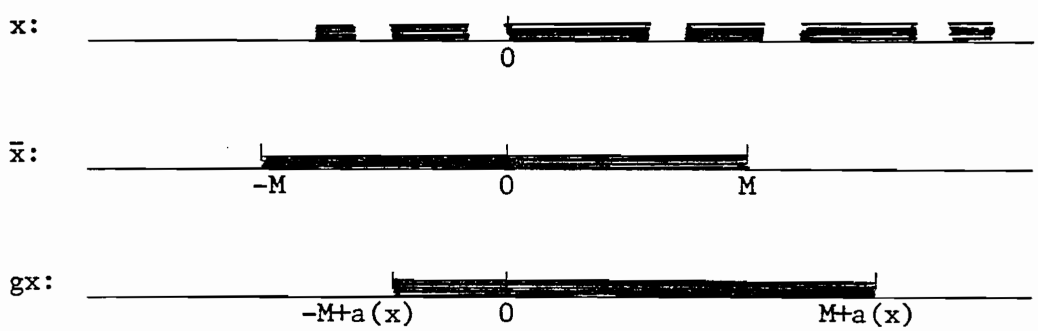
coordinates to the left of zero and $P_M(x)$ coordinates to the right of zero. For example, darkening these coordinates, we might have the following picture.




The following lemma will let us prove that g commutes with the shift. Given a sequence y, we let $[y_i \ldots y_j]$ be the set of sequences x in the ambient space such that $x_i \ldots x_j = y_i \ldots y_j$.

(2.4) Lemma

Let $A_k$ and $B_k$ be as in lemma 2.3. Suppose that $A_k = X$, $B_k$ is empty and $f_j(x) = 1$.

Then $a(Sx) = a(x) - j + 1$.

Proof

We will assume that j is positive; the argument for negative j is essentially the same. The case $j = 0$ cannot occur, because $f_o(x) = 0$. Pick $M > k + N$. Let $P = P_M(x)$, $N = N_M(x)$. So, $a(x) = \frac{1}{2}(N - P)$.

Since $f_j(x) = 1$, we know $\overline{Sx} = \bar{T}^j\bar{x}$, and

$[\bar{x}_{-M} \ldots \bar{x}_M] = [(\bar{T}^j\bar{x})_{-M-j} \ldots (\bar{T}^j x)_{M-j}] = [(\overline{Sx})_{-M-j} \ldots (\overline{Sx})_{M-j}] \equiv B.$

That is, B is defined by $x_o$ and by N negative and P positive coordinates of x (including $x_1$, since $M > k$). So, B is defined by $(Sx)_o$ and by $(N + 1)$ negative and $(P - 1)$ positive coordinates of Sx. Since $M + j > k$, we may compute a(Sx) by determining the number of positive and negative coordinates of Sx used to define $[(\overline{Sx})_{-(M+j)} \cdots (\overline{Sx})_{(M+j)}] \equiv C$. C differs from B by the additional specification of the 2j coordinates $(\overline{Sx})_{M-j+1}, \ldots, (\overline{Sx})_{M+j}$. Because $M-j \geqq M-N > k$, lemma 2.3 guarantees that these coordinates of $(\overline{Sx})$ are determined by positive coordinates of Sx. So,

$$
\begin{aligned}
a(Sx) &= \tfrac{1}{2}[N_{M+j}(Sx) - P_{M+j}(Sx)] \\
&= \tfrac{1}{2}[(N+1) - (P-1+2j)] \\
&= \tfrac{1}{2}[N-P] - j + 1 \\
&= a(x) - j + 1.
\end{aligned}
$$

□

(2.5) Lemma

S is conjugate to $\bar{T}$ by the map $gx = \bar{T}^{a(x)}\bar{x}$, or to $\bar{T}^{-1}$ by the map $x \to \bar{T}^{-a(x)}\bar{x}$, where a(x) is the continuous function

$$a(x) = \lim_{M} \tfrac{1}{2}[N_M(x) - P_M(x)].$$

Proof

If necessary replacing $\bar{T}$ with $\bar{T}^{-1}$, we may assume that lemma 2.3 gives S and T with the same orientation ($A_k = X$, $B_k$ is empty). Clearly g is continuous.

