# The wild world of multidimensional shifts of finite type

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Copenhagen Lec. 8 June 2008

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## Outline of the talk

**I. Background on shifts**

**II. Overview: $\mathbb{Z}^d$ SFTs and sofic shifts, $d = 1$ vs. $d \geq 2$**

**III. Entropy**

**IV. Subsystems**

**V. Quotients (factors)**

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## I. Background on shifts

### $\mathbb{Z}^d$ Subshifts

Notations: given positive integers $d, N$:

- $\mathcal{A} := \{0, 1, \ldots, N-1\}$, a finite alphabet
- An element of $\mathcal{A}^{\mathbb{Z}^d}$ is pictured as a way of filling the $\mathbb{Z}^d$ lattice with symbols
- For $d = 1$, $x$ in $\mathcal{A}^{\mathbb{Z}^d}$ is $x = \ldots x_{-1}x_0x_1\ldots$, with all $x_i$ in $\mathcal{A}$ and $x_i = x(i)$
- For $d = 2$, $x$ in $\mathcal{A}^{\mathbb{Z}^d}$ is a planar array of symbols, etc.

- $\mathcal{A}^{\mathbb{Z}^d}$ is a metric space, $\text{dist}(x,y) = \frac{1}{k+1}$, where $k = \min\{||v|| : x(v) \neq y(v)\}$

- There is a shift action $\sigma$ of $\mathbb{Z}^d$ on $\mathcal{A}^{\mathbb{Z}^d}$ by homeomorphisms. For $v \in Z^d$, the shift homeomorphism $\sigma^v$ is defined by
$(\sigma^v x)(u) = x(u+v)$, $u \in \mathbb{Z}^d$

- For example, for $d = 1$ and $y = \sigma^1 x$:
if $\dots x_{-1}x_0x_1 \dots = \dots 003 \dots$, then
$\dots y_{-2}y_{-1}y_0 \dots = \dots 003 \dots$.

- If $Y$ is a closed $\sigma$-invariant subset of $\mathcal{A}^{\mathbb{Z}^d}$, then $(Y, \sigma|_Y)$ is a *subshift* (or just *shift*). I may use the same letter for the shift or its domain.

- For any subshift $Y$, there exists a set $\mathcal{L}$ of finite configurations such that $Y = X_{\mathcal{L}}$: the set of all $x$ in $\mathcal{A}^{\mathbb{Z}^d}$ satisfying:
  for every finite subset $C$ of $\mathbb{Z}^d$ and $u \in \mathbb{Z}^d$,

  $$x|_{(u+C)} \notin \mathcal{L}.$$

  If it is possible to choose $\mathcal{L}$ to be some finite set $\mathcal{F}$ then $Y$ is a *shift of finite type* (SFT).

- Example: $d = 1$, $\mathcal{A} = \{0, 1\}$, $\mathcal{F} = \{00\}$.

## Block codes

Let $Y$ be a subshift on alphabet $\mathcal{A}$.
Let $Y'$ be a subshift on alphabet $\mathcal{A}'$.
Let $U_n = \{v \in \mathbb{Z}^d : ||v|| \leq n\}$.

- A block code is a function $\phi : Y \to Y'$ for which there exists $\Phi : \mathcal{A}^{U_n} \to \mathcal{A}'$ with $(\phi x)(v) = \Phi(x|_{v+U_n})$ for all $v \in \mathbb{Z}^d$.

- Example: $Y = Y' = \{0, 1\}^Z$ and $(\phi x)_i = x_i + x_{i+1}$ (mod 2)

- $\phi$ is a 1-block code if $(\phi x)(v) = \Phi(x|_v)$

- The continuous shift-commuting maps between subshifts are exactly the block codes.

## Sofic Shifts

- A sofic shift is a subshift which is a quotient of an SFT.

- For example: SFT $= X_{\mathcal{F}}$, on alphabet $\{a, b, c\}$ with $\mathcal{F} = \{ba, bb, cc\}$. (I.e.a point in $X$ is an arbitrary concatatenation of the words $a$ and $bc$.) Define $Y$ as the image under the one-block code $a \mapsto 1, b \mapsto b, c \mapsto b$. If $ab^n a$ occurs in $Y$, then $n$ must be even. $Y = X_{\mathcal{L}}$ where

$$\mathcal{L} = \{ab^{2n+1}a : n \in \mathbb{N}\} \ .$$

  $Y$ (the "even system") is sofic and not SFT.

