple of all the least f -periods of the jointly periodic points of shift period k . In contrast, a very special case of a powerful theorem of Ashley [1] has the following statement: for any K, N and any shift-commuting map g from $\cup_{1 \leq k \leq K} P_k(S_N)$ to itself, there will exist surjective c.a. on N symbols whose restriction to $\cup_{1 \leq k \leq K} P_k(S_N)$ equals g .

The following remark is another indication of the difficulty of understanding joint periodicity of even injective c.a. For a map T , $\text{Fix}(T)$ denotes $P_1(T)$, the set of fixed points of T .

Remark 2.1. Given $N \geq 2$, let S denote S_N , and suppose ϕ is an injective one-dimensional c.a. on N symbols. Suppose N is prime. Then there will exist some integer m and some $\kappa > 0$ such that for all $k \in \mathbb{N}$,

$$\left| \text{Fix}((S^a \phi^b)^k) \right| = N^{(a+mb)k} = \left| \text{Fix}((S^a (S^m)^b)^k) \right| = \left| \text{Fix}((S^{a+mb})^k) \right|$$

whenever $|b/a| < \kappa$ (this follows from [6, Theorem 2.17]). That is, for the two \mathbb{Z}^2 actions generated respectively by S, ϕ and S, S^m , the periodic point counts for actions by individual elements (a, b) of \mathbb{Z}^2 are the same for all (a, b) in some open cone around the positive horizontal axis. Nevertheless, with the given m fixed, the sequence $(|P_k(\phi)|)$ can still vary tremendously with ϕ . (For a dramatic example in the setting of shifts of finite type, see [27, Example 10.1]).

Lastly, we note that the invariant ν , defined in the introduction, has an unusual robustness, as follows.

Remark 2.2. Fix N and let $S = S_N$. Suppose $x \in \text{Per}(S_N)$ and f is a c.a. on Σ_N . Then x is in $\text{Per}(f)$ if and only if for some $i > 0$, $f^i x$ and x are in the same S -orbit. It follows that for all integers i, j, k with k, i positive and j nonnegative, we have $\nu_k(f, S) = \nu_k(f^i S^j, S)$, and thus $\nu(f, S) = \nu(f^i S^j, S)$.

3. SOME MECHANISMS FOR PERIODICITY

Throughout this section f denotes a c.a. map on N symbols. In this section, we discuss four ways to prove $\nu(f, S_N)$ is large:

- (1) find a large shift fixed by f (or more generally by a power of f)
- (2) let f be linear (i.e., a group endomorphism of Σ_N , where addition on the compact group Σ_N is defined coordinatewise mod N)
- (3) use the algebra of a polynomial presenting f
- (4) find equicontinuity points.

After discussing these, we offer a random maps heuristic and a question.

- (1) We will exhibit the first mechanism in some generality. Two c.a. f, g are isomorphic if there is an invertible c.a. ϕ such that $f = \phi g \phi^{-1}$ (where e.g. ϕg is the composition, $(\phi g)(x) = \phi(g(x))$). The c.a. f, g are equivalent as quotient maps

if there are invertible c.a. ϕ, ψ such that $f = \psi g \phi$. We prove Theorem 3.2 below to show that for a c.a. f , no property defined on equivalence classes of quotient maps can prevent $\nu(f, S_N)$ from being arbitrarily close to N . To avoid a lengthy digression to background, we give a proof assuming familiarity with symbolic dynamics; however the statement of Theorem 3.2 is self-contained. Below, $h(T)$ denotes the topological entropy of T .

Lemma 3.1. *Suppose (Σ_A, S) is a mixing shift of finite type (SFT) of positive entropy, and $f : \Sigma_A \rightarrow \Sigma_A$ is a surjective block code, and $\delta > 0$. Then there is an automorphism ϕ of (Σ_A, S) and a mixing SFT (Σ, T) contained in S such that $h(T) > h(S) - \delta$ and the fixed point set of ϕf contains Σ .*

Proof. First, pick a periodic point y in Σ_A such that y and $f(y)$ have the same least period. (Such y must exist: otherwise, f would map the periodic points of prime least period to fixed points, and this would imply that $f(\Sigma_A)$ is a single point, contradicting surjectivity of f and positive entropy of (Σ_A, S) .)

Next, e.g. using [11, Lemma 26.17], find a mixing SFT (Σ'_1, T'_1) in (Σ_A, S) such that $h(T'_1) > h(S) - \delta$ and $y \notin \Sigma'_1$. Let $\Sigma_1 = f^{-1}(\Sigma'_1)$. Now easily construct a mixing SFT (Σ_2, T_2) with $(\Sigma_A, S) \supset (\Sigma_2, T_2) \supset \Sigma_1 \cup \{y\}$ and $\Sigma_2 \cap f^{-1}\{f(y)\} = \{y\}$.

The restriction of f to Σ_2 is finite to one with degree 1. Let W be a magic word for this restriction. Let X_M denote the set of points x of Σ_2 such that for every i the word $x[i, i + M]$ contains an occurrence of W . Then the restriction of f to X_M will be one-to-one, and for large enough M the restriction of S to X_M will still have entropy greater than $h(S) - \delta$.

Now pick K such that for every $n \geq K$, S has at least two orbits of length n which are not in T_1 . Then choose (Σ, T) to be an SFT inside $(X_M, S|_{X_M})$ such that T has no orbits of length smaller than K and also such that still $h(T) > h(S) - \delta$. The image of (Σ, T) under f is an SFT (Σ', T') isomorphic to (Σ, T) , and there is a block code $g : \Sigma' \rightarrow \Sigma$ such that on Σ , gf is the identity map. By [7, Theorem 1.5], there is an automorphism ϕ of S whose restriction to T' equals g . Clearly the restriction of ϕf to T is the identity map. \square

Theorem 3.2. *Suppose f is a surjective c.a. on N symbols and $\epsilon > 0$. Then there is an invertible c.a. ϕ such that $\nu(\phi f, S_N) > N - \epsilon$.*

Proof. If T is a mixing shift of finite type with $h(T) = \log \lambda$, then $\lim_k |\text{Fix}(T^k)|^{1/k} = \lambda$. If this T is a set of fixed points for a c.a. ψ on N symbols, it follows that $\nu(\psi, S_N) \geq \lambda$. Now the theorem follows from Lemma 3.1. \square

Remark 3.3. The statements of Lemma 3.1 and Proposition 3.2 remain true if ϕf is replaced by $f\phi$. One way to see this is to notice that the systems $(f\phi, S)$ and $(\phi(f\phi)\phi^{-1}, \phi S\phi^{-1}) = (\phi f, S)$ are topologically conjugate.

(2) Now we turn to algebra. Σ_N is a group under coordinatewise addition (mod N), and some c.a. are group endomorphisms of this group; these are the *linear* cellular automata whose jointly periodic points were studied in [24] and later in a number of papers (see [9, 10, 15, 21, 26, 31] and their references). The algebraic structure allowed a number theoretic description of the way that f -periods of jointly periodic points of S_N period n vary (irregularly) with n . We show now that when f is a linear c.a., it is easy to see that $\nu(f, S_N) = \log N$.

Proposition 3.4. *Suppose a c.a. map f is a linear map on S_N . Then for all large primes p , $\nu_p(f, S_N) \geq N^{p-1}$. Therefore $\nu(f, S_N) = N$.*

Proof. We use an argument from the proof of a related result, Proposition 3.2 of [5]. Let M be the cardinality of the kernel of f . Suppose $p > M$ and p is prime; then f must map orbits of length p to orbits of length p (otherwise, some orbit of length p would be collapsed to an orbit of length dividing p , i.e. to a fixed point, which would contradict the fact that every point has M preimages, with $M < p$). The fixed points of $(S_N)^p$ form a subgroup H which is mapped into itself by f , and for all $k > 0$, the kernel of f^k contains no point in an orbit of length p , so $H \cap \ker(f^k) \subset \text{Fix}(S_N)$. For some $k > 0$, the restriction of f^k to $f^k H$ is injective, and all points in the set $f^k H$ are f -periodic. Because the kernel of $(f^k)|_H$ contains at most N points, it follows that at least $1/N$ of the fixed points of S^p are periodic for f . \square

(3) Algebra can be used in another way. Frank Rhodes [28], using properties of certain families of polynomials presenting c.a. maps, exhibited a family of noninvertible c.a. f for which there exists $k \in \mathbb{N}$ such that f is injective on $P_{kn}(S_N)$ for all $n \in \mathbb{N}$. Clearly in this case $\nu(f, S_N) = N$. We will not review that argument.

(4) We now turn to equicontinuity. A point x is equicontinuous for f if for every positive integer M there exists a positive integer K such that for all points y , if $x[-K, K] = y[-K, K]$ then $(f^n x)[-M, M] = (f^n y)[-M, M]$ for all $n > 0$. If the surjective c.a. f has x as a point of equicontinuity, and M is chosen larger than the span of the block code f , and W is the corresponding word $x[-K, K]$, then the following holds: if z is a point in which W occurs with bounded gaps, then z is f -periodic. Thus $\lim_n (1/n) \log \nu_n(f, S_N) = \log(N)$, and moreover the convergence is exponentially fast [3]. Points of equicontinuity may occur in natural examples [4, 20].

For many (probably “most”) surjective c.a., the criteria above are not applicable. This leads to the experimental investigations discussed in the next section, and to the possibility raised in Questions 1.2 and 1.3 of a general plenitude of jointly periodic points. Question 1.2 arises because in the experimental data, the restrictions of the c.a. f to $P_k(S_N)$ are somewhat reminiscent of a random map on a finite set. Since f is a surjective one dimensional c.a. map, there is an M such that no point has more than M preimages under f . Suppose for example k is a prime greater than M and let $\mathcal{O}_k(S_N)$ denote the set of S_N orbits of size k . Then f defines an at most M -to-1 map f_k from $\mathcal{O}_k(S_N)$ into itself, and we see a possible heuristic: (1) in the absence of some additional structure, the sequence (f_k) will reflect some properties of random maps, and (2) an “additional structure” such as existence of equicontinuity points for f will tend to produce more rather than fewer periodic points. The beautiful and extensive theory of random maps on finite sets contains precise asymptotic distributions answering various natural questions [29]. Here we simply note that for a random map on a set of K elements, asymptotically on the order of \sqrt{K} of the elements will lie in cycles (whether the map is bounded-to-one [12, Theorem 2] or not [29]), and there will be few big cycles.

The maps f_k derived from the surjective c.a. f are nonrandom not only in being bounded-to-one, but also in that most points have the minimal possible number of preimages [14, 19, 22]. To the extent it matters, this seems to work in favor of the random maps heuristic behind Question 1.2. In particular, it seems that the qualification to the random maps analogy offered in [24, p.252], regarding large in-degrees for cellular automata, does not hold for the class of surjective c.a.

4. THE MAPS

We examine with our programs several cellular automata on N symbols, having or not having various properties as indicated below. Except for Tables 15 and 16, all c.a. examined are on $N = 2$ symbols.

The c.a. A is the addition map $x_0 + x_1 \pmod{N}$. This c.a. is linear, bipermutative, and everywhere N to one.

The c.a. B is $x_0 + x_1x_2$. This c.a. is left permutative, degree 1, not right closing.

The c.a. C is $B \circ B_{rev}$, where $B_{rev} = x_0x_1 + x_2$. This c.a. is degree 1, and it is nonclosing, as it is the composition of a not-left-closing c.a. and a not-right-closing c.a.

The c.a. D is the map C composed with $(S_2)^{-2}$, i.e., D is the composition of $x_0 + x_1x_2$ with $x_{-2}x - 1 + x_0$. All periodic points for the golden mean shift (the sequences x in which the word 11 does not occur) become fixed points for D (vs. being periodic of varying periods for C).

The c.a. E is the composition A followed by B . This c.a. on $N = 2$ symbols has degree 2, and is left permutative but not right closing.

The c.a. J on 2 symbols is A precomposed with the automorphism U of S_2 which applies the flip to the symbol in the $*$ space of the frame $10 * 11$. This U is $x_0 + x_{-2}(1 + x_{-1})x_1x_2$, which equals $x_0 + x_{-2}x_1x_2 + x_{-2}x_{-1}x_1x_2$. The c.a. J has degree N and is biclosing, but is neither left permutative nor right permutative.

The c.a. G is $x_{-1} + x_0x_1 + x_2$. This c.a. on 2 symbols is bipermutative, degree 2, and is not linear.

