Multidimensional shifts of finite type and sofic shifts

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Considering the emphasis of last week, I'll mostly neglect the dramatic recursion theoretic constructive advances of Hochman, Meyerovitch and others.

One theme in that work is the consideration of what subshifts can occur as dynamics of a single directional shift in a Z^d SFT or sofic shift, for $d \geq 3$ (Hochman) or $d \geq 2$ (Durand, Romaschenko and Shen; Aubrn and Sablik).

In this first talk we'll consider aspects of how the one dimensional subactions vary within a Z^2 SFT, in the case of a Z^2 SFT generated by an automorphism of a mixing Z SFT.

There is an elaboration of this introduction in my "Open problems in symbolic dynamics", with references.

Outline

1. Some Definitions

2. Expansiveness and the regularity of subdynamics

3. Constructing commuting Z-SFTs from commuting matrices

4. Commuting Z-SFTs without commuting matrices

5. Nasu's textile systems: a graphical calculus for Wang tilings

I. Some Definitions

- An *automorphism* of a continuous map f is a homeomorphism U commuting with f $(Uf = fU)$.

- Continuous maps f, F are *topologically conjugate* $(f \sim F)$ if there exists a homeomorphism h such that $hF = fh$.

- Continuous maps f, g *can commute* if $\exists F \sim f, G \sim g$ with $FG = GF$.

- σ_A is the twosided edge shift of finite type (SFT) defined by the square $\mathbb{Z}_+$ matrix A.

- "S is SFT" means $S \sim \sigma_A$, for some A.

If ϕ is a continuous map commuting with the shift map σ on some subshift X, then to x in X we can associate the Z^2 array $\widehat{x}$ which assigns to (i, j) in Z^2 the symbol $(\phi^j(x))_i$.

The set X_ϕ of all $\widehat{x}$ is a Z^2 subshift. Row $j+1$ of the array is the image under ϕ of row j. The map $x \mapsto \widehat{x}$ defines a topological conjugacy of σ to $\sigma_{1,0}$ and ϕ to $\sigma_{0,1}$.

If the Z shift X is a Z SFT, then X_ϕ is a Z^2 SFT.

II. Expansiveness and the regularity of subdynamics

A Z^d action by homeomorphisms α_v, $v \in Z^d$, on a compact metric space X is *expansive* if there exists $\epsilon > 0$ such that for every pair of distinct points x, y in X there exists $v \in Z^d$ such that dist$(\alpha_v x, \alpha_v y) > \epsilon$.

For example, every Z^d subshift is expansive. An expansive action of Z^d on a zero dimensional compact metrizable space is topologically conjugate to a subshift.

Expansiveness arises in various ways in dynamics. Here we will see it as a regularity condition. We'll consider Z^2 subshifts. This is a case of a more general theory for Z^d actions on compact metric spaces (B-Lind Expansive Subdynamics).

Let X be a Z^2 subshift. A configuration x in X is a function which assigns to each u in Z^2 an element $x[u]$ from a given finite alphabet. An element v in Z^d acts by the directional shift σ_v, where $(\sigma_v x)[u] = x[u+v]$.

Say ℓ is an *expansive line*, or *expansive direction*, for X if there exists $M > 0$ such that distinct points in X must have distinct restrictions to $U(\ell, M) \cap Z^2$.

In words: a configuration in X is determined by its restriction to the lattice points in the strip $U(\ell, M)$.

For example, if $X = X_\phi$ for an automorphism ϕ of a Z-subshift, then the horizontal direction is expansive.

More generally, if ℓ has rational slope, then $\ell \cap Z^2 \neq \{(0,0)\}$. In this case, ℓ is an expansive line iff σ_v is an expansive homeomorphism for any/every nonzero v in $\ell \cap Z^2$.

The space of directions is topologized as PS^1: two lines through the origin are close if their two-point intersections with the unit circle are close.

FACTS.

- For a Z^2 subshift X, its set E_X of expansive directions is open.

- If X is a finite set (trivial case), then every direction is expansive.

- If X is infinite, then the set of nonexpansive directions can be any nonempty closed subset of directions (by [B-Lind 1997] + [Hochman to appear]).

Why care?

For an infinite Z^2 subshift X, set $E_1(X) = \{v \in R^2 : Rv$ is an expansive line for $X\}$, the set of "expansive vectors" of X. Suppose $E_1(X)$ is nonempty.

Let $\mathcal{C}$ be a connected component of $E_1(X)$ (an open cone or an open half-space).

The meta principle:
As v varies in $E_1(X) \cap Z^2$, qualitative dynamical properties of σ_v tend to be constant, and qualitative properties tend to vary regularly.

Heuristically, the dynamics of $\{\sigma_v\}$ comes in oceans of regularity (the components $\mathcal{C}$) separated by ... something.

Example. Suppose for some $v \in \mathcal{C} \cap Z^2$ that σ_v is MSFT. Then

- For every $u \in \mathcal{C} \cap Z^2$, σ_u is MSFT

- There are numbers α, β such that for $u = (i,j) \in \mathcal{C} \cap Z^2$, $h(\sigma_u) = \alpha i + \beta j$

- There are integral matrices A, B such that for $u = (i,j) \in \mathcal{C} \cap Z^2$, the shift equivalence class of the mixing SFT $\sigma_{(i,j)}$ is given by $A^i B^j$. (This class determines a mixing SFT up to topological conjugacy of all large powers.)