Suppose x is in X. By lemma 2.1, pick the unique j such that $f_j(x) = 1$. Then $\bar{T}gx = \bar{T}(\bar{T}^{a(x)}\bar{x}) = \bar{T}^{1+a(x)}\bar{x}$, while $gSx = \bar{T}^{a(Sx)}(\bar{T}^{j}\bar{x}) = \bar{T}^{a(Sx)+j}\bar{x}$. By lemma 2.4, we conclude that g commutes with the shift. Therefore, the image under g of an S-orbit is a full $\bar{T}$-orbit, and g is surjective. We now show that g is injective, hence a conjugacy.

Suppose $gx = gy$. Then $y = S^i x$ for some i. Then $gx = gS^i x = \bar{T}^i gx$, so gx is periodic for $\bar{T}$, so $\bar{x}$ is periodic for $\bar{T}$, so x is periodic for S. So, given x with least period $j > 1$, it suffices to prove that $x, Sx, \ldots, S^{j-1}x$ have distinct images. Suppose not. Then there are integers i and k, with $0 \leq i < k < j$, such that $gS^i x = gS^k x$, so $\bar{T}^i gx = \bar{T}^k gx$, so $gx = \bar{T}^{k-i} gx$, with $|k-i| < j$. This is a contradiction, because gx is in the orbit of $\bar{x}$ and therefore has least period j. □

<u>Remark</u> If in (2.5), $gx = \bar{T}^{a(x)}\bar{x}$ is a conjugacy of S and $\bar{T}$, then $x \to T^{a(x)}x$ is a conjugacy of S and T.

(2.6) <u>Theorem</u>

Suppose S and T are transitive homeormorphisms of a compact metric space X such that $Sx = T^{n(x)}x$, where n(x) is continuous, and S and T have the same orbits.

Then S is conjugate to T or $T^{-1}$.

Moreover, the conjugating map g may be given the form $gx = T^{a(x)}x$, where a(x) is continuous.

<u>Proof</u>

If S is a permutation, then the theorem is trivial, so we suppose not. Then by transitivity, there is a point y in X with a dense orbit.

Given x in X, define a doubly infinite sequence $\bar{x} = \gamma(x)$ by

$\bar{x}_i = n(S^i x)$. Let $\gamma X = \bar{X}$. Give $\bar{X}$ the usual topology, and let $\bar{S}$ be the shift on $\bar{X}$: then $\gamma$ is a continuous factor map, $\gamma\colon S \downarrow \bar{S}$. Define $\bar{T}$ on $\bar{X}$ by $\bar{T}\gamma x = \gamma T x$. Then $\gamma\colon T \downarrow \bar{T}$, and $\bar{S}$ and $\bar{T}$ are orbit conjugate by continuous jumps: $\bar{T}\bar{x} = \bar{S}^{n(\bar{x})}\bar{x}$, where $n(\bar{x}) = \bar{x}_o$, equal to $n(x)$ if $\gamma x = \bar{x}$.

Case I: $\bar{y}$ is periodic.

Since $y$ is dense in $X$, $\bar{y}$ is dense in $\bar{X}$, so $\bar{X}$ is a finite orbit $\{\bar{y}, \bar{S}\bar{y}, \ldots, \bar{S}^{k-1}\bar{y}\}$ for some minimal $k$. We let $\Sigma$ denote the sum $\bar{y}_o + \bar{y}_1 + \ldots + \bar{y}_{k-1}$. Then $|\Sigma| = k$: otherwise the orbits of $S$ and $T$ would differ. Since we only need to show flip conjugacy, we may assume that $\Sigma = k$.