The c.a. H is the composition $A \circ A \circ U$. It has the properties of F , except that the degree is now $2^2 = 4$.

The c.a. K is the composition $B \circ U$. This c.a. is left closing degree 1; it is not left permutative and it is not right closing.

In addition we use a library of surjective span 4 and span 5 c.a. due to Hedlund, Appel and Welch, who conducted the early investigation [13] in which they found all surjective c.a. on two symbols of span at most five. (This was not trivial, especially in 1963, because there are 2^{32} c.a. on two symbols of span at most five.) Among these onto maps of span four, there are exactly 32 which are not linear in an end variable (i.e., neither left nor right permutative) and which send the point $\dots 0000 \dots$ to itself. These 32 are listed in Table 1. Any other span four onto map which is not linear in an end variable is one of these 32 maps g precomposed or postcomposed with the flip map $F = x_0 + 1$. Because $gF = F(Fg)F = F^{-1}(Fg)F$, the jointly periodic data for Fg and gF will be the same. Altogether, then, we can handle all surjective span 4 maps not linear in an end variable by examining 64 maps.

According to [13], there are 141,792 surjective c.a. of span 5. These are arranged in [13] into classes – linear in end variables, compositions of lower-span maps, and the remainder. The remainder class (11,388 maps) is broken down into subclasses by patterns of generation, and a less regular residual class of 200 maps. These 200 are generated by 26 maps [13, Table XII] and various operations. We list the codes for this irregular class of 26 maps in Table 2, and use it as a modest sample of span 5 maps.

5. FDENSE

The program FDense takes as its input a c.a. f , an integer $N \geq 2$, a positive integer m and a finite set \mathcal{K} of positive integers k . (FDense can also handle sets of maps as inputs, producing output for all the maps, and suppressing various data.) The input f can be given by a polynomial or a tabular rule. For a given f and each k in \mathcal{K} , FDense determines whether the set $\text{Per}(f) \cap P_k(S_N)$ is m -dense (in which case we say that f is m -dense at k). If not, then FDense will separately list all the S_N words of length m which do not appear in any periodic point of f in $P_k(S_N)$, in a lexicographically truncated form potentially useful for seeing patterns. (For example, if m is ten and the word 011 does not occur in the examined points, then FDense would list 011* as excluded rather than listing all words of length ten beginning with 011.)

The underlying algorithm for FDense lists all words of length m and k in tagged form and operates on tags as it moves through the words of length m with f . Memory is the fundamental constraint on FDense. With m considerably smaller than k , the essential demand on memory is the tagged list of N^k words of length k . With $N = 2$, roughly $m = 13$ and $k = 27$ was a practical limit for our machine, and this was also quite slow. We restricted our investigations almost entirely to the case of $N = 2$ symbols for two reasons: with $N = 2$ we can examine longer periods; and we would be astonished to find any relation between the questions at hand and N .

The following proposition follows from the data of Tables 3 and 4.

Proposition 5.1. *For every span 4 surjective cellular automaton on two symbols, the set of jointly periodic points is (at least) 13-dense.*

In Tables 5-7, we applied FDense, for $N = 2$ symbols, to check for which $k \leq 24$ various other surjective c.a. f are 10-dense at k .

Table 5. After postcomposition with the map $A = x_0 + x_1$, the 32 onto span 4 c.a. of Table 1 remain 10-dense at some $k \leq 24$.

Table 6. The 26 irregular span 5 maps of Table 2 are 10-dense at some $k \leq 24$.

Table 7. For each of the 32 span 4 maps j of Table 1, let $p_j(x_0, x_1, x_2, x_3)$ denote its defining polynomial. Construct a c.a. f_j with defining polynomial $x_0 + p_j(x_1, x_2, x_3, x_4)$. These f_j are demonstrated to be 10-dense at some $k \leq 24$.

For the c.a. in Tables 5-7, often the least k at which 10-density is achieved lies in the range 19–24. (This is the point of Table 7, as we know already from [5] that the jointly periodic points of permutative c.a. are dense.) This is consistent with the heuristic that apart from possible extra structure the c.a. map on points of least period k looks something like a random map. For a random map f from a set of 2^k points into itself, on the order of $\sqrt{2^k}$ points are expected to lie in f -cycles. For $k = 20$, we have $\sqrt{2^k} = 2^{10}$. (Of course, $10 < 24/2$. A point of S_N -period 20 will contain up to 20 distinct words of length 10; the words aren't expected to occur with complete uniformity; specific codes are not random. For the heuristic of randomness, it is perhaps striking to find the rough agreement we do see.)

We also checked 10-denseness for several c.a. on 2 symbols with specified properties, described in Section 4.

Example 5.2. [Linear] The c.a. $A = x_0 + x_1$ is 10-dense at $k = 11, 13 - 24$ out of [10,24].

Example 5.3. [Permutative, not biclosing] The c.a. B is 10-dense at $k = 22 - 24$ out of $[10, 24]$. It is 13-dense at only $k = 25$ out of $[13, 25]$.

Example 5.4. [Not closing] The c.a. C (and likewise D) is 10-dense at $k = 17 - 24$ out of $[1, 24]$, and 13-dense for $k = 23, 24$ out of $[13, 24]$.

Example 5.5. [Degree 2, biclosing, not permutative] The c.a. J is 10-dense at $k = 23 - 25$ out of $[10, 25]$. It is 13-dense at only $k = 25$ out of $[13, 25]$.

In summary, there is reasonable supporting evidence for the Conjecture 1.1, and the counts seen seem consistent with the random maps heuristic.

6. FPERIOD

Recall $P_k(S_N)$ denotes the set of points fixed by the k th power of the full shift on N symbols. Each such point x is determined by the word $x_0x_1 \dots x_{k-1}$.

The FPeriod program takes as input a c.a. f , an integer $N \geq 2$ and a finite set of positive integers k . For each k , the program then determines data including the following (included in tables cited below):

- P := the number of points in $P_k(S_N)$ which are periodic for f .
- L := the length of the longest f -cycle in $P_k(S_N)$.

The program does much more; for the points in $P_k(S_N)$, it can produce a complete list of f cycle lengths and preperiods with multiplicities, and related data such as ν_k and averages. It can also do this for points in $P_k^o(S_N)$ rather than $P_k(S_N)$ (i.e. for points of least shift period k). The program also has an option for producing truncated and assembled data for a collection of maps.

The basic algorithm idea of FPeriod is the following. FPeriod takes the given c.a. f and a given shift-period length k ; stores all 2^k words of length 2^k ; and then changes various tags on these words as f moves through the corresponding periodic points. The tags in particular are changed to keep track of how long f iterates before returning. When the program returns to a previously visited point, it can deduce the corresponding f period and preperiod. The essential limit of FPeriod is that for large k it becomes a horrendous memory hog. We could conveniently reach period $k = 23$, and with patience we could reach $k = 25$ or 26 , before our memory resources were exhausted. In practice, running the program using $N = 2$ and $k = 26$ required 1.8 gigabytes of memory.

In this section we apply FPeriod to various maps from Section 4 with specific properties, and also to many maps of span 4 and 5. The main message is that for nonlinear maps, we generally see $\nu_k(f, S_N)$ compatible with affirmative answers to Questions 1.2 and 1.3, and frequently the data suggestion strongly that the limit $\nu(f, S_N)$ is smaller than N . Below, unless otherwise indicated, f is defined on the full shift S_N with $N = 2$, and the symbol set is $\{0, 1\}$.

Table 8 [Linear]. We exhibit results for the c.a. $A = x_0 + x_1$; here $\nu_k(A, S_2)$ is large, consistent with the fact $\nu(A, S_2) = 2$.

Table 9 [Biclosing]. We exhibit results for the c.a. J , which is A composed with an invertible c.a. The composition significantly reduces the numbers ν_k .

Table 10 [Linear composed with degree 1 permutative]. We exhibit results for the c.a. E .

Table 11 [Bipermutative]. We exhibit results for the c.a. G .

Table 12 [Permutative, not biclosing]. We exhibit results for the c.a. B .

Table 13 [Closing, not permutative, not biclosing]. We exhibit results for the c.a. K .

Table 14 [Not closing]. We exhibit results for the c.a. C .

Tables 15 and 16. We give our only examples for a c.a. on more than 2 symbols (they are c.a. on 3 symbols). The pattern is the same but we are able to investigate only up to shift period 13.

Tables 17 and 18 [Span 5 irregular]. We display data for the 26 irregular maps of span 5 given in Table 2 and discussed in Section 4.

Tables 19 and 21 [Span 4]. We exhibit data for the 32 maps g of Table 1. (This addresses all span 4 surjective c.a. on 2 symbols not linear in an end variable, as discussed in Section 4.)

Tables 20 and 22. [Span 4 composed with flip]. We exhibit data for the 32 maps of Table 1 postcomposed with the flip involution $F = x_0 + 1$.

Table 23 [Permutative comparison]. ν_k^o is computed for 16 left permutative span 5 maps, to make a rough comparison of a sample of maps which are and are not linear in an end variable. We see no particular difference.

Table 26. For the map B , periods with multiplicity are probed for $k \leq 30$ for two samples, of size 10 and size 30. The maximum period is the same except for two values of k .

Table 24. For $B = x_0 + x_1x_2$, complete data for B -periods with multiplicity are found by FPeriod (not FProbPeriod) for points in $P_k(S_2)$ for $k \leq 22$.

7. FPROBPERIOD

The k for which the program FPeriod can explore f -periodicity of points in $P_k(S_N)$ is limited on account of the memory demands of FPeriod. This begs for a probabilistic approach. For large k it is generally useless to sample points of shift period k for f -periodicity (commonly, this will be a fraction of the shift periodic points exponentially small in k). Instead, FProbPeriod randomly samples points of period k and computes for them the length of the f -cycle into which they eventually fall. This extends the range of k which can be investigated, depending on the map; for different maps we've seen practical limits at $k = 33$ to $k = 37$ (typical), to past 50 (for the linear $x_0 + x_1$ on two symbols). In any case, we can search larger k than are accessible to us with FPeriod. The program FProbPeriod again works by listing and tagging, but now only needs to keep in memory words for the points visited along an iteration. As long as the preperiod and period of the forward orbit aren't too large, the program won't crash.

The input data for FProbPeriod then are the c.a. f ; a finite set of periods k ; the number N of symbols; and the number m of points to be randomly sampled for each k . The program will for each k take m random samples of points from $P_k(S_N)$, and find the corresponding periods and preperiods with multiplicity. Given k , L denotes the largest f -period found in the sample. For any sequence of samples, clearly $\limsup_k L^{1/k} \leq \limsup_k \nu_k(f, S_N) \leq N$, and inequalities must become sharp in some cases (f linear or f of finite order). Still, the data we see seems consistent with positive answers to Questions 1.2 and 1.3.

The specific maps cited below are described in Section 4.

Table 25. For sample size $m = 10$, for the (degree one, left permutative, not right closing) map $B = x_0 + x_1x_2$, the (eventual) periods are listed with their multiplicities in the sample, for $1 \leq k \leq 37$.

Table 26. For the map B , periods with multiplicity are probed for $k \leq 30$ for two samples, of size 10 and size 30. The maximum period is the same except for two values of k . By comparison with Table 24, one sees that the size 30 sample in Table 26 found the largest period except at $k = 12$ (where it found period 56 but not the maximum period 60).

Table 31. For the linear c.a. A , periods with multiplicity are probed for $k \leq 49$ for two sample sizes, 10 and 30. The results are almost identical.

Table 27. For sample size $m = 10$, for $1 \leq k \leq 37$, the numbers $L^{1/k}$ are computed for several c.a. described in Section 4: A, B, C, E, G, H, J, K . The corresponding preperiod data is displayed in *Table 28*.

Table 29. For sample size $m = 10$, for $1 \leq k \leq 32$, the sampled periods for the nonclosing c.a. C are listed with their multiplicities in the sample.

Table 30. For sample size $m = 10$, for $1 \leq k \leq 32$, the sampled periods for the nonclosing c.a. D are listed with their multiplicities in the sample.

Table 32. This table lists the preperiods found for B by FProbPeriod for the sample size 10 in the range $18 \leq k \leq 35$.

Table 33. This table lists the preperiods found for C by FProbPeriod for the sample size 10 in the range $18 \leq k \leq 35$.

APPENDIX A. TABLES OF SOME SPAN 4 AND 5 C.A.