## II. Overview: $\mathbb{Z}^d$ SFTs and sofic shifts, $d = 1$ vs. $d \geq 2$

For $\mathbb{Z}$ SFTs:

- There are computable, fine invariants.

- Invariants and structure have an algebraic quality.
  (Any SFT is topologically isomorphic to $X_B$ for some matrix $B$. Various algebraic invariants of $B$ give invariants of topological isomorphism for $X_B$.)

- Qualitatively, mixing $\mathbb{Z}$ SFTs have a homogeneous structure, rich in subsystems and quotients.

- Generally problems of SFTs reduce easily to problems of mixing SFTs.

For $\mathbb{Z}^d$ SFTs, $d \geq 2$:

- $\mathbb{Z}^d$ SFTs, including mixing $\mathbb{Z}^d$ SFTs, are qualitatively heterogeneous.

- $\mathbb{Z}^d$ SFT problems do not reduce to problems of mixing $\mathbb{Z}^d$ SFTs.

- Essentially nothing can be computed for a general arbitrary $\mathbb{Z}^d$ SFT if $d \geq 2$.
  (Berger, Robinson, Kari, ... )

- The landscape of general possibilities is recursion-theoretic.
  Hochman-Meyerovitch and Hochman give constructive results and techniques of generality unprecedented in this area.

## III. Entropy

Let us pause to consider two examples of the power of the new constructive techniques

Given a $\mathbb{Z}^d$ subshift $(X, \sigma|X)$:
let $B_n = \{v \in Z^d : 0 \leq v_i \leq n-1, 1 \leq i \leq d\}$.
The *entropy* of this shift (as a $\mathbb{Z}^d$ action) is

$$h(\sigma|X) = \lim_n \frac{1}{n^d} \log \text{card}\{x|_{B_n} : x \in X\} \ .$$

This is the main numerical measure of the complexity of a subshift.

For $X = \{0, \ldots, N-1\}^{Z^d}$, the entropy is simply $\log N$.

It is easy to compute the entropy of a $\mathbb{Z}$ SFT. For $d \geq 2$: there are not many exact computations of entropy for systems "in nature"; and no Turing machine can compute $h(\sigma_{\mathcal{F}})$ for arbitrary finite $\mathcal{F}$.

DEFN: A recursive sequence is the output of a Turing machine – intuitively, a sequence produced by any kind of algorithm you could imagine a computer implementing.

The classes of possible entropies:

- For $\mathbb{Z}$ SFTs: the logs of a well understood class of algebraic integers. (Lind)

- For $\mathbb{Z}^d$ SFTs, $d \geq 2$: the numbers $\alpha = \lim_n \alpha_n$, where $(\alpha_n)_{n\in\mathbb{N}}$ is a decreasing recursive sequence of rational numbers. (Hochman-Meyerovitch)

- For $d$-dimensional cellular automata maps: the numbers $\alpha = \liminf_n \alpha_n$, where $(\alpha_n)_{n\in\mathbb{N}}$ is a recursive sequence of rational numbers. (Hochman)

## IV. Subsystems

## Mixing

- A $\mathbb{Z}^d$ subshift $(X, \sigma)$ is *mixing* if any two legal finite configurations can occur at all but finitely many separations.

- That is, for all $x, y$ in $X$, for all $n$ we have for all but finitely many $u \in Z^d$ there exists $w \in X$ such that
$w|_{B_n} = x|_{B_n}$ and $w|_{u+B_n} = y|_{B_n}$.

- Generally problems of $\mathbb{Z}$ SFTs reduce easily to problems of mixing $\mathbb{Z}$ SFTs.

- For $d \geq 2$, the mixing condition splinters into a host of different conditions.

