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Lecture 7

Three open problems

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The problems:

- Which SFTs can commute?
- Must an expansive automorphism of an irreducible SFT be itself an SFT?
- Must the periodic points of a surjective one dimensional cellular automaton map be dense?

Definitions

- An *automorphism* of a continuous map f is a homeomorphism U commuting with f $(Uf = fU)$.

- Continuous maps f, F are *topologically conjugate* $(f \sim F)$ if $\exists F \sim f, G \sim g$ with $FG = GF$.

- Continuous maps f, g *can commute* if $\exists F \sim f, G \sim g$ with $FG = GF$.

- σ_A is the twosided edge shift of finite type (SFT) defined by the square $\mathbb{Z}_+$ matrix A.

- S is SFT if $S \sim \sigma_A$, for some A.

Which maps can commute with SFTs? [Nasu]

Which SFTs can commute?

CONJECTURE: Suppose S and T are mixing SFTs. Then for all large i, j, S^i and T^j can commute.

If S, T are commuting bijections and $\forall n > 0$ $|\text{Fix}(S^n)| < \infty$ and $|\text{Fix}(T^n)| < \infty$, then $\text{Per}(S) = \text{Per}(T)$. Thus low-order periodic point obstructions sometimes imply two maps cannot commute: e.g.,

- if $|\text{Fix}(\sigma_A)| = 1$ and $|\text{Fix}(\sigma_B)| = 0$, then σ_A and σ_B cannot commute

- $\sigma_{[2]}$ and $\sigma_{[3]}$ cannot commute

However, there is no set theoretic periodic point obstruction to the conjecture.

Commuting SFTs from commuting matrices

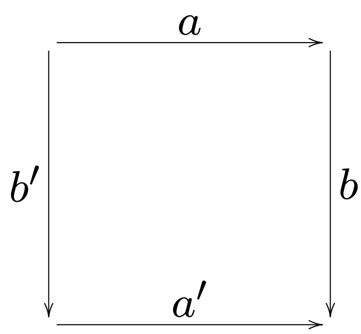
Note: $AB = BA$ does not guarantee that σ_A, σ_B can commute. (E.g. $[A] = 2, B = [3]$). However:

Proposition. Suppose A, B are commuting $\mathbb{Z}_+$ matrices. Then there are homeomorphisms S, T such that $ST = TS$ and $S^iT^j \sim \sigma_{A^iB^j}$ for $i, j > 0$.

The proposition follows from remarks on a construction of Nasu in his 1995 AMS Memoir, which created an elaborate "textile systems" apparatus for studying endomorphisms and automorphisms of an SFT. In this memoir and successor papers, Nasu achieved major results, especially on automorphisms of onesided SFTs.

We go on to explain the Proposition.

Suppose A and B are $n \times n$ matrices over $\mathbb{Z}_+$, with $AB = BA$. View A and B as adjacency matrices for two directed graphs, with disjoint edge sets and a common vertex set $\{1, 2, \ldots n\}$. Say e.g. an ab path from i to j is an A edge from i to some k followed by a B edge from that k to j. "$AB = BA$" means that for each pair i, j the number of ab paths from i to j equals the number of ba paths from i to j. Thus we can build a set $\mathcal{W}$ of Wang tiles



such that each ab path is the top/right of exactly one tile and each ba path is the left/bottom of exactly one tile. (In the tile pictured, a, a' are A-edges and b, b' are B-edges.)