Map	Tabular rule	Map	Tabular rule
1	0000 1111 0010 1101	17	0011 1001 1100 1100
2	0000 1111 0100 1011	18	0011 1010 0011 1100
3	0001 1100 0011 1110	19	0011 1010 1100 0011
4	0001 1110 0101 1010	20	0011 1100 0101 0011
5	0010 1001 0110 1101	21	0011 1100 0101 1100
6	0010 1101 0000 1111	22	0011 1100 1010 0011
7	0011 0011 0110 0011	23	0011 1100 1010 1100
8	0011 0011 0110 1100	24	0011 1110 0001 1100
9	0011 0011 1001 0011	25	0100 1001 0110 1011
10	0011 0011 1001 1100	26	0100 1011 0000 1111
11	0011 0101 0011 1100	27	0101 1010 0001 1110
12	0011 0101 1100 0011	28	0101 1010 0111 1000
13	0011 0110 0011 0011	29	0110 1011 0100 1001
14	0011 0110 1100 1100	30	0110 1101 0010 1001
15	0011 1000 0111 1100	31	0111 1000 0101 1010
16	0011 1001 0011 0011	32	0111 1100 0011 1000

TABLE 1. The 32 span 4 onto c.a. of the 2 shift which fix ...000... and are not linear in an end variable [13, Table I]. Maps 2, 6, 7 and 16 are one-to-one. The rule above for map 30 corrects a misprint in [13, Table I].

Map	Tabular rule	Map	Tabular rule
1	0001 0111 1110 1000 0001 0111 1111 0000	14	0100 1101 1111 0000 0100 1101 1011 0010
2	0001 1011 0111 0100 1110 0100 1111 0000	15	0110 0001 1010 1011 0110 0001 0110 0111
3	0010 0010 1111 0011 0010 1110 0000 1111	16	0110 1000 0111 1001 0110 0001 1110 1001
4	0010 1001 0110 1101 0100 1001 0110 1011	17	0110 1011 1100 0010 0100 1011 0001 1101
5	0010 1110 0000 1111 0010 1110 1111 0000	18	0111 0001 1011 0010 0111 0001 1000 1110
6	0100 0111 0001 0111 1011 1000 0000 1111	19	0111 0010 1011 0100 0111 0010 0111 1000
7	0100 0111 0100 1011 1000 1011 0100 1011	20	0111 1000 0100 1011 0111 1000 0111 1000
8	0100 1011 1000 0111 0100 1011 0100 1011	21	0111 1000 0100 1011 0111 1000 1011 0100
9	0100 1101 1011 0010 1000 1110 1011 0010	22	0111 1000 0100 1011 0111 1000 1111 0000
10	0100 1101 1011 0010 1100 1100 1011 0010	23	0111 1000 0100 1101 0111 1000 1000 1110
11	0100 1101 1101 0010 0011 0011 1101 0010	24	0111 1011 1000 0100 0100 1011 0000 1111
12	0100 1101 1101 0010 0111 0001 1101 0010	25	0111 1011 1100 0000 0100 1011 0000 1111
13	0100 1101 1101 0010 1111 0000 1101 0010	26	0111 1011 1100 0000 0100 1011 0100 1011

TABLE 2. 26 irregular span 5 onto maps of the 2 shift which fix ...000... and are not linear in an end variable [13, Table XII].

APPENDIX B. FDENSE TABLES

Map	10-dense at	13-dense at	Map	10-dense at	13-dense at
1	11-24	13-24	17	17,18,20-24	24
2	10-24	13-24	18	17,19-24	23-24
3	18-24	24	19	19-24	23-24
4	21-24	(27)	20	17,19-23	(25)
5	17,19-23	(25)	21	19-24	23-24
6	10-24	13-24	22	19-24	23-24
7	10-24	13-24	23	19-24	21-22
8	21-24	(27)	24	17,19-24	23-24
9	11-24	13-24	25	17,19-23	(25)
10	19,21-24	24	26	11-24	13-24
11	18-24	24	27	19, 21-24	24
12	17,19-23	(25)	28	22-24	(25)
13	11-24	13-24	29	19-24	23-24
14	22-24	(25)	30	19-24	23-24
15	19-24	23-24	31	17,18,20-24	24
16	10-24	13-24	32	19-24	21-22

TABLE 3. The map numbers refer to the 32 span 4 maps of Table 1. Table 3 shows for the given sample of maps, and for $m = 10$ and $m = 13$, for which k in the range $[m, 24]$ the jointly periodic points in $P_k(S_2)$ are m -dense. If the map is not m -dense for any k in this range, then the number listed in parentheses is the smallest k for which the jointly periodic points in $P_k(S_2)$ are m -dense.

Map	10-dense at	13-dense at	Map	10-dense at	13-dense at
F◦1	11-24	13,15-24	F◦17	19,21-24	24
F◦2	11-24	13-24	F◦18	19-24	23-24
F◦3	19-24	21-22	F◦19	19-24	(27)
F◦4	22-24	(25)	F◦20	21-24	23
F◦5	21-24	23	F◦21	17,19-24	23-24
F◦6	10-24	13-24	F◦22	19-24	(27)
F◦7	20-24	13-24	F◦23	18-24	24
F◦8	22-24	(25)	F◦24	19-24	23-24
F◦9	11-24	13,15-24	F◦25	21-24	23
F◦10	17-18,21-24	24	F◦26	11-24	13,15-24
F◦11	19-24	21,22	F◦27	17-18,20-24	24
F◦12	21-24	23	F◦28	21-24	(27)
F◦13	11-24	13,15-24	F◦29	19-24	(27)
F◦14	21-24	(27)	F◦30	19-24	(27)
F◦15	17-19,20-24	23-24	F◦31	19,21-24	24
F◦16	10-24	13-24	F◦32	18-24	24

TABLE 4. The map numbers refer to the 32 span 4 maps of Table 1. F is the involution $F = x_0 + 1$. Table 4 shows for the given sample of maps, and for $m = 10$ and $m = 13$, for which k in the range $[m, 24]$ the jointly periodic points in $P_k(S_2)$ m -dense. If the map is not m -dense for any k in this range, then the number listed in parentheses is the smallest k for which the jointly periodic points in $P_k(S_2)$ are m -dense.

Map	10-dense at	Map	10-dense at	Map	10-dense at	Map	10-dense at
F \circ 1	11-24	F \circ 17	19,21-24	D \circ 1	20-24	D \circ 17	21-24
F \circ 2	11-24	F \circ 18	19-24	D \circ 2	19,22-24	D \circ 18	21-24
F \circ 3	19-24	F \circ 19	19-24	D \circ 3	20-24	D \circ 19	22-24
F \circ 4	22-24	F \circ 20	21-24	D \circ 4	21,23-24	D \circ 20	20,24
F \circ 5	21-24	F \circ 21	17,19-24	D \circ 5	20,24	D \circ 21	21,23-24
F \circ 6	10-24	F \circ 22	19-24	D \circ 6	20,22,24	D \circ 22	24
F \circ 7	20-24	F \circ 23	18-24	D \circ 7	19,22-24	D \circ 23	21-24
F \circ 8	22-24	F \circ 24	19-24	D \circ 8	21,23-24	D \circ 24	21-24
F \circ 9	11-24	F \circ 25	21-24	D \circ 9	22-24	D \circ 25	22-24
F \circ 10	17-18,21-24	F \circ 26	11-24	D \circ 10	21,23-24	D \circ 26	22-24
F \circ 11	19-24	F \circ 27	17-18,20-24	D \circ 11	22-24	D \circ 27	21,23-24
F \circ 12	21-24	F \circ 28	21-24	D \circ 12	20-21,23-24	D \circ 28	22-24
F \circ 13	11-24	F \circ 29	19-24	D \circ 13	20-24	D \circ 29	22-24
F \circ 14	21-24	F \circ 30	19-24	D \circ 14	22-24	D \circ 30	24
F \circ 15	17-19,20-24	F \circ 31	19,21-24	D \circ 15	21,23-24	D \circ 31	21-24
F \circ 16	10-24	F \circ 32	18-24	D \circ 16	20,22,24	D \circ 32	21-24

TABLE 5. The c.a. listed are compositions, e.g. $D \circ j$ is map j followed by D . The map numbers j refer to the 32 span 4 maps of Table 1. The map D is given by $x_0 + x_1$. The map F is the flip involution $F = 1 + x_0$. The data on $F \circ j$ are copied in from Table 4 for contrast with $D \circ j$.

j	10-dense at	j	10-dense at	j	10-dense at	j	10-dense at
1	18-24	9	17-19,21-24	17	18-20,22-24	25	11,13-24
2	19,21-24	10	19-24	18	17,21-24	26	11,13-24
3	17-24	11	17,19-24	19	17-24		
4	18-19,21-24	12	18-24	20	18-24		
5	17-24	13	16-23	21	19-24		
6	15-24	14	16-24	22	19-24		
7	16-24	15	18-24	23	20-24		
8	11,13-24	16	18-19,21-24	24	11,13-24		

TABLE 6. The map numbers refer to the 26 “irregular” span 5 maps of [13][Table XII], copied in Table 2. 1.

j	10-dense at	j	10-dense at	j	10-dense at	j	10-dense at
1	20-24	9	22-23	17	21,24	25	20,23-24
2	18-24	10	21,24	18	22,24	26	19,22-24
3	20,22-24	11	21-24	19	23,24	27	21,24
4	21,23-24	12	21-24	20	22-24	28	21,23-24
5	21,23-24	13	21-24	21	22,24	29	20,22-23
6	19-24	14	21-24	22	23,24	30	19-24
7	20,22-24	15	21-24	23	21-24	31	21,24
8	21-24	16	20-21,23	24	21-24	32	20,22-24

TABLE 7. As in Table 5, the map numbers refer to the 32 span 4 maps of Table 1. For such a map j , let $p_j(x_0, x_1, x_2, x_3)$ be the polynomial such that $(jx)_0 = p_j(x_0, x_1, x_2, x_3)$. Then a row j of Table 7 refers to the map f_j such that $(f_jx)_0 = x_0 + p_j(x_1, x_2, x_3, x_4)$. Equivalently, $f_j = x_0 + (j \circ S_2)$. For the map f_j , all k in the range $[10, 24]$ at which f_j is 10-dense are listed.

By comparison to Table 5, we see that altering the rules j as we have produces maps which look more “random” in the sense that all the 10-density is being achieved at k on the order of 2×10 . Note that the maps in Table 7 are by construction left permutative, and for these the jointly periodic points are known to be dense [5].

APPENDIX C. FPERIOD TABLES

Output obtained from the FPeriod Program, discussed in Section 6, is compiled in the Tables below. In Tables 8-16, for a c.a. f , the numbers given for a row k are computed with respect to $P_k(S_N)$, the set of points fixed by the shift S_N . Except for Tables 15 and 16, the number of symbols N is 2. For a given k , L denotes the maximum f -period of a point in $P_k(S_N)$; P denotes the number of points in $P_k(S_N)$ which are f -periodic (so, $P + \text{Not-}P = N^k$); and ν_k denotes $\nu_k(f, S_N) = P^{1/k}$. In some later tables, ν_k^o is used to denote the k th root of the number of points of *least* S_n -period k which are periodic for f . The preperiod of a point x is the smallest nonnegative integer j such that $f^j(x)$ is f -periodic.

k	Fraction Periodic	ν_k	$L^{1/k}$	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	0.500000	1.00	1.00	1	1	1	1.00	0.50	1
2	0.250000	1.00	1.00	1	1	3	1.00	1.25	2
3	0.500000	1.58	1.44	4	3	4	2.50	0.50	1
4	0.062500	1.00	1.00	1	1	15	1.00	3.06	4
5	0.500000	1.74	1.71	16	15	16	14.12	0.50	1
6	0.250000	1.58	1.34	16	6	48	5.12	1.25	2
7	0.500000	1.81	1.32	64	7	64	6.91	0.50	1
8	0.003906	1.00	1.00	1	1	255	1.00	7.00	8
9	0.500000	1.85	1.58	256	63	256	62.05	0.50	1
10	0.250000	1.74	1.40	256	30	768	29.01	1.25	2
11	0.500000	1.87	1.69	1,024	341	1024	340.67	0.50	1
12	0.062500	1.58	1.23	256	12	3840	11.57	3.06	4
13	0.500000	1.89	1.67	4,096	819	4096	818.80	0.50	1
14	0.250000	1.81	1.20	4,096	14	12,288	13.89	1.25	2
15	0.500000	1.90	1.19	16,384	15	16,384	14.99	0.50	1
16	0.000015	1.00	1.00	1	1	65535	1.00	15.00	16
17	0.500000	1.92	1.38	65,536	255	65,536	254.33	0.50	1
18	0.250000	1.85	1.30	65,536	126	196,608	125.73	1.25	2
19	0.500000	1.92	1.62	262,144	9,709	262,144	9708.96	0.50	1
20	0.062500	1.74	1.22	65,536	60	983,040	59.88	3.06	4
21	0.500000	1.93	1.21	1,048,576	63	1,048,576	62.99	0.50	1
22	0.250000	1.87	1.34	1,048,576	682	3,145,728	681.67	1.25	2
23	0.500000	1.94	1.39	4,194,304	2,047	4,194,304	2047.00	0.50	1