## Subsystems of $\mathbb{Z}$ SFTs

Nontrivial mixing $\mathbb{Z}$ SFTs have a homogeneous structure, rich in subsystems and quotients:

- Krieger Embedding Theorem $\implies$ if $(X, \sigma_X)$ is a mixing $\mathbb{Z}$ SFT and $(Y, \sigma_Y)$ is a $\mathbb{Z}$ subshift with no periodic points and $h(\sigma_Y) < h(\sigma_X)$, then $(Y, \sigma_Y)$ is topologically conjugate to a subshift contained in $(X, \sigma_X)$.

- Jewett-Krieger Theorem: every finite entropy measurable $\mathbb{Z}$-system is realized by a uniquely ergodic $\mathbb{Z}$-subshift

- Given a proper subsystem of a mixing $\mathbb{Z}$ SFT, all the embeddings above can be chosen with images missing that subsystem.

"There's always room
at the Krieger Hotel."

## Subsystems of $\mathbb{Z}$ Sofic Shifts

- A mixing $\mathbb{Z}$ sofic shift $X$ contains an increasing union of mixing $\mathbb{Z}$ SFTs $X_n$ with $\lim_n h(X_n) = h(X)$.

- So the subsystem results of the last page for mixing $\mathbb{Z}$ SFTs also hold for mixing $\mathbb{Z}$ sofic shifts.

So: a nontrivial mixing $\mathbb{Z}$ SFT or sofic shift contains a vast family of pairwise disjoint minimal subsystems with entropy close to $h(\sigma_X)$.

## Subsystems of $\mathbb{Z}^d$ Shifts, $d \geq 2$

- For a large subclass of the mixing $\mathbb{Z}^2$ SFTs $S$: any smaller entropy $\mathbb{Z}^2$ subshift without periodic points can be embedded into $S$.

- Again for a large subclass of the mixing $\mathbb{Z}^d$ SFTs $S$, Robinson and Sahin have proved existence of subshifts carrying completely positive entropy or Bernoulli measures.

- There is also an analogue of the Jewett Krieger Theorem for $\mathbb{Z}^d$ shifts, $d \geq 2$.

Moreover there is a result on richness of subsystems which applies to ALL $\mathbb{Z}^d$ SFT/sofic shifts:

THEOREM (Desai) For $d \geq 2$ and a given $\mathbb{Z}^d$ shift $S$:

- If $S$ is SFT, then $S$ contains SFTs with entropies dense in [0,$h(S)$].
- If $S$ is sofic, then $S$ contains SFTs with entropies dense in [0,$h(S)$].
- In either case every number in $[0, h(S)]$ is the entropy of a subsystem.

So there are a lot of subsystems.

But they cannot in general be separated:

THEOREM (B-Pavlov-Schraudner) Given $d \geq 2$ and $M \in \mathbb{R}$, there is a mixing $\mathbb{Z}^d$ sofic shift $S$ with the following properties.

- $h(S) > M$ and $S$ is mixing
- $S$ contains a unique minimal subsystem, which is a fixed point for the shift action.

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THEOREM (B-Pavlov-Schraudner) Given $d \geq 2$ and $M \in \mathbb{R}$, there is a mixing $\mathbb{Z}^d$ SFT $S$ with the following properties.

- $h(S) > M$ and $S$ is mixing
- $S$ contains a zero entropy sofic shift which intersects every subshift in $S$.

## V. Quotient (Factor) Maps

$\mathbb{Z}$ SFTs and $\mathbb{Z}$ sofic shifts also enjoy a rich supply of quotient maps.

THEOREM

1. If $X$ is a $\mathbb{Z}$ sofic shift, and $Y$ is a mixing $\mathbb{Z}$ SFT with $h(X) > h(Y)$, then $Y$ is a quotient of $X$.

2. If $Y$ is a $\mathbb{Z}$ SFT and $h(Y) \geq \log N$, then $Y$ has as a factor the full shift on $N$ symbols.

Johnson and Madden asked whether (2) generalizes to $\mathbb{Z}^d$ SFTs. Their work, as extended by Desai, proved the conclusion of (2) for a $\mathbb{Z}^d$ SFT $Y$ with the "corner gluing" mixing condition when $h(Y) > \log N$.