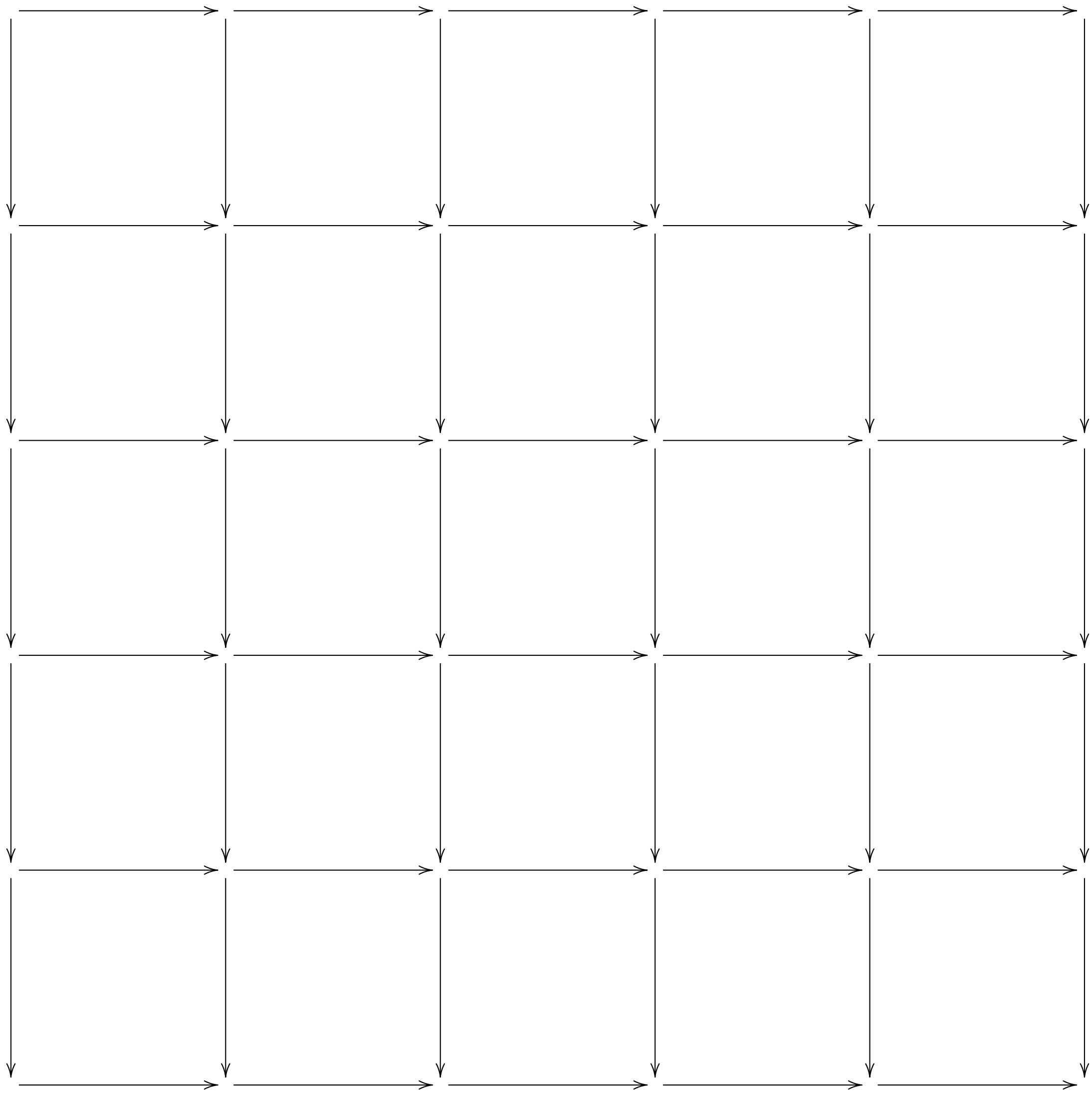
Thus each Wang tile

$$
\begin{tikzcd}
{} \arrow[r, "a"] \arrow[d, "b'"'] & {} \arrow[d, "b"] \\
{} \arrow[r, "a'"] & {}
\end{tikzcd}
$$