TABLE 8. The c.a. is $A = x_0 + x_1$ on the 2-shift: a linear, two-to-one map. If $k = q2^j$ with q odd and j a nonnegative integer, then there are exactly $q2^{-(j+1)}$ points in $P_k(S_2)$ which are f -periodic. Thus, $\nu_k = 2^{(k-1)/k}$ if k is odd.

k	Fraction Periodic	ν_k	$L^{1/k}$	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	.5000	1.00	1.00	1	1	1	1.00	0.50	1
2	.2500	1.00	1.00	1	1	3	1.00	1.25	2
3	.5000	1.58	1.44	4	3	4	2.50	0.50	1
4	.3125	1.49	1.41	5	4	11	2.50	1.31	3
5	.3437	1.61	1.58	11	10	21	9.44	0.97	2
6	.4375	1.74	1.61	28	18	36	11.31	0.69	2
7	.0625	1.34	1.32	8	7	120	6.91	4.00	7
8	.0195	1.22	1.18	5	4	251	3.25	6.58	12
9	.1484	1.61	1.58	76	63	436	58.26	3.17	7
10	.0888	1.57	1.52	91	70	933	18.17	7.77	17
11	.0703	1.57	1.46	144	66	1,904	65.35	5.52	14
12	.0576	1.57	1.36	236	42	3,860	24.44	10.98	34
13	.0350	1.54	1.53	287	273	7,905	217.65	11.93	29
14	.0201	1.51	1.39	330	105	16,054	12.65	36.60	74
15	.0123	1.49	1.44	404	255	32,364	179.68	35.36	91
16	.0232	1.58	1.54	1,525	1,008	64,011	272.23	33.28	98
17	.0286	1.62	1.52	3,758	1,377	127,314	913.23	31.04	114
18	.0091	1.54	1.53	2,386	2,250	259,758	2,026.85	55.23	152
19	.0039	1.49	1.47	2,091	1,672	522,197	1,658.11	91.44	251
20	.0015	1.44	1.31	1,635	240	1,046,941	14.16	279.12	575
21	.0046	1.54	1.48	9,650	4,326	2,087,502	461.24	244.11	638
22	.0011	1.47	1.40	4,896	1,848	4,189,408	1,158.45	274.42	647
23	.0027	1.54	1.53	23,461	19,297	8,365,147	18,849.71	269.70	824

TABLE 9. The c.a. is J , the composition $x_0 + x_1$ followed by the involution $U = x_0 + x_{-2}x_1x_2 + x_{-2}x_{-1}x_1x_2$. This invertible c.a. U is the involution of the 2-shift which replaces x_0 with $x_0 + 1$ when $x[-2, 2] = 10x_011$. J is biclosing but not permutative.

k	Fraction Periodic	ν_k	$L^{1/k}$	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	0.5000	1.00	1.00	1	1	1	1.00	0.50	1
2	0.2500	1.00	1.00	1	1	3	1.00	0.75	1
3	0.1250	1.00	1.00	1	1	7	1.00	1.62	2
4	0.3125	1.49	1.41	5	4	11	2.50	0.94	2
5	0.3437	1.61	1.58	11	10	21	9.44	0.66	1
6	0.0156	1.00	1.00	1	1	63	1.00	3.52	5
7	0.2812	1.66	1.54	36	21	92	17.62	1.05	2
8	0.0195	1.22	1.18	5	4	251	3.91	5.65	11
9	0.0546	1.44	1.22	28	6	484	5.82	5.18	11
10	0.1767	1.68	1.46	181	45	843	29.75	2.14	7
11	0.0703	1.57	1.55	144	132	1904	98.08	6.76	19
12	0.0012	1.14	1.12	5	4	4091	1.01	19.25	36
13	0.0556	1.60	1.49	456	182	7736	162.94	18.54	49
14	0.0261	1.54	1.35	428	70	15956	28.54	18.35	55
15	0.0342	1.59	1.45	1121	285	31647	138.58	21.60	58
16	0.0074	1.47	1.47	485	480	65051	430.96	71.09	146
17	0.0160	1.56	1.55	2109	1734	128963	1633.83	51.36	169
18	0.0060	1.50	1.41	1594	549	260550	334.44	70.40	233
19	0.0046	1.50	1.45	2452	1197	521836	834.45	92.00	227
20	0.0058	1.54	1.50	6165	3640	1042411	2700.37	70.21	211
21	0.0017	1.47	1.36	3627	693	2093525	585.86	356.39	817
22	0.0033	1.54	1.46	14004	4147	4180300	3305.59	251.62	864
23	0.0022	1.53	1.53	18746	18538	8369862	18491.96	262.30	900

TABLE 10. The c.a. E is the composition $x_0 + x_1$ followed by $x_0 + x_1x_2$, a linear 2-to-1 map followed by a degree 1 left permutative map.

k	Fraction Periodic	ν_k	$L^{1/k}$	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	1.0000	2.00	1.00	2	1	0	1.00	0.00	0
2	.5000	1.41	1.00	2	1	2	1.00	0.50	1
3	.2500	1.25	1.00	2	1	6	1.00	1.12	2
4	.1250	1.18	1.00	2	1	14	1.00	1.62	3
5	.2187	1.47	1.37	7	5	25	1.62	2.03	4
6	.4062	1.72	1.51	26	12	38	4.56	1.20	3
7	.0703	1.36	1.32	9	7	119	6.91	4.65	9
8	.0703	1.43	1.41	18	16	238	1.94	6.98	12
9	.1796	1.65	1.48	92	36	420	15.52	2.55	7
10	.0263	1.39	1.17	27	5	997	1.07	9.08	15
11	.1782	1.70	1.53	365	110	1,683	77.16	3.79	16
12	.0122	1.38	1.30	50	24	4,046	17.51	10.57	26
13	.1049	1.68	1.53	860	260	7,332	199.90	6.20	21
14	.0056	1.38	1.37	93	84	16,291	70.69	22.86	48
15	.0340	1.59	1.43	1,117	225	31,651	117.64	13.52	42
16	.0154	1.54	1.40	1,010	224	64,526	111.24	27.58	68
17	.0135	1.55	1.45	1,770	612	129,302	558.46	41.02	112
18	.0037	1.46	1.33	980	180	261,164	52.93	32.45	107
19	.0078	1.54	1.50	4,125	2,242	520,163	824.24	52.35	168
20	.0011	1.42	1.32	1,227	280	1,047,349	88.00	77.69	196
21	.0008	1.42	1.39	1,731	1,092	2,095,421	29.02	180.81	480
22	.0006	1.43	1.27	2,829	220	4,191,475	85.05	134.13	399
23	.0008	1.46	1.44	6,833	4,462	8,381,775	4,148.57	209.22	699

TABLE 11. The c.a. is $G = x_{-1} + x_0 x_1 + x_2$, which is bipermutative but not linear.

k	Fraction Periodic	ν_k	$L^{1/k}$	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	0.500000	1.00	1.00	1	1	1	1.00	0.50	1.00
2	0.750000	1.73	1.00	3	1	1	1.00	0.25	1.00
3	0.500000	1.58	1.00	4	1	4	1.00	0.88	2.00
4	0.687500	1.82	1.41	11	4	5	2.50	0.31	1.00
5	0.812500	1.91	1.71	26	15	6	9.75	0.19	1.00
6	0.281250	1.61	1.00	18	1	46	1.00	3.95	8.00
7	0.609375	1.86	1.74	78	49	50	37.75	0.94	5.00
8	0.667969	1.90	1.81	171	120	85	75.94	0.86	6.00
9	0.482422	1.84	1.55	247	54	265	44.93	2.83	12.00
10	0.535156	1.87	1.82	548	410	476	345.04	2.85	17.00
11	0.183105	1.71	1.60	375	176	1673	158.91	28.00	73.00
12	0.176270	1.73	1.40	722	60	3374	6.38	37.95	85.00
13	0.200073	1.76	1.59	1639	416	6553	220.46	19.73	76.00
14	0.212524	1.79	1.62	3482	882	12902	483.97	42.97	153.00
15	0.231598	1.81	1.59	7589	1095	25179	523.90	42.69	191.00
16	0.117599	1.74	1.63	7707	2688	57829	1422.26	159.56	457.00
17	0.078995	1.72	1.60	10354	3230	120718	2481.50	371.77	938.00
18	0.078449	1.73	1.37	20565	324	241579	302.81	350.15	1155.00
19	0.061646	1.72	1.64	32320	13471	491968	12128.71	404.87	1233.00
20	0.065800	1.74	1.64	68996	21240	979580	15870.41	285.87	1063.00
21	0.032823	1.69	1.56	68835	11865	2028317	816.87	1050.92	3506.00
22	0.021364	1.67	1.60	89609	32428	4104695	20280.02	1335.34	5030.00
23	0.011244	1.64	1.48	94324	9108	8294284	7929.18	4869.70	10024.00

TABLE 12. The c.a. is $B = x_0 + x_1x_2$ on the 2-shift. B is degree 1, left permutative and not right closing.

k	Fraction Periodic	ν_k	$L^{1/k}$	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	.5000	1.00	1.00	1	1	1	1.00	0.50	1
2	.7500	1.73	1.00	3	1	1	1.00	0.25	1
3	.5000	1.58	1.00	4	1	4	1.00	0.88	2
4	.6875	1.82	1.00	11	1	5	1.00	0.31	1
5	.8125	1.91	1.37	26	5	6	2.56	0.19	1
6	.6562	1.86	1.61	42	18	22	7.38	0.86	4
7	.6093	1.86	1.60	78	28	50	17.35	1.16	5
8	.5117	1.83	1.62	131	48	125	27.50	2.08	10
9	.4296	1.82	1.58	220	63	292	43.61	3.39	11
10	.4082	1.82	1.31	418	15	606	5.45	6.91	21
11	.4355	1.85	1.57	892	143	1,156	99.90	12.91	53
12	.3608	1.83	1.36	1,478	42	2,618	11.61	15.59	53
13	.3270	1.83	1.66	2,679	754	5,513	577.86	33.42	123
14	.2167	1.79	1.26	3,552	28	12,832	23.06	79.16	191
15	.2503	1.82	1.48	8,204	385	24,564	303.28	69.75	232
16	.3152	1.86	1.63	20,659	2,528	44,877	1,197.54	48.40	281
17	.1784	1.80	1.55	23,393	1,853	107,679	1,538.93	168.75	464
18	.1821	1.81	1.59	47,760	4,464	214,384	3,208.77	172.00	697
19	.1357	1.80	1.49	71,175	1,957	453,113	1,685.66	352.58	1082
20	.1620	1.82	1.56	169,886	7,976	878,690	5,604.39	258.96	953
21	.1032	1.79	1.52	216,612	7,056	1,880,540	6,344.22	2,389.64	4,363
22	.0902	1.79	1.35	378,612	740	3,815,692	633.16	2,315.42	6,465
23	.0858	1.79	1.58	720,246	39,353	7,668,362	36,059.28	1,760.56	5,984

TABLE 13. The c.a. is K , the composition which is the automorphism U of Table 9 followed by the left permutative, not right-closing map $x_0 + x_1x_2$. The c.a. K is left closing, not right-closing, and not permutative on either side.