Let $A_i = \gamma^{-1}(\bar{S}^i\bar{y})$, for $0 \le i \le k-1$. Then $S$ maps $A_i$ homeomorphically onto $A_{i+1}$ ($A_{k-1}$ onto $A_0$). The $A_i$ are clopen and disjoint. There is a transitive permutation $\tau$ on $k$ elements such that $T$ maps each $A_i$ homeomorphically to $A_{\tau(i)}$ (in fact $\tau(i) = i + \bar{y}_i$). That is, we have a diagram

$$\begin{array}{ccccccccc}
A_0 & \xrightarrow{S} & A_1 & \xrightarrow{S} & \cdots & \xrightarrow{S} & A_{k-1} & \xrightarrow{S} & A_0 \\
\downarrow g & & \downarrow g & & & & \downarrow g & & \downarrow g \\
A_0 & \xrightarrow{T} & A_{\tau(0)} & \xrightarrow{T} & \cdots & \xrightarrow{T} & A_{\tau^{k-1}(0)} & \xrightarrow{T} & A_0
\end{array} .$$

We define $g = \mathrm{id}$ on $A_0$; to make the diagram commute, define $g = T^i S^{-i}$ on $A_i$ if $1 \le i \le k$. The definitions are consistent on $A_0$ because $T^k = S^\Sigma = S^k$. So $g$ is a conjugacy. Clearly $g$ can be given the form $T^{a(x)}$ for a (continuous) function constant on each $A_i$.

<u>Case II</u>: $\bar{y}$ is aperiodic.

By lemma 2.5, we may assume (if necessary replacing T with $T^{-1}$) that there is a continuous function $\bar{a}(\bar{x})$ such that $\bar{g}$, given by $\bar{g}(\bar{x}) = \bar{T}^{a(\bar{x})}\bar{x}$, is a homeomorphism on $\bar{X}$ with $\bar{g}\bar{S} = \bar{T}\bar{g}$. We define a(x) on X by $a(x) = \bar{a}(\gamma x)$, and we define $gx = T^{a(x)}x$. We claim that g is a conjugacy.

The map g is bijective, because $\bar{g}$ and S are bijective. Then because $g = T^k$ on the clopen set $\{x: a(x) = k\}$, g is a homeomorphism. Thus $g^{-1}Tg$ is a homeomorphism. Since $g^{-1}Tg$ also takes T-orbits into T-orbits, it must take T-orbits onto T-orbits, hence $g^{-1}Tg$ is orbit conjugate to S. Since $\bar{g}^{-1}\bar{T}\bar{g}\bar{x} = \bar{S}\bar{x}$, it follows that $g^{-1}Tgx$ is in the fiber $\gamma^{-1}(\bar{S}\bar{x})$. If $\bar{x}$ is aperiodic under $\bar{S}$, then the fibers $\gamma^{-1}(\bar{S}^i\bar{x})$ are disjoint in X, so $\gamma^{-1}(\bar{S}x) \cap (\text{orbit of } x) = \{Sx\}$, and therefore $g^{-1}Tgx = Sx$. Therefore $S = g^{-1}Tg$ on the dense orbit of y, hence everywhere. □

Straightforward adjustments to the proof of (2.6) yield the following result.

(2.7) <u>Theorem</u> (Bounded jumps)

Suppose S and T are transitive homeomorphisms of a compact metric space X, S and T have the same orbits, and $Tx = S^{n(x)}x$ with n(x) bounded.

Let $N = \max |n(x)|$, and suppose that S is not a finite permutation. Let P be the closure of the set of all orbits which include a point at which n(x) is discontinuous. Then

(1) P is a nowhere dense set of periodic points, whose periods are all less than or equal to 2N;

(2) the restrictions of S and T to the complement of P are flip conjugate; and

(3) the flip conjugacy on the complement of P may be given the form $x \to T^{a(x)}x$, where a(x) is continuous on the complement of P.