is determined by either of the paths

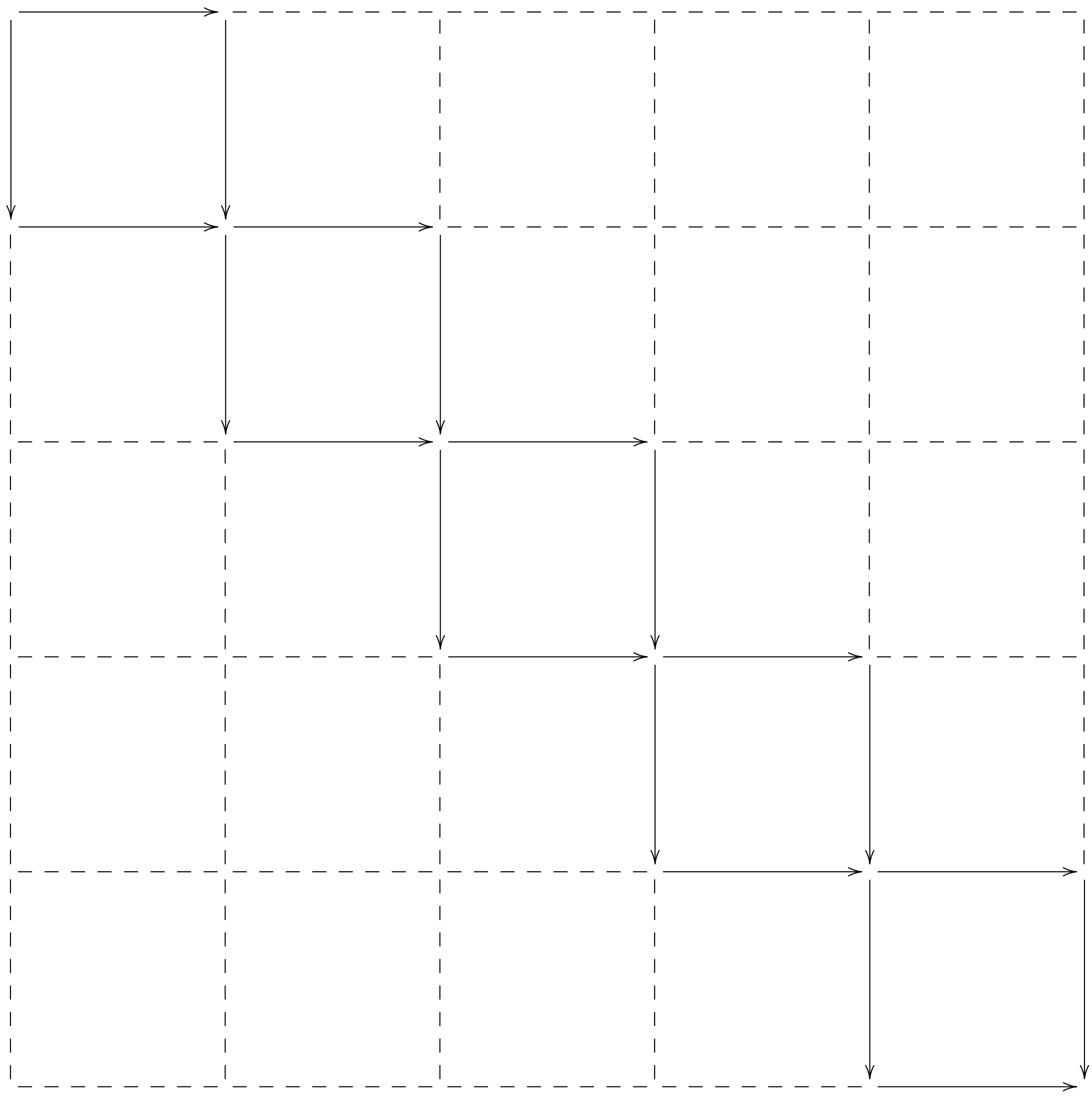
$$
\begin{tikzcd}
{} \arrow[r, "a"] & {} \arrow[d, "b"] \\
& {}
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
{} \arrow[d, "b'"'] & \\
{} \arrow[r, "a'"] & {}
\end{tikzcd}
$$

Now let the tile sides be unit length and let W be the space of infinite Wang tilings of the plane with $\mathcal{W}$, with tile corners on $\mathbb{Z}^2$.