If $S$ is a quotient of $T$, then disjoint subsystems of $T$ pull back to disjoint subsystems of $S$. So we see that if $S$ has poorly separated subsystems, then it cannot factor onto a $T$ with well separated subsystems, such as a full shift. The BPS examples for subsystems thus not surprisingly have some pathological properties with regard to their possible factors.

THEOREM (BPS) Given $d \geq 2$ and $M \in \mathbb{R}$, there is a $\mathbb{Z}^d$ sofic shift $S$ with the following properties.

- $h(S) > M$
- $S$ contains a unique minimal subsystem, which is a fixed point for the shift action.
- And with regard to quotients:

- Any subshift quotient $Y$ of $S$ satisfies the following:
  - If $Y$ is SFT, then $Y$ is a fixed point.
  - $Y$ is not block-gluing (i.e. not mixing on block shapes with uniform separation).
  - $Y$ supports no $\sigma$-invariant measure which is of completely positive entropy.
  - $Y$ does have a subshift factor of topologically completely positive entropy.

- $S$ can be chosen mixing, except that in this case we can't control top. c.p.e. of non-trivial factors for $d \geq 3$.

There are similar slightly weaker results for $S$ SFT. The key to the proof for $d \geq 3$ is a theorem of Hochman.

# Effective Systems

Hochman has defined an Effective Symbolic System (ESS) to be a $\mathbb{Z}^d$ subshift $\sigma_{\mathcal{L}}$ such that the defining set $\mathcal{L}$ of forbidden finite configurations is the output of a Turing machine.

THEOREM (Hochman) For each $d \geq 3$, up to topological conjugacy the following classes of $\mathbb{Z}$ subshift are the same:

- $\mathbb{Z}$ subshifts isomorphic to $\sigma^{e_1}$ for some $\mathbb{Z}^d$ sofic shift $\sigma$

- The class of ESS's.

(With SFT in place of sofic, the class of ESS's becomes just slightly more narrow.)

Heuristically: not only are we faced with many bad examples: in fact every (bad) thing we could possibly imagine happening, does happen.

## VI. A Hint of Proof

We can indicate how Hochman's subdynamics theorem lets one easily construct a nonmixing sofic example $S$ for $d = 3$.

- Construct an effective $\mathbb{Z}$ subshift $W$ such that arbitrarily large blocks of 0's occur syndetically in all points, and every $W$ word occurs with positive frequency in every point.

- By Hochman: for $i = 1, 2, 3$, pick a $\mathbb{Z}^3$ sofic shift $T_i$ for which $\sigma^{e_i}$ is a copy of $W$.

- Then in each coordinate of $T_1 \times T_2 \times T_3$, for each $M \in \mathbb{N}$, strings of $M$ consecutive zeros occur syndetically.

- Let $W$ be the quotient of $T_1 \times T_2 \times T_3$ by the map which replaces a symbol $(a, b, c)$ with 0 if any of $a, b, c$ is zero, and otherwise replaces $(a, b, c)$ with 1.

- For every $M$ and $w \in W$, every finite configuration in $W$ occurs inside some large block configuration on which the boundary is covered by $M$-thick slabs of zeros.

- Define $S$ by freely allowing the replacement of 1 in a configuration with symbols from $\{1, 2, ..., k\}$ for $k$ large.

- Easily:

  - $h(\sigma_X) > M$ (for large enough $k$)

  - $\sigma_X$ has a unique minimal subsystem, $0^{Z^3}$.

  - The only SFT which is in $X$ or in a quotient of $X$ is a fixed point.

The nonexistence of factors with meas.th. c.p.e. measures uses the disjointness result of Glasner, Thouvenot and Weiss.

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## VI. The Playground

We have entered a certain period in this topic where one can imagine some wildly general recursion theoretic obstruction to $\mathbb{Z}^d$ SFT/sofic behavior, and then try to show there is no other obstruction.

The Hochman-Meyerovitch/Hochman techniques are very concrete — given the oracle Turing machine.

However: while the very general landscape of $\mathbb{Z}^d$ SFTs and sofic shifts seems quite recursion theoretic, this will not hold for special important classes, at least in the same generality.