k	Fraction Periodic	ν_k	$L^{1/k}$	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	0.500	1.00	1.00	1	1	1	1.00	0.50	1
2	0.750	1.73	1.00	3	1	1	1.00	0.25	1
3	0.500	1.58	1.44	4	3	4	1.75	0.50	1
4	0.687	1.82	1.18	11	2	5	1.75	0.31	1
5	0.812	1.91	1.71	26	15	6	11.00	0.19	1
6	0.468	1.76	1.20	30	3	34	1.66	1.28	3
7	0.500	1.81	1.66	64	35	64	28.34	0.99	4
8	0.667	1.90	1.63	171	52	85	30.39	0.52	3
9	0.306	1.75	1.27	157	9	355	8.89	2.35	8
10	0.261	1.74	1.49	268	55	756	20.18	6.75	18
11	0.387	1.83	1.57	793	143	1255	53.53	3.00	13
12	0.088	1.63	1.16	362	6	3734	1.39	20.61	48
13	0.150	1.72	1.63	1236	611	6956	259.75	20.15	78
14	0.126	1.72	1.51	2068	329	14316	119.61	33.22	132
15	0.091	1.70	1.50	3014	465	29754	414.94	44.45	138
16	0.092	1.72	1.50	6043	728	59493	650.33	101.66	282
17	0.107	1.75	1.68	14145	6783	116927	3918.82	48.16	196
18	0.060	1.71	1.58	15753	4095	246391	3406.78	110.78	396
19	0.072	1.74	1.60	38191	7619	486097	6336.19	142.98	406
20	0.038	1.69	1.54	40396	5780	1008180	1691.96	279.69	780
21	0.018	1.65	1.48	37867	4011	2059285	3961.81	705.45	1777
22	0.017	1.66	1.51	75309	9658	4118995	4527.64	605.57	1770
23	0.017	1.67	1.57	144096	34477	8244512	26857.88	1191.56	2687

TABLE 14. The c.a. is C , the composition $x_0x_1 + x_2$ followed by $x_0 + x_1x_2$, on the 2-shift. The c.a. C is neither left nor right closing. It is not known whether the periodic points of C are dense.

k	Fraction Periodic	ν_k	$L^{1/k}$	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	1.00	3.00	2.00	3	2	0	1.67	0.00	0.00
2	1.00	3.00	2.44	9	6	0	4.56	0.00	0.00
3	1.00	3.00	2.46	27	15	0	9.30	0.00	0.00
4	0.11	1.73	1.56	9	6	72	4.56	0.89	1.00
5	1.00	3.00	2.77	243	165	0	121.13	0.00	0.00
6	1.00	3.00	2.80	729	486	0	334.54	0.00	0.00
7	1.00	3.00	2.57	2187	742	0	401.62	0.00	0.00
8	0.01	1.73	1.58	81	40	6480	33.50	4.24	9.00
9	1.00	3.00	2.24	19683	1469	0	1185.85	0.00	0.00
10	1.00	3.00	2.76	59049	25865	0	22737.63	0.00	0.00
11	1.00	3.00	2.92	177147	131857	0	109208.21	0.00	0.00
12	0.00	1.76	1.67	909	486	530532	239.27	45.51	133.00
13	1.00	3.00	2.79	1594323	631605	0	291222.95	0.00	0.00

TABLE 15. This map on the 3-shift is an automorphism W followed by the degree 9 linear map $x_0 + x_2$, where $W = x_0 + 2x_0x_1x_1 + 2x_0x_1 + x_1 * x_1 + x_1$. Let π denote the permutation on $\{0, 1, 2\}$ which transposes 0 and 2. Then $(Wx)_0 = x_0$ if $x_1 \neq 1$ and $(Wx)_0 = \pi(x_0)$ if $x_1 = 1$.

k	Fraction Periodic	ν_k	$L^{1/k}$	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	.3333	1.00	1.00	1	1	2	1.00	1.00	2
2	.5555	2.23	1.00	5	1	4	1.00	0.56	2
3	.2592	1.91	1.00	7	1	20	1.00	1.78	3
4	.2098	2.03	1.00	17	1	64	1.00	3.17	8
5	.4362	2.54	2.09	106	40	137	27.65	1.08	4
6	.2208	2.33	1.51	161	12	568	6.79	2.71	9
7	.0932	2.13	1.66	204	35	1,983	5.89	13.89	38
8	.0391	2.00	1.00	257	1	6,304	1.00	27.02	67
9	.1667	2.45	1.60	3,283	72	16,400	48.89	13.41	52
10	.0299	2.11	1.62	1,770	130	57,279	62.67	55.38	163
11	.0224	2.12	1.89	3,972	1122	173,175	593.34	99.23	297
12	.0164	2.13	1.40	8,729	60	522,712	12.56	88.45	222
13	.0076	2.06	1.81	12,117	2366	1,582,206	2,228.50	676.85	1,504

TABLE 16. The map $x_0 + x_1x_2$ on the 3-shift: still degree 1, left permutative, not right closing.

k	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1.00	1.00	2.00	2.00	1.00	2.00	2.00	2.00	1.00	1.00	1.00	1.00	1.00
2	0.00	0.00	0.00	1.41	1.41	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	1.81	1.44	1.81	1.81	0.00	1.44	1.44	1.81	1.81	1.44	1.44	0.00	0.00
4	1.41	0.00	0.00	1.86	0.00	1.41	1.68	0.00	0.00	0.00	1.68	1.68	1.41
5	1.90	1.82	1.90	1.82	1.71	1.97	1.71	1.97	1.90	1.37	1.71	1.71	1.82
6	1.61	0.00	1.51	1.69	0.00	1.69	1.76	1.86	1.81	1.61	1.51	1.61	0.00
7	1.77	1.80	1.80	1.77	1.74	1.88	1.90	1.99	1.45	1.85	1.90	1.70	1.92
8	1.62	1.70	1.75	1.70	1.48	1.72	1.81	1.90	1.80	0.00	1.78	1.72	1.70
9	1.89	1.76	1.74	1.90	1.90	1.91	1.91	1.99	1.79	1.58	1.76	1.77	1.80
10	1.80	1.58	1.25	1.83	1.78	1.81	1.91	1.95	1.67	0.00	1.76	1.68	1.90
11	1.66	1.69	1.75	1.73	1.78	1.85	1.92	1.99	1.67	1.66	1.80	1.51	1.48
12	1.71	1.75	1.78	1.84	1.68	1.85	1.84	1.97	1.77	1.70	1.68	1.73	1.69
13	1.73	1.72	1.79	1.73	1.72	1.87	1.93	2.00	1.72	1.69	1.80	1.75	1.84
14	1.66	1.61	1.73	1.73	1.63	1.81	1.91	1.98	1.57	1.54	1.69	1.68	1.74
15	1.66	1.71	1.60	1.73	1.74	1.85	1.92	1.99	1.78	1.67	1.70	1.68	1.76
16	1.68	1.64	1.74	1.71	1.72	1.79	1.93	1.98	1.54	1.65	1.67	1.49	1.75
17	1.69	1.53	1.73	1.68	1.72	1.84	1.91	2.00	1.73	1.59	1.73	1.65	1.69
18	1.68	1.46	1.69	1.68	1.71	1.83	1.91	1.99	1.65	1.59	1.57	1.65	1.67
19	1.67	1.61	1.68	1.67	1.69	1.81	1.93	2.00	1.71	1.63	1.66	1.71	1.73

TABLE 17. $\nu_k^o(\cdot, S_2)$ for the span five maps 1-13 of Table 2.

k	14	15	16	17	18	19	20	21	22	23	24	25	26
1	1.00	2.00	2.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00
2	0.00	1.41	1.41	0.00	1.41	1.41	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	1.81	1.44	0.00	1.44	1.44	1.44	0.00	0.00	0.00	0.00	1.81	1.81	1.81
4	0.00	1.68	1.86	0.00	1.41	1.68	0.00	1.41	0.00	1.68	0.00	0.00	0.00
5	1.82	1.82	1.82	1.71	1.71	1.90	1.71	1.58	1.71	1.58	1.97	1.97	1.97
6	1.81	1.86	1.76	1.61	1.51	1.51	0.00	1.69	1.81	1.81	1.69	1.69	1.69
7	1.80	1.85	1.92	1.60	1.70	1.83	1.83	1.54	1.70	1.54	1.99	1.99	1.99
8	1.41	1.83	1.70	1.72	1.83	1.86	1.68	1.68	1.54	1.68	1.81	1.81	1.86
9	1.84	1.84	1.76	1.76	1.71	1.78	1.78	1.74	1.84	1.87	1.99	1.99	1.99
10	1.86	1.85	1.83	1.65	1.76	1.82	1.79	1.52	1.67	1.72	1.90	1.90	1.93
11	1.57	1.78	1.81	1.88	1.81	1.75	1.77	1.64	1.79	1.54	1.99	1.99	1.99
12	1.57	1.83	1.84	1.66	1.64	1.80	1.62	1.73	1.73	1.67	1.94	1.94	1.94
13	1.71	1.71	1.68	1.73	1.72	1.85	1.61	1.78	1.79	1.65	2.00	2.00	2.00
14	1.76	1.73	1.72	1.67	1.66	1.80	1.57	1.73	1.75	1.60	1.95	1.95	1.96
15	1.71	1.76	1.79	1.72	1.77	1.78	1.72	1.65	1.66	1.52	1.99	1.99	1.99
16	1.71	1.71	1.71	1.73	1.69	1.78	1.71	1.51	1.65	1.55	1.97	1.97	1.97
17	1.74	1.66	1.63	1.64	1.71	1.78	1.64	1.65	1.67	1.68	2.00	2.00	2.00
18	1.71	1.68	1.68	1.67	1.62	1.73	1.65	1.61	1.52	1.62	1.98	1.98	1.98
19	1.68	1.69	1.70	1.70	1.63	1.75	1.70	1.66	1.65	1.60	2.00	2.00	2.00

TABLE 18. $\nu_k^o(\cdot, S_2)$ for the span five maps 14-26 of Table 2.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2.00	2.00	1.00	1.00	2.00	2.00	2.00	1.00	2.00	1.00	1.00	2.00	2.00	1.00	1.00	2.00
2	0.00	1.41	0.00	1.41	1.41	1.41	1.41	1.41	0.00	0.00	0.00	1.41	0.00	1.41	1.41	1.41
3	1.81	1.81	1.81	1.44	0.00	1.81	1.81	1.44	1.81	1.44	1.81	0.00	1.81	1.44	1.44	1.81
4	1.68	1.86	1.68	1.41	1.41	1.86	1.86	1.41	1.68	1.41	1.68	1.41	1.68	0.00	1.68	1.86
5	1.97	1.97	1.82	1.90	1.58	1.97	1.97	1.90	1.97	1.82	1.82	1.58	1.97	1.58	1.90	1.97
6	1.90	1.94	1.81	1.61	1.76	1.94	1.94	1.61	1.90	1.34	1.81	1.76	1.90	1.86	1.81	1.94
7	1.99	1.99	1.83	1.70	1.80	1.99	1.99	1.70	1.99	1.74	1.83	1.80	1.99	1.83	1.70	1.99
8	1.95	1.98	1.81	1.48	1.75	1.98	1.98	1.48	1.95	1.78	1.81	1.75	1.95	1.54	1.88	1.98
9	1.99	1.99	1.86	1.68	1.82	1.99	1.99	1.68	1.99	1.82	1.86	1.82	1.99	1.73	1.86	1.99
10	1.98	1.99	1.76	1.70	1.82	1.99	1.99	1.70	1.98	1.75	1.76	1.82	1.98	1.56	1.65	1.99
11	1.99	1.99	1.70	1.65	1.68	1.99	1.99	1.65	1.99	1.89	1.70	1.68	1.99	1.60	1.90	1.99
12	1.99	1.99	1.51	1.65	1.61	1.99	1.99	1.65	1.99	1.65	1.51	1.61	1.99	1.34	1.75	1.99
13	2.00	2.00	1.70	1.57	1.63	2.00	2.00	1.57	2.00	1.73	1.70	1.63	2.00	1.54	1.68	2.00
14	1.99	1.99	1.74	1.65	1.70	1.99	1.99	1.65	1.99	1.81	1.74	1.70	1.99	1.66	1.74	1.99
15	1.99	1.99	1.71	1.68	1.70	1.99	1.99	1.68	1.99	1.73	1.71	1.70	1.99	1.47	1.77	1.99
16	1.99	1.99	1.74	1.67	1.70	1.99	1.99	1.67	1.99	1.76	1.74	1.70	1.99	1.59	1.67	1.99
17	2.00	2.00	1.67	1.53	1.71	2.00	2.00	1.53	2.00	1.75	1.67	1.71	2.00	1.59	1.61	2.00
18	1.99	1.99	1.71	1.56	1.65	1.99	1.99	1.56	1.99	1.71	1.71	1.65	1.99	1.52	1.63	1.99
19	2.00	2.00	1.73	1.54	1.72	2.00	2.00	1.54	2.00	1.77	1.73	1.72	2.00	1.57	1.69	2.00