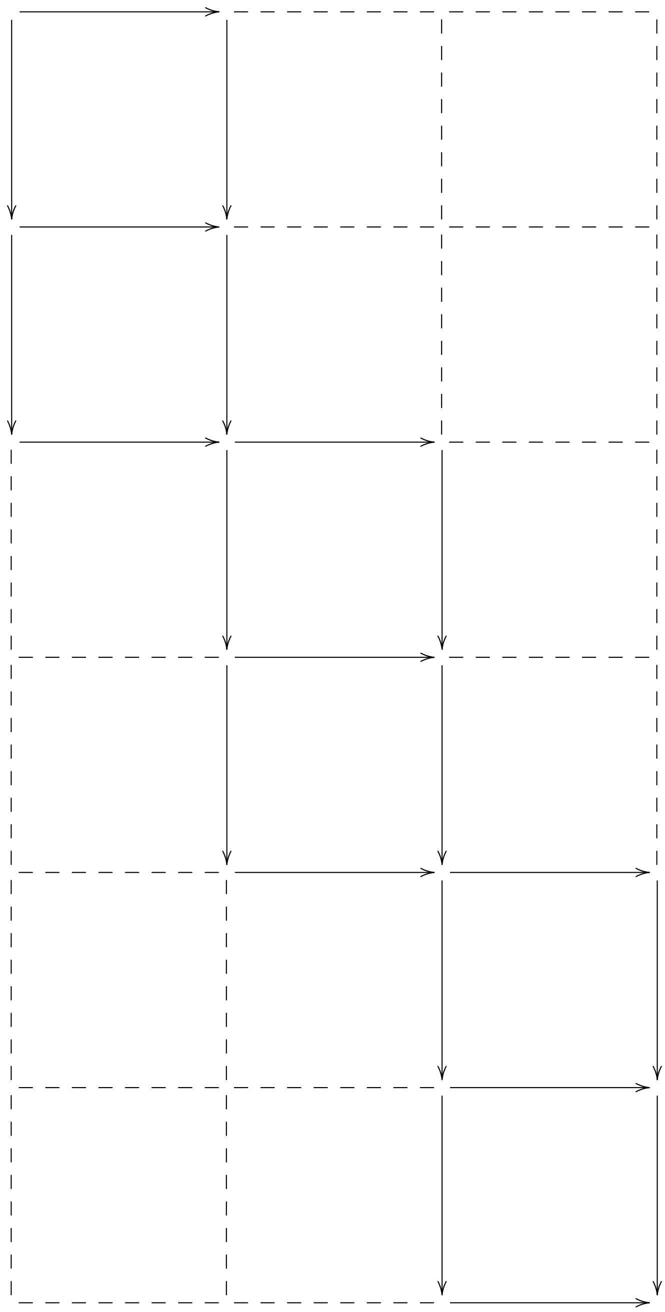


E.g., above is a finite piece of a point in W, with edge-name labels suppressed. For $\mathbf{v} \in \mathbb{Z}^2$, let $\alpha_{\mathbf{v}}$ denote the shift map on W in direction $\mathbf{v}$.

The bijections cited two slides back show the dashed-line sides below are determined by the solid diagonal squares. Thus $\alpha_{(1,-1)}$ is expansive and conjugate to the SFT σ_{AB}.



Likewise the solid squares below determine the rest, and $\alpha_{(1,-2)} \sim \sigma_{AB^2}$.



Given the commuting matrices A, B we showed how to embed $\sigma_{A^i B^j}$ into a commuting family of maps when $(i, j) = (1, 1)$ or $(i, j) = (1, 2)$. The argument is the same for $i > 0, j > 0$. The proof also works for onesided SFTs, for which Nasu has a converse: commuting onesided SFTs can be presented by commuting $\mathbb{Z}_+$ matrices.

Now we turn to algebraic invariants which can be realized by such commuting A, B, modulo passing to higher powers.

For a $k \times k$ $\mathbb{Z}_+$-matrix A, set $G_A = \varinjlim_{A} \mathbb{Z}^k$.

Regard G_A as an ordered group, with the natural order: G_A is the *dimension group* of A.

Proposition. Suppose σ_A is a mixing SFT and $\phi : G_A \to G_A$ is an isomorphism commuting with $\hat{A}$ all of whose eigenvalues are algebraic integers. There is a $\mathbb{Z}$ matrix B presenting the action of ϕ such that $BA = AB$. Suppose the spectral radius λ_B is a simple root of χ_B; $\lambda_B > 1$; and λ_B is the number by which B multiplies the Perron eigenvector of A. Then for all large i, B^i is positive, commutes with A.