(In particular, if S is expansive, then P is finite.) We only briefly sketch the adjustments. Let $A_i = \{x: n(x) = i\}$; then X is the union of those $A_i$ for which $|i| \leq N$. Clearly n(x) is continuous on the interior of each $A_i$. If $|i| \leq N$ and x is in the boundary of $A_i$, then for some j not equal to i with $|j| \leq N$, the set $A_j$ must include x. Then $Tx = S^i x = S^j x$, with $|i-j| \leq 2N$; this shows (1) above. The map $\gamma x = \bar{x}$ of (2.6) maps the complement of P continuously onto a cofinite subset of the domain of a subshift. On this set, factors $\bar{S}$ and $\bar{T}$ of S and T may be defined as before. With a little care, one can obtain the same map $\bar{x} \to \bar{T}^{a(x)}$ conjugating $\bar{S}$ to $\bar{T}$ (or to $\bar{T}^{-1}$); but now a(x), though continuous on the domain of $\bar{S}$ and $\bar{T}$, need not be uniformly continuous. The flip conjugacy of $\bar{S}$ and $\bar{T}$ lifts to S and T as before. □

We close this section with a technical observation.

(2.8) <u>Proposition</u>

Suppose S is a subshift, T is a homeomorphism with the same orbits as S, and $Tx = S^{n(x)}x$ with n(x) bounded. Then T is topologically conjugate to a subshift.

Proof

Let $N = \max|n(x)|$. Let $B_{16N+1}(S)$ be the alphabet for a subshift $\bar{T}$ defined as a factor of T by the map $\psi$: $x \to \bar{x}$, where $\bar{x}_i = (T^i x)_{-8N} \ldots (T^i x)_{8N}$. It is enough to show that $\psi$ is injective. That is, suppose that $\bar{x} = \bar{y}$. We must show that $x = y$. (We remark that a smaller number than 16N would suffice; a large number was chosen to make the arguments more obvious.)

Suppose $k = n(x) - n(y)$. Then

$x_i = y_i$ , $-8N \leq i \leq 8N$, since $\bar{x} = \bar{y}$;

$x_{i+n(x)} = y_{i+n(y)}$ , $-8N \leq i \leq 8N$, since $\bar{T}\bar{x} = \bar{T}\bar{y}$; so

$x_{i+n(x)-n(y)} = x_i$ , $-8N + n(y) \leq i \leq 8N + n(y)$. In particular,

$x_i = x_{i+k}$ , $-7N \leq i \leq 7N$, and $|k| \leq 2N$.

Consequently, if $\bar{x}$ has least period greater than 2N, then $n(T^i x) = n(T^i y)$ for all integers i, and x must equal y. Suppose $\bar{x}$ is periodic with period k less than or equal to 2N. Then for every $x_i$, the word $x_{i-7N} \ldots x_{i+7N}$ would be k-periodic. Then x (likewise y) would have period k, and $x = y$ could be read off from $\bar{x}_0 = x_{-8N} \ldots x_{8N}$. So, from here we may assume that $\bar{x}$ (and therefore x and y) is aperiodic.

Suppose (in the terminology of (2.1) of Chapter 1) that $x(\alpha \ldots \beta)$ is A-maximal in x, where A is a block of length at most 2N and $\alpha - \beta \geq 16N + 2$. Then we call $x(\alpha \ldots \beta)$ a periodic stretch (in x); if $\alpha$or $\beta$ is finite, then we call $x_\alpha$ or $x_\beta$ a seam. Then x contains a periodic stretch if and only if, for some i and some $k \leq 2N$, $\bar{x}_i$ is a k-periodic block of X. Visualize x as a succession of interrupted periodic stretches.