TABLE 19. $\nu_k^o(\cdot, S_2)$ for the span four maps 1-16 of Table 1.

k	$F1$	$F2$	$F3$	$F4$	$F5$	$F6$	$F7$	$F8$	$F9$	$F10$	$F11$	$F12$	$F13$	$F14$	$F15$	$F16$
1	2.00	2.00	1.00	1.00	2.00	2.00	2.00	1.00	2.00	1.00	1.00	2.00	2.00	1.00	1.00	2.00
2	0.00	1.41	0.00	1.41	1.41	1.41	1.41	1.41	0.00	0.00	0.00	1.41	0.00	1.41	1.41	1.41
3	1.81	1.81	1.81	1.44	1.44	1.81	1.81	1.44	1.81	0.00	1.81	1.44	1.81	1.44	0.00	1.81
4	1.41	1.86	1.68	0.00	1.41	1.86	1.86	0.00	1.41	0.00	1.68	1.41	1.41	1.41	1.68	1.86
5	1.97	1.97	1.82	1.58	1.71	1.97	1.97	1.58	1.97	1.71	1.82	1.71	1.97	1.90	1.90	1.97
6	1.86	1.94	1.69	1.86	1.69	1.94	1.94	1.86	1.86	1.69	1.69	1.69	1.86	1.61	0.00	1.94
7	1.99	1.99	1.66	1.83	1.70	1.99	1.99	1.83	1.99	1.60	1.66	1.70	1.99	1.70	1.92	1.99
8	1.93	1.98	1.81	1.54	1.80	1.98	1.98	1.54	1.93	1.41	1.81	1.80	1.93	1.48	1.75	1.98
9	1.99	1.99	1.71	1.73	1.62	1.99	1.99	1.73	1.99	1.77	1.71	1.62	1.99	1.68	1.68	1.99
10	1.95	1.99	1.70	1.56	1.44	1.99	1.99	1.56	1.95	1.79	1.70	1.44	1.95	1.70	0.00	1.99
11	1.99	1.99	1.74	1.60	1.46	1.99	1.99	1.60	1.99	1.59	1.74	1.46	1.99	1.65	1.65	1.99
12	1.97	1.99	1.65	1.34	1.55	1.99	1.99	1.34	1.97	1.63	1.65	1.55	1.97	1.65	1.69	1.99
13	2.00	2.00	1.75	1.54	1.65	2.00	2.00	1.54	2.00	1.53	1.75	1.65	2.00	1.57	1.67	2.00
14	1.98	1.99	1.74	1.66	1.53	1.99	1.99	1.66	1.98	1.72	1.74	1.53	1.98	1.65	1.51	1.99
15	1.99	1.99	1.74	1.47	1.64	1.99	1.99	1.47	1.99	1.68	1.74	1.64	1.99	1.68	1.74	1.99
16	1.99	1.99	1.66	1.59	1.55	1.99	1.99	1.59	1.99	1.66	1.66	1.55	1.99	1.67	1.68	1.99
17	2.00	2.00	1.67	1.59	1.57	2.00	2.00	1.59	2.00	1.74	1.67	1.57	2.00	1.53	1.69	2.00
18	1.99	1.99	1.63	1.52	1.61	1.99	1.99	1.52	1.99	1.70	1.63	1.61	1.99	1.56	1.61	1.99
19	2.00	2.00	1.69	1.57	1.51	2.00	2.00	1.57	2.00	1.54	1.69	1.51	2.00	1.54	1.63	2.00

TABLE 20. $\nu_k^o(\cdot, S_2)$ for the span 4 maps 1-16 of Table 1, postcomposed with the flip map $F = x_0 + 1$.

k	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
1	1.00	1.00	2.00	2.00	1.00	2.00	1.00	1.00	2.00	2.00	1.00	1.00	2.00	2.00	1.00	1.00
2	0.00	1.41	0.00	1.41	1.41	0.00	0.00	1.41	1.41	0.00	0.00	1.41	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	1.44	0.00	1.81	0.00	0.00	1.81	1.44	1.44	0.00	0.00	0.00	1.81
4	0.00	1.68	1.68	1.41	1.68	1.68	1.68	1.68	1.41	1.68	1.41	0.00	1.68	1.68	0.00	1.68
5	1.71	1.90	1.71	1.58	1.90	1.71	1.82	1.90	1.58	1.97	1.82	1.58	1.71	1.71	1.71	1.82
6	1.69	0.00	1.34	1.76	1.81	1.34	1.69	0.00	1.76	1.90	1.34	1.86	1.34	1.34	1.69	1.69
7	1.60	1.92	1.54	1.80	1.70	1.54	1.66	1.92	1.80	1.99	1.74	1.83	1.54	1.54	1.60	1.66
8	1.41	1.75	1.41	1.75	1.88	1.41	1.81	1.75	1.75	1.95	1.78	1.54	1.41	1.41	1.41	1.81
9	1.77	1.68	1.74	1.82	1.86	1.74	1.71	1.68	1.82	1.99	1.82	1.73	1.74	1.74	1.77	1.71
10	1.79	0.00	1.72	1.82	1.65	1.72	1.70	0.00	1.82	1.98	1.75	1.56	1.72	1.72	1.79	1.70
11	1.59	1.65	1.41	1.68	1.90	1.41	1.74	1.65	1.68	1.99	1.89	1.60	1.41	1.41	1.59	1.74
12	1.63	1.69	1.59	1.61	1.75	1.59	1.65	1.69	1.61	1.99	1.65	1.34	1.59	1.59	1.63	1.65
13	1.53	1.67	1.66	1.63	1.68	1.66	1.75	1.67	1.63	2.00	1.73	1.54	1.66	1.66	1.53	1.75
14	1.72	1.51	1.44	1.70	1.74	1.44	1.74	1.51	1.70	1.99	1.81	1.66	1.44	1.44	1.72	1.74
15	1.68	1.74	1.58	1.70	1.77	1.58	1.74	1.74	1.70	1.99	1.73	1.47	1.58	1.58	1.68	1.74
16	1.66	1.68	1.64	1.70	1.67	1.64	1.66	1.68	1.70	1.99	1.76	1.59	1.64	1.64	1.66	1.66
17	1.74	1.69	1.59	1.71	1.61	1.59	1.67	1.69	1.71	2.00	1.75	1.59	1.59	1.59	1.74	1.67
18	1.70	1.61	1.46	1.65	1.63	1.46	1.63	1.61	1.65	1.99	1.71	1.52	1.46	1.46	1.70	1.63
19	1.54	1.63	1.60	1.72	1.69	1.60	1.69	1.63	1.72	2.00	1.77	1.57	1.60	1.60	1.54	1.69

TABLE 21. $\nu_k^2(\cdot, S_2)$ for the span 4 onto maps 17-32 of Table 1.

k	$F17$	$F18$	$F19$	$F20$	$F21$	$F22$	$F23$	$F24$	$F25$	$F26$	$F27$	$F28$	$F29$	$F30$	$F31$	$F32$
1	1.00	1.00	2.00	2.00	1.00	2.00	1.00	1.00	2.00	2.00	1.00	1.00	2.00	2.00	1.00	1.00
2	0.00	1.41	0.00	1.41	1.41	0.00	0.00	1.41	1.41	0.00	0.00	1.41	0.00	0.00	0.00	0.00
3	1.44	1.44	1.44	1.44	0.00	1.44	1.81	1.44	1.44	1.81	0.00	1.44	1.44	1.44	1.44	1.81
4	1.41	1.68	1.68	1.41	1.68	1.68	1.68	1.68	1.41	1.41	0.00	1.41	1.68	1.68	1.41	1.68
5	1.82	1.90	1.58	1.71	1.90	1.58	1.82	1.90	1.71	1.97	1.71	1.90	1.58	1.58	1.82	1.82
6	1.34	1.81	1.61	1.69	0.00	1.61	1.81	1.81	1.69	1.86	1.69	1.61	1.61	1.61	1.34	1.81
7	1.74	1.70	1.70	1.70	1.92	1.70	1.83	1.70	1.70	1.99	1.60	1.70	1.70	1.70	1.74	1.83
8	1.78	1.88	1.65	1.80	1.75	1.65	1.81	1.88	1.80	1.93	1.41	1.48	1.65	1.65	1.78	1.81
9	1.82	1.86	1.60	1.62	1.68	1.60	1.86	1.86	1.62	1.99	1.77	1.68	1.60	1.60	1.82	1.86
10	1.75	1.65	1.61	1.44	0.00	1.61	1.76	1.65	1.44	1.95	1.79	1.70	1.61	1.61	1.75	1.76
11	1.89	1.90	1.69	1.46	1.65	1.69	1.70	1.90	1.46	1.99	1.59	1.65	1.69	1.69	1.89	1.70
12	1.65	1.75	1.49	1.55	1.69	1.49	1.51	1.75	1.55	1.97	1.63	1.65	1.49	1.49	1.65	1.51
13	1.73	1.68	1.66	1.65	1.67	1.66	1.70	1.68	1.65	2.00	1.53	1.57	1.66	1.66	1.73	1.70
14	1.81	1.74	1.59	1.53	1.51	1.59	1.74	1.74	1.53	1.98	1.72	1.65	1.59	1.59	1.81	1.74
15	1.73	1.77	1.53	1.64	1.74	1.53	1.71	1.77	1.64	1.99	1.68	1.68	1.53	1.53	1.73	1.71
16	1.76	1.67	1.69	1.55	1.68	1.69	1.74	1.67	1.55	1.99	1.66	1.67	1.69	1.69	1.76	1.74
17	1.75	1.61	1.63	1.57	1.69	1.63	1.67	1.61	1.57	2.00	1.74	1.53	1.63	1.63	1.75	1.67
18	1.71	1.63	1.55	1.61	1.61	1.55	1.71	1.63	1.61	1.99	1.70	1.56	1.55	1.55	1.71	1.71
19	1.77	1.69	1.65	1.51	1.63	1.65	1.73	1.69	1.51	2.00	1.54	1.54	1.65	1.65	1.77	1.73

TABLE 22. $\nu_k^2(\cdot, S_2)$ for the span 4 maps 17-32 of Table 1, post-composed with the flip map $F = x_0 + 1$.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1.00	1.00	2.00	2.00	1.00	1.00	1.00	2.00	1.00	2.00	2.00	1.00	1.00	2.00	2.00	1.00
2	1.41	0.00	1.41	0.00	0.00	0.00	0.00	0.00	1.41	1.41	1.41	0.00	1.41	0.00	0.00	0.00
3	1.44	0.00	1.44	0.00	0.00	1.44	1.44	1.81	0.00	0.00	1.44	1.81	0.00	1.81	1.44	1.44
4	0.00	1.41	1.41	1.68	1.68	0.00	1.86	1.41	0.00	0.00	1.41	1.41	1.41	1.41	1.68	1.86
5	1.37	1.37	1.58	1.71	1.82	1.90	1.37	1.71	1.90	1.71	1.58	1.82	1.37	1.71	1.37	1.90
6	1.51	1.51	1.51	1.34	1.51	1.34	1.34	0.00	0.00	1.51	0.00	1.61	1.34	0.00	1.34	0.00
7	1.74	1.88	1.54	1.60	1.66	1.60	1.77	1.54	1.70	1.66	0.00	1.74	1.80	1.54	1.60	1.32
8	1.48	1.58	1.65	1.62	1.41	1.65	0.00	1.62	1.62	1.58	1.68	1.54	0.00	1.62	1.68	0.00
9	1.62	1.78	1.74	1.58	1.74	1.58	1.68	0.00	1.48	1.48	1.44	1.73	1.44	0.00	1.64	1.60
10	1.34	1.62	1.40	1.54	1.66	1.58	1.52	1.68	1.61	1.62	1.34	1.47	1.34	1.68	1.34	1.50
11	1.58	1.69	1.43	1.64	1.51	1.76	1.67	1.60	1.55	1.53	1.60	1.66	1.58	1.60	1.57	1.67
12	1.54	1.44	1.60	1.51	1.63	1.74	1.46	1.56	0.00	1.54	1.54	1.44	1.52	1.56	1.38	1.38
13	1.61	1.66	1.62	1.65	1.56	1.77	1.59	1.63	1.32	1.66	1.61	1.47	1.57	1.63	1.68	1.60
14	1.64	1.62	1.52	1.57	1.53	1.80	1.55	1.40	1.55	1.41	1.60	1.55	1.48	1.40	1.55	1.65
15	1.60	1.74	1.60	1.38	1.62	1.73	1.58	1.62	1.52	1.52	1.50	1.49	1.66	1.62	1.66	1.61
16	1.48	1.73	1.47	1.53	1.50	1.60	1.51	1.59	1.49	1.44	1.29	1.58	1.57	1.59	1.49	1.57
17	1.46	1.69	1.58	1.59	1.62	1.64	1.61	1.47	1.60	1.58	1.56	1.55	1.46	1.47	1.63	1.36
18	1.56	1.72	1.55	1.49	1.55	1.52	1.46	1.50	1.51	1.52	1.54	1.45	1.56	1.50	1.52	1.45
19	1.56	1.66	1.56	1.55	1.52	1.64	1.45	1.60	1.47	1.51	1.55	1.56	1.52	1.60	1.59	1.48

TABLE 23. ν_k^o for 16 left permutative span 5 maps.