The proof is routine dimensiongroupology and generalizes to finitely many commuting ϕ_j. This gives many families of commuting SFTs.

When commuting matrices produce commuting SFTs, their dimension groups are the same; so, modulo determination of lower powers which commute, we won't get further commuting SFTs *directly* from commuting matrices.

SFTs σ_A and σ_B can commute without being algebraically related in any way I see:

EXAMPLE (Nasu 95): $\sigma_A T = T\sigma_A$, $T \sim \sigma_B$,

- $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

- $\chi_B(x) = (x+1)^2(x^3 - 2x^2 + x + 1)$.

(σ_A and T do not even have the same measure of maximal entropy.)

Nasu gave a complicated algorithm which, given an automorphism U of an irreducible SFT, will find a matrix B such that $\sigma_B \sim U$, *IF* U is SFT. The example above came from applying the algorithm to a particular automorphism.

It would be interesting to see any systematic construction of commuting SFTs which need not be algebraically related.

Question (Nasu 1989). Must an expansive automorphism of an irreducible SFT be itself SFT?

EXAMPLE (D. Fiebig, 1996) A reducible SFT S with an expansive automorphism U which is not SFT.

Here, S consists of two fixed points p, q and two connecting orbits from p to q. Concretely, the fixed points and connecting orbits are

$$\begin{aligned} p &= \dots 000 \dots \quad , \qquad && \dots 0002111 \dots \quad , \\ q &= \dots 111 \dots \quad , \qquad && \dots 0003111 \dots \quad . \end{aligned}$$

$U = S$, except that $U = S^{-1}$ on one of the connecting orbits. U is expansive and totally chain transitive but not SFT. D.F. (easily) also elaborated this example to positive entropy.

We have a result which at least, after all this time, addresses a meaningful case of the question.

THEOREM (B. 2004) A strictly sofic AFT (almost finite type) shift S cannot commute with a mixing SFT T.

Above, "mixing" can be replaced by "chain recurrent". A sofic shift is AFT if it a factor of an irreducible SFT by a biclosing map, that is, a map which is one to one outside a proper closed set. (Krieger showed this map is canonical). The AFT sofic shifts enjoy various properties and seem to be the one big, natural class of nice sofic shifts.

Periodic points of onto cellular automata.

Let f denote a surjective endomorphism of a full shift $\sigma_{[N]}$, i.e., an onto one-dimensional cellular automata.

Question.
Are the periodic points of f dense?
The answer is yes if f is right or left closing (B-Kitchens) or if f has a point of equicontinuity (Blanchard-Tisseur). Otherwise nothing is known.

The sequel follows experimental mathematics with Bryant Lee (paper on my web page, along with the computer program for exploring), looking at periodic and preperiodic data for the action of f on points of given σ_N period.

(Martin, Odlyzko and Wolfram [1984] explained the pattern of jointly periodic points when f is a group endomorphism.)

Definition.
$\nu_k(f, \sigma_N) = |\{x \in \mathsf{Fix}(\sigma_N)^k : x \text{ is } f-\text{periodic}\}|$, and
$\nu(f, \sigma_N) = \overline{\lim}_k \, \nu_k(f, \sigma_N)^{1/k}$.

Above, $\log N$ is the growth rate of the periodic points of the full shift (on N symbols), and $\log \nu$ (where $\nu = \nu(f, \sigma_N)$) is the growth rate of the jointly periodic points. Trivially $\nu \leq N$. Our experimental evidence (suggestive but not compelling) leads to ...

Question.
Is $\nu(f, \sigma_N) > 1$ for every onto c.a. f?

Question.
Is $\nu(f, \sigma_N) \geq \sqrt{N}$ for every onto c.a. f?

For all large primes p, an onto c.a. f maps the set of points of period p into itself. So, the last question reflects a random maps heuristic: if a pattern doesn't force more periodicity, then we see i.o. at least about the periodicity we'd expect of a random map. An answer yes is consistent with our data, which are suggestive but (with the bound 26) certainly not compelling.

Conjecture.
There exist f such that $\nu(f, \sigma_N) < N$.

From our data, it seems obvious that the conjectured inequality is typical. (Equality holds in the algebraic case and some other classes.) But we can't give a proof for any example.