If x contains no periodic stretch, then $n(T^i x) = n(T^i y)$ for all i, so $x = y$. So, we assume x contains a periodic stretch, and then that $x_0$ is a seam (if not, replace x with some $T^i x$, y with $T^i y$: $x = y$ if and only if $T^i x = T^i y$). Now x (likewise y) is determined by $\bar{x}$.

To see this, think of beginning from $\bar{x}_0 = x_{-8N} \ldots x_{8N}$ to construct x, and having the successive $\bar{x}_i$ tell you where to jump in the orbit of x (and then giving you more coordinates of x). There is no ambiguity until a periodic stretch of $\bar{x}$ is entered. Even here, one can tell from the $\bar{x}_i$ when a periodic stretch is entered/exited, and on which side; the problem is to determine the length of the periodic stretch (a given periodic stretch may be "visited" several times). By keeping track of the entries and exits, one can determine exactly which coordinates $\bar{x}_i$ determine coordinates in a given periodic stretch of x. This determines the length of the periodic stretch in x, because $k \to f_k(x)$ is a bijection of the integers. So $\psi$ is injective. □

## 3. Sofic Shifts

In this section we consider orbit conjugacies between sofic shifts, in particular between subshifts of finite type. If S is intrinsically ergodic, then we denote the unique measure of maximal entropy by $\max_S$. We write C(S) for the space of continuous functions on S. The pressure of a continuous function f we denote as P(f); see [16] for properties of pressure. Given a measurable map h: X → Y and a measure m on X, we write hm for the measure on Y defined on each measurable set E by $hm(E) = m(h^{-1}E)$.

(3.1) <u>Proposition</u>

Suppose S and T are TPPD (transitive, with periodic points dense) sofic shifts and h is a homeomorphism which takes orbits of S onto orbits of T.

Then the following hold:

(1) $P(f) = P(fh)$, for all f in C(T);

(2) $h_m(S) = h_{hm}(T)$, for all m in M(S);

(3) for any f in C(T):

hm is an equilibrium state of f if and only if

m is an equilibrium state of fh;

(4) if S and T are subshifts of finite type,

then m in M(S) is Markov with support if and only if hm is;

(5) the range of $\max_S$ on cylinder sets equals

the range of $\max_T$ on cylinder sets.

Proof

(1) Suppose f is in C(T). Let $P_n = \{x: T^n x = x\}$. Then define

$$P_n(f) = \frac{1}{n} \log \sum_{x \in P_n} \exp\{\sum_{i=0}^{n-1} fS^i x\}.$$

Because T is TPPD sofic, $P(f) = \lim_n P_n(f)$. Now (1) follows because S is also TPPD sofic, and h takes periodic points to points of equal period.

(2) Recall from Section 1 that h takes M(S) bijectively to M(T). We have

$$\begin{aligned} h_{hm}(T) &= \inf_{f \in C(T)} \{P(f) - \int fd(hm)\}, \quad \text{since T is expansive,} \\ &= \inf_{f \in C(T)} \{P(fh) - \int (fh)dm\}, \quad \text{by (1),} \\ &= \inf_{g \in C(S)} \{P(g) - \int gdm\}, \quad \text{since h is a homeomorphism,} \\ &= h_m(S), \quad \text{since S is expansive.} \end{aligned}$$

(3) A measure m in M(S) is an equilibrium state of fh

if and only if $\int (fh)dm + h_m(S) = P(fh)$, by definition,

if and only if $\int fd(hm) + h_{hm}(T) = P(f)$, by (1) and (2),

if and only if $\int hm$ is an equilibrium state of f, by definition.

(4) On a subshift of finite type, every Markov measure with support is the unique equilibrium state of a locally constant function, and every locally constant function has a unique equilibrium state which is a Markov measure with support. Apply (3).

(5) The measure of maximal entropy on a TPPD sofic shift is the unique equilibrium state of any constant function. So h must take $max_S$ to $max_T$. Since h is a homeomorphism, $max_S$ and $max_T$ must have the same range of values on cylinder sets. □