Let $p_n(x[0, 4])$ denote the polynomial rule for the map n in Table 23 and let $q_n(x[0, 3])$ denote the polynomial rule for the span 4 map n in Table 1. Then p_n is defined by $p_n(x[0, 4]) = x_0 + q_n(x[1, 4])$.

The purpose of Table 23 is to allow a rough comparison of a sample of maps which are linear in an end variable to maps which are not (Table 23 vs. Table 1). We see no particular difference.

Table 24 gives complete cycle data for the c.a. B through shift period $k = 22$. of Table 26. For each k , all B -periods p of points from $P_k(S_2)$ are listed. The multiplicities given are the number μ_{orb} of B -cycles in $P_k(S_2)$ with the given size p ; the number μ_{per} of points in all these cycles; and the number μ_{ev} of points in $P_k(S_2)$ with eventual period p .

k	p	μ_{orb}	μ_{per}	μ_{ev}	k	p	μ_{orb}	μ_{per}	μ_{ev}
1	1	1	1	2	16	1	2207	2207	2208
2	1	3	3	4		4	1	4	6192
3	1	4	4	8		120	1	120	23520
4	1	7	7	8		2688	2	5376	33616
	4	1	4	8	17	1	3571	3571	3572
5	1	11	11	12		1020	1	1020	1530
	15	1	15	20		2533	1	2533	119357
6	1	18	18	64	3230	1	3230	6613	
7	1	29	29	30	18	1	5778	5778	5824
	49	1	49	98		9	1	9	9
8	1	47	47	48		38	36	1368	3834
	4	1	4	48		54	3	162	4815
	120	1	120	160	108	6	648	648	
9	1	76	76	80	216	6	1296	7740	
	9	1	9	9	296	36	10656	10656	
	54	3	162	423	324	2	648	228618	
10	1	123	123	124	19	1	9349	9349	9350
	15	1	15	40		76	1	76	76
	410	1	410	860		133	2	266	14421
11	1	199	199	200		171	1	171	171
	176	1	176	1848	646	1	646	2755	
12	1	322	322	3692	4161	1	4161	25156	
	4	1	4	8	4180	1	4180	12122	
	56	6	336	336	13471	1	13471	460237	
	60	1	60	60	20	1	15127	15127	15128
13	1	521	521	522		4	1	4	8
	10	13	130	650		15	1	15	1000
	26	1	26	117		132	5	660	1560
	143	1	143	3900		140	4	560	560
	403	1	403	845		410	1	410	9420
	416	1	416	2158		5240	2	10480	306300
14	1	843	843	844	20500	1	20500	197240	
	49	1	49	602	21240	1	21240	517360	
	161	2	322	2212	21	1	24476	24476	24480
	448	2	896	7686		14	3	42	42
	490	1	490	882		21	2	42	42
	882	1	882	4158		49	1	49	98
						57	21	1197	1197
15	1	1364	1364	1368	266	3	798	1949766	
	15	1	15	20	2618	6	15708	15708	
	180	1	180	180	4886	3	14658	14658	
	399	5	1995	8625	11865	1	11865	91161	
	450	3	1350	15705	22	1	39603	39603	39604
	530	3	1590	1590		132	2	264	264
	1095	1	1095	5280		176	1	176	910272
						660	1	660	1100
						1067	2	2134	112948
						14344	1	14344	924814
					32428	1	32428	2205302	

TABLE 24. Table for $B = x_0 + x_1x_2$ constructed from FPeriod: complete data for shift-periods through $k = 22$. See previous page for definitions.

APPENDIX D. FPROBPERIOD TABLES

For a given map f , a given positive integer N and a given positive integer m and given set of positive integers k , the program FProbPeriod will for each k randomly sample m blocks of length k on alphabet $\{0, 1, \dots, N - 1\}$, and compute the period and preperiod under f of the point x in $P_k(S_N)$ such that $x[0, k - 1]$ is the chosen block. Here, by definition the period p of x is the eventual period: the smallest $j > 0$ such that for some $k \geq 0$, $f^k(x) = f^{k+j}(x)$. So, p is the length of the f -cycle into which x is mapped by some f^k , $k \geq 0$. The preperiod of x is the smallest nonnegative k such that $f^k(x)$ is f -periodic, that is $f^k(x) = f^{k+p}(x)$. For a given sample of m points from $P_k(S_2)$, the multiplicity μ of p is the number of times the sampled point has the (eventual) period p . In the following tables, we restrict to cellular automata f on $N = 2$ symbols.

k	p	μ	k	p	μ	k	p	μ
1	1	10	15	1	1	27	3402	4
2	1	10		399	3		12096	3
3	1	10		450	3		218835	2
4	1	4		530	1		242352	1
	4	6		1095	2	28	882	1
5	1	4	16	120	7		32144	2
	15	6		2688	3		57036	7
6	1	10	17	2533	9	29	98223	2
7	1	4		3230	1		193256	1
	49	6	18	296	1		340286	3
8	1	2		324	9		504252	4
	4	2	19	1	1	30	17580	1
	120	6		13471	9		161721	8
9	1	2	20	5240	3		212670	1
	9	1		20500	1	31	2228621	10
	54	7		21240	6	32	473792	10
10	410	10	21	266	10	33	74439	7
11	1	1	22	176	2		313984	3
	176	9		14344	4	34	2533	10
12	1	8		32428	4	35	1635074	6
	56	2	23	1	2		4485250	1
13	1	1		622	1		14840595	3
	143	3		9108	7	36	152	2
	403	2	24	1	2		324	2
	416	4		12432	4		22974	1
14	49	1		20256	4		1700772	1
	448	7	25	61830	4		4191696	4
	882	2		104425	6	37	1365226	1
			26	143	1		7065594	5
				6994	9		39209196	4

TABLE 25. FProbPeriod output for the c.a. $B = x_0 + x_1x_2$ on two symbols, with sample size $m = 10$.

k	p_{30}	μ_{30}	p_{10}	μ_{10}	k	p_{30}	μ_{30}	p_{10}	μ_{10}	k	p_{30}	μ_{30}	p_{10}	μ_{10}	
1	1	30	1	10	15	1	4			23	1	1			
2	1	30	1	10		180	1				622	2	622	1	
3	1	30	1	10		399	10	399	6		9108	27	9108	9	
4	1	13	1	5		450	8	450	1	24	1	4			
	4	17	4	5		530	2				184	1			
5	1	12	1	4	1095	5	1095	3	2330		1	2330	1		
	15	18	15	6	16	1	2				7440	6	7440	1	
6	1	30	1	10		4	1				12432	10	12432	5	
7	1	9	1	3		120	12	120	3	25	4325	1			
	49	21	49	7		2688	15	2688	7		13015	1			
8	1	2	1	3		1	2				61830	14	61830	4	
	4	6	4	2	2533	26	2533	9	73175		4	73175	1		
	120	22	120	5	3230	2	3230	1	104425		10	104425	5		
9	1	3			18	1	1	54	1	26	6994	30	6994	10	
	54	27	54	10		216	2								
10	1	2				296	5				27	3402	9	3402	3
	15	3	1	2		324	22	324	9			12096	9	12096	3
	410	25	410	8		19	4161	1	1			1	218835	10	218835
11	1	9	1	2	4180		2	4180	1	28		242352	2	242352	3
	176	21	176	8	13471		27	13471	8			448	5		
12	1	27	1	10	20	15	1				882	5	882	4	
	56	3				5240	5	5240	5		32144	6	32144	1	
13	1	1				20500	7	20500	1		29	57036	14	57036	5
	10	2	10	1		21240	17	21240	4	98223		4			
	143	13	143	2		21	1	1				193256	7		
	403	5	403	1	266		26	266	10	340286		13	340286	3	
416	9	416	6	11865	3				504252	6		504252	7		
14	49	2			22	176	6	176	4	30	1	1			
	161	4	161	3		1067	1				450	5			
	448	11	448	4		14344	9				1800	1	1800	1	
	490	1	490	1		32428	14	32428	6		127995	2			
	882	12	882	2								132720	1		
												161721	19	161721	7
									212670	1	212670	2			

TABLE 26. Table for $B = x_0 + x_1x_2$ constructed as for Table 29, for sample sizes 10 and 30 for FProbPeriod. The longest orbit length p found is the same for both sample sizes, except for $k = 12$ and $k = 21$. Exact cycle data for the map B , through shift period $k = 22$, is given in Table 24. Note that for $k \leq 22$, only at $k = 12$ did the size-30 probabilistic sample miss the largest B period.

map	A	B	C	E	G	H	J	K
k	$L^{1/k}$	$L^{1/k}$	$L^{1/k}$	$L^{1/k}$	$L^{1/k}$	$L^{1/k}$	$L^{1/k}$	$L^{1/k}$
1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3	1.44	1.00	1.44	1.00	1.00	1.44	1.44	1.00
4	1.00	1.41	1.18	1.41	1.00	1.00	1.41	1.00
5	1.71	1.71	1.71	1.58	1.37	1.37	1.58	1.37
6	1.34	1.00	1.20	1.00	1.51	1.20	1.61	1.61
7	1.32	1.74	1.66	1.54	1.32	1.60	1.32	1.60
8	1.00	1.81	1.63	1.18	1.00	1.29	1.18	1.62
9	1.58	1.55	1.27	1.22	1.48	1.29	1.58	1.58
10	1.40	1.82	1.49	1.46	1.00	1.31	1.25	1.31
11	1.69	1.60	1.57	1.55	1.53	1.43	1.46	1.57
12	1.23	1.39	1.16	1.00	1.30	1.34	1.30	1.36
13	1.67	1.59	1.63	1.49	1.53	1.48	1.53	1.66
14	1.20	1.62	1.38	1.35	1.37	1.38	1.39	1.26
15	1.19	1.59	1.50	1.39	1.40	1.42	1.44	1.48
16	1.00	1.63	1.50	1.47	1.40	1.13	1.54	1.63
17	1.38	1.60	1.68	1.55	1.45	1.52	1.52	1.55
18	1.30	1.37	1.58	1.41	1.33	1.40	1.53	1.59
19	1.62	1.64	1.60	1.45	1.50	1.51	1.47	1.49
20	1.22	1.64	1.54	1.50	1.32	1.18	1.12	1.56
21	1.21	1.30	1.48	1.36	1.15	1.29	1.48	1.52
22	1.34	1.60	1.51	1.46	1.23	1.49	1.40	1.35
23	1.39	1.48	1.57	1.53	1.44	1.55	1.53	1.58
24	1.14	1.51	1.54	1.45	1.39	1.14	1.46	1.46
25	1.50	1.58	1.48	1.50	1.23	1.53	1.44	1.56
26	1.32	1.40	1.56	1.46	1.33	1.44	1.43	1.29
27	1.42	1.58	1.54	1.50	1.43	1.46	1.28	1.58
28	1.12	1.47	1.43	1.47	1.30	1.31	1.48	1.44
29	1.56	1.57	1.54	1.49	1.31	1.35	1.51	1.63
30	1.12	1.50	1.59	1.44	1.33	1.37	1.45	1.64
31	1.11	1.60	1.59	1.41	1.47	1.32	1.49	1.63
32	1.00	1.50	1.60	1.26	1.31	1.46	1.48	1.54
33	1.23	1.46	1.63	1.44	1.37	1.43	1.49	1.54
34	1.20	1.25	1.45	1.48	1.35	1.48	1.48	1.56
35	1.26	1.60	1.44	1.48	1.42	1.43	1.48	1.59
36	1.16	1.52	1.50	1.41	1.27	1.29	1.38	1.46
37	1.49	1.60	1.55	1.46	1.30	1.44	1.45	1.57

TABLE 27. This table is based at each k on a random sample of 10 blocks of length k on two symbols. L is the maximum period from the sample.

$A = x_0 + x_1$, linear, and $B = x_0 + x_1x_2$, permutative.

$C = B \circ B_{rev}$, nonclosing, where $B_{rev} = x_0x_1 + x_2$.

$G = x_{-1} + x_0x_1 + x_2$, bipermutative, nonlinear.

$E = B \circ A$, degree 2, left closing, not right closing.

$J = A \circ U$, where $U = x_0 + x_{-2}(1 + x_{-1})x_1x_2$ is invertible.

$H = A \circ A \circ U$, and $K = B \circ U$.

map	A	B	C	E	G	H	J	K
k	$M^{1/k}$	$M^{1/k}$	$M^{1/k}$	$M^{1/k}$	$M^{1/k}$	$M^{1/k}$	$M^{1/k}$	$M^{1/k}$
1	1.00	1.00	1.00	1.00	0.00	1.00	1.00	1.00
2	1.41	1.00	1.00	1.00	1.00	1.00	1.41	1.00
3	1.00	1.25	1.00	1.25	1.25	1.00	1.00	1.25
4	1.41	1.00	1.00	1.18	1.31	1.18	1.31	1.00
5	1.00	1.00	1.00	1.00	1.31	1.24	1.14	1.00
6	1.12	1.41	1.20	1.25	1.20	1.12	1.12	1.25
7	1.00	1.21	1.16	1.10	1.34	1.16	1.32	1.25
8	1.29	1.09	1.00	1.33	1.36	1.18	1.36	1.31
9	1.00	1.25	1.25	1.27	1.19	1.24	1.24	1.24
10	1.07	1.14	1.32	1.11	1.30	1.21	1.30	1.30
11	1.00	1.44	1.23	1.26	1.28	1.25	1.25	1.42
12	1.12	1.44	1.37	1.34	1.26	1.20	1.32	1.38
13	1.00	1.34	1.27	1.32	1.21	1.25	1.23	1.41
14	1.05	1.41	1.37	1.32	1.30	1.25	1.35	1.42
15	1.00	1.39	1.37	1.30	1.26	1.34	1.32	1.41
16	1.18	1.46	1.39	1.34	1.28	1.25	1.29	1.32
17	1.00	1.47	1.34	1.33	1.28	1.26	1.26	1.41
18	1.03	1.46	1.34	1.31	1.23	1.22	1.30	1.40
19	1.00	1.44	1.34	1.32	1.25	1.34	1.32	1.42
20	1.07	1.39	1.37	1.28	1.28	1.27	1.36	1.37
21	1.00	1.46	1.41	1.36	1.33	1.35	1.33	1.48
22	1.03	1.44	1.39	1.34	1.26	1.23	1.31	1.48
23	1.00	1.47	1.40	1.33	1.29	1.34	1.31	1.44
24	1.09	1.43	1.43	1.35	1.32	1.28	1.29	1.44
25	1.00	1.44	1.41	1.33	1.30	1.27	1.30	1.47
26	1.02	1.45	1.43	1.31	1.28	1.26	1.31	1.45
27	1.00	1.45	1.43	1.33	1.31	1.37	1.37	1.45
28	1.05	1.46	1.42	1.32	1.30	1.29	1.34	1.46
29	1.00	1.42	1.41	1.31	1.31	1.33	1.32	1.44
30	1.02	1.41	1.44	1.34	1.31	1.28	1.35	1.47
31	1.00	1.44	1.43	1.38	1.30	1.36	1.34	1.45
32	1.11	1.46	1.43	1.37	1.30	1.31	1.34	1.45
33	1.00	1.46	1.42	1.35	1.27	1.34	1.33	1.46
34	1.03	1.47	1.24	1.34	1.33	1.32	1.34	1.47
35	1.00	1.45	1.56	1.37	1.31	1.34	1.37	1.46
36	1.04	1.44	1.46	1.36	1.29	1.32	1.36	1.47
37	1.01	1.47	1.46	1.35	1.32	1.34	1.35	1.46

TABLE 28. This table is based at each k on a random sample of 10 blocks of length k on two symbols. M is the maximum preperiod from the sample. The maps are the same as used in Table 27 (Remark: For $x \in P_k(S_2)$ with $k = q^{2^j}$ with q odd, the point $A^{j+1}(x)$ must be periodic under f , so $M \leq j + 1$.)

k	p	μ	k	p	μ	k	p	μ	k	p	μ
1	1	10	12	1	10	20	3	1	30	31	1
2	1	10	13	47	6		2790	9		82531	9
3	1	10		52	4	21	573	10	31	57747	10
4	1	10	14	1	1	22	11	3	32	85	2
5	1	1		5	3		519	3		91	2
		3		13	3		9658	4		234649	6
6	1	5		47	2	23	1499	8	33	3452570	6
		3		49	1		9384	2		10357710	4
7	1	3	15	31	7	24	1	1	34	1717	7
		5		145	3		35160	9		10574	1
8	1	5	16	29	1	25	20475	10		999056	2
		13		85	9	26	441	1	35	572068	1
9	1	1	17	101	3		9401	9		2860340	1
		9		399	7	27	4543	5		3262280	8
10	1	1	18	1	1		19710	1	36	56	2
		3		455	9		113643	4		4095	1
		5	19	1	1	28	5	1		729537	3
		11		401	4		1260	9		908910	1
11	11	6		2755	3	29	277298	10		2188611	3
	143	4		7125	2				37	5881335	2
										12081277	8

TABLE 29. This table is based at each k on a random sample by FProbPeriod of 10 blocks of length k on symbols 0, 1. It is a table of the resulting periods p with multiplicities μ for the nonclosing map $C = B \circ B_{rev}$ where $B = x_0 + x_1x_2$ and $B_{rev} = x_0x_1 + x_2$. Here, a sampled block $x[0, k - 1]$ determines a point x of period k , and the period p is by definition the eventual period of x under iteration by C . The multiplicity μ for a given period is the number of samples for which x has that period.

k	p	μ	k	p	μ	k	p	μ	k	p	μ	k	p	μ
1	1	1	11	11	7	19	2755	3	27	113643	6	34	1717	10
2	1	1		143	3		7125	5		122661	4	35	3262280	8
3	1	7	12	1	9		7619	2	28	35	2		6886355	2
	3	3		6	1	20	1395	9		180	5	36	56	2
4	1	6	13	13	2		5780	1		1260	2		504	2
	2	4		52	5	21	4011	10		26124	1		729537	3
5	1	2		611	3	22	11	1	29	277298	10		2188611	3
	5	3	14	7	2		878	5	30	5205	1	37	3768043	1
	15	5		35	2		5709	2		137190	1		5881335	2
6	1	7		91	6		9658	2		1237965	8		12081277	7
	3	3	15	87	2	23	9384	2	31	457777	1			
7	7	2		465	8		34477	8		1790157	9			
	35	8	16	680	7	24	1	2	32	680	1			
8	1	1		728	3		11720	5		728	3			
	4	2	17	1717	5		35160	3		267824	2			
	52	7		6783	5	25	4095	8		3754384	4			
9	1	1	18	9	1		20475	2	33	3452570	5			
	9	9		56	1	26	52	2		10357710	5			
10	15	7		4095	8		122213	8						
	55	3												

TABLE 30. This table is constructed just as Table 29 was, except for the following: the data is for the map D which is the map C composed with $(S_2)^{-2}$, i.e., D is the composition of $x_0 + x_1x_2$ with $x_{-2}x - 1 + x_0$.

k	p	μ_{30}	μ_{10}	k	p	μ_{30}	μ_{10}	k	p	μ_{30}	μ_{10}
1	1	30	10	17	255	30	10	36	252	30	10
2	1	30	10	18	126	30	10	37	3233097	30	10
3	1	7	3	19	9709	30	10	38	19418	30	10
	3	23	7	20	30	1	1	39	4095	30	10
4	1	30	10		60	29	9	40	120	30	10
5	15	30	10	21	63	30	10	41	41943	30	10
6	1	3	0	22	682	30	10	42	126	30	10
	3	9	3	23	2047	30	10	43	5461	30	10
	6	18	7	24	24	30	10	44	1364	30	10
7	7	30	1	25	25575	30	10	45	4095	30	10
8	1	30	9	26	1638	30	10	46	4094	30	10
9	63	30	10	27	13797	30	10	47	8388607	30	10
10	15	1	10	28	28	30	10	48	48	30	10
	30	29	10	29	475107	30	10	49	2097151	30	10
11	341	30	10	30	30	30	10				
12	12	30	10	31	31	30	10				
13	819	30	10	32	1	30	10				
14	14	30	10	33	1023	30	10				
15	15	30	10	34	510	30	10				
16	1	30	10	35	4095	30	10				

TABLE 31. Table for $A = x_0 + x_1$ constructed as for Table 29, for sample sizes 10 and 30 for FProbPeriod. Both sample sizes work well for this linear map.

k	Pre.	k	Pre.	k	Pre.	k	Pre.	k	Pre.	k	Pre.
18	0	21	216	24	394	27	2669	30	7988	33	100664
	45		232		1032		3329		8537		108279
	170		254		1627		5641		10545		146726
	226		578		1834		9244		12479		149017
	293		595		2880		9697		13204		157529
	362		756		3145		11036		13676		161009
	506		1174		3396		11583		16210		188071
	556		1313		3905		13921		16315		196758
	751		3058		5215		14745		24373		240207
	102		3186		5517		23357		31106		270439
19	0	22	73	25	57	28	6000	31	4184	34	74577
	9		350		404		7790		11132		80674
	145		381		463		14999		16211		161429
	201		522		750		18569		30108		193935
	291		587		1361		23067		30661		209842
	301		1163		1547		25108		32789		283852
	490		1480		1671		28943		39001		360139
	547		1788		2491		31291		57399		400913
	658		3131		9415		37745		70339		452695
	105		3247		9531		47285		99228		521740
20	0	23	0	26	428	29	2702	32	8955	35	38461
	44		0		7866		9704		14007		51383
	181		671		9538		10239		19792		71921
	230		2543		11744		10783		48308		82039
	257		2791		12056		11194		49195		122647
	290		2835		13070		13472		65168		261207
	445		2900		15762		19429		65824		416233
	556		3123		16164		22208		72902		492662
	622		6492		16999		31392		126309		529999
	755		7854		17346		31856		184781		555899

TABLE 32. This table is based at each k on a random sample of 10 blocks of length k on symbols 0, 1. It is a table of the preperiods (Pre.) seen by FProbPeriod in a sample for the map $B = x_0 + x_1x_2$.

k	Pre.	k	Pre.	k	Pre.	k	Pre.	k	Pre.	k	Pre.
18	0	21	134	24	316	27	2016	30	1862	33	1790
	15		159		969		2046		2020		6742
	17		286		1012		2543		2159		8823
	26		305		1027		3793		6736		19037
	53		551		1541		5981		15523		19158
	98		600		1954		10577		16289		19234
	113		696		2800		13024		29590		33862
	177		748		3054		14611		32253		56053
	184		1379		3371		15185		56898		105560
	198		1550		5430		17428		59246		119315
19	15	22	7	25	1215	28	982	31	2176	34	28492
	35		14		1379		5042		3552		176703
	48		202		3305		5438		3716		184042
	49		241		3429		5783		16352		203938
	108		389		3486		5890		30302		204854
	119		436		3673		8235		53475		261574
	211		481		3865		9723		55921		282167
	264		1003		4716		10118		58664		295105
	271		1119		4719		15474		60608		374994
	275		1477		6056		19004		65768		384473
20	7	23	177	26	289	29	7160	32	2686	35	15491
	20		198		327		8519		4683		68409
	136		341		1127		9150		9120		76254
	138		366		1301		16361		9808		134451
	273		793		4720		16679		18985		168960
	280		1036		5146		21158		19562		205781
	422		1043		5817		22279		32081		223889
	474		1402		8610		24581		76390		247831
	478		2389		11410		25450		89515		351861
	614		2613		12155		25558		98677		385646

TABLE 33. This table is based at each k on a random sample of 10 blocks of length k on symbols 0, 1. It is a table of the preperiods (Pre.) seen by FProbPeriod in a sample for the nonclosing map $C = B \circ B_{rev}$, where $B = x_0 + x_1x_2$ and $B_{rev} = x_0x_1 + x_2$.

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