- All the σ_u, $u \in \mathcal{C} \cap Z^2$, have the same measure of maximal entropy.

There is a dynamical relation which underlies this regularity. Suppose X is a Z^2 subshift and $\mathcal{C}$ is a connected component of $E_1(X)$. Suppose u, v are in $\mathcal{C} \cap Z^2$, and x, y are in X. Then

$$\lim_{n \to +\infty} \text{dist}(\sigma_u x, \sigma_u y) = 0$$

if and only if

$$\lim_{n \to +\infty} \text{dist}(\sigma_v x, \sigma_v y) = 0 \ .$$

I.e., the stable relation is the same for all σ_u with $u \in \mathcal{C}$. (So is the unstable relation.)

Example. Suppose ϕ is an automorphism of a Z-subshift X. Choose k such that for all x in X, $x[-k,k]$ determines $(\phi x)[0]$ and $(\phi^{-1}x)[0]$. In R^2, let ℓ be the line of slope $1/(k+1)$ through the origin, and let V be the strip of points whose horizontal distance to ℓ is at most k.

On each integer row of a point in X_ϕ, V sees a configuration which codes a symbol in coordinates directly above and below. Taking the union of these, we see that the configuration in V codes symbols one unit to the right and one unit to the left. Continuing this forever, we see the configuration in V determines the entire configuration.

Similarly, every line with slope between $-1/(k+1)$ and $1/(k+1)$ is also expansive.

Note: in the example, $x[-k,\infty)$ determines the configuration on the cone bounded by the half lies with slopes $-1/(k+1)$ and $1/(k+1)$]. Likewise, any half line in this cone lies in a strip which together with a finite configuration around the origin. In particular, the stable relation is the same for all the σ_v with v a nonzero lattice point in this cone.

Problems

We still know very little about the possible sets of expansive components in the key case:

Problem [B-Lind 1997]. Suppose ϕ is an automorphism of a mixing SFT. Can $E_1(X_\phi)$ have infinitely many components? Can the boundary of a component be a ray of irrational slope?

With the idea of seeing islands of regularity in the subdynamics, we naturally focus on what islands of regularity can coexist. The most fundamental question is over 20 years old.

Problem [Nasu 1989] If ϕ is an expansive automorphism of an irreducible SFT, must ϕ be SFT?

Nasu's "textile systems" theory gives an algorithm which, if an expansive automorphism ϕ is SFT, will eventually produce a matrix A such that $\phi \sim \sigma_A$.

We can at least say something:

THEOREM (B. 2004) A strictly sofic AFT (almost finite type) shift S cannot commute with a mixing SFT T.

A sofic shift is AFT if it a factor of an irreducible SFT by a biclosing map. (Krieger showed this map is canonical). The AFT sofic shifts enjoy various properties and seem to be the one big, natural class of nice sofic shifts.

We consider next when two SFTs can commute.

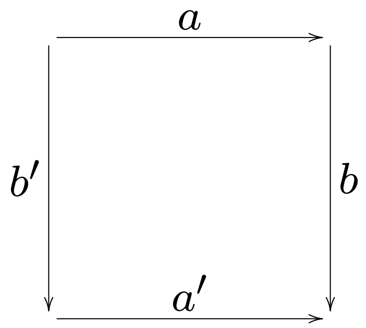
III. Constructing Commuting Z-SFTs from commuting matrices

Note: $AB = BA$ does not guarantee that σ_A, σ_B can commute. (E.g. for $[A] = 2, B = [3]$, an examination of low order periodic points shows σ_A and σ_B cannot commute.) However:

Proposition. Suppose A, B are commuting $\mathbb{Z}_+$ matrices. Then there are homeomorphisms S, T such that $ST = TS$ and $S^i T^j \sim \sigma_{A^i B^j}$ for $i, j > 0$.

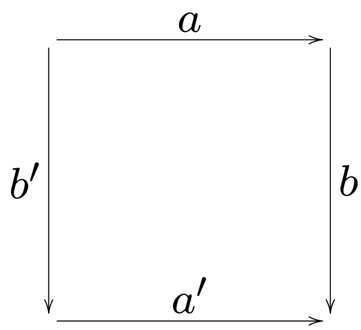
The proof is a simple and elegant construction of Nasu (constructing an "LR textile system"). It is next.

Suppose A and B are $n \times n$ matrices over $\mathbb{Z}_+$, with $AB = BA$. View A and B as adjacency matrices for two directed graphs, with disjoint edge sets and a common vertex set $\{1, 2, \ldots n\}$. Say e.g. an ab path from i to j is an A edge from i to some k followed by a B edge from that k to j. "$AB = BA$" means that for each pair i, j the number of ab paths from i to j equals the number of ba paths from i to j. Thus we can build a set $\mathcal{W}$ of Wang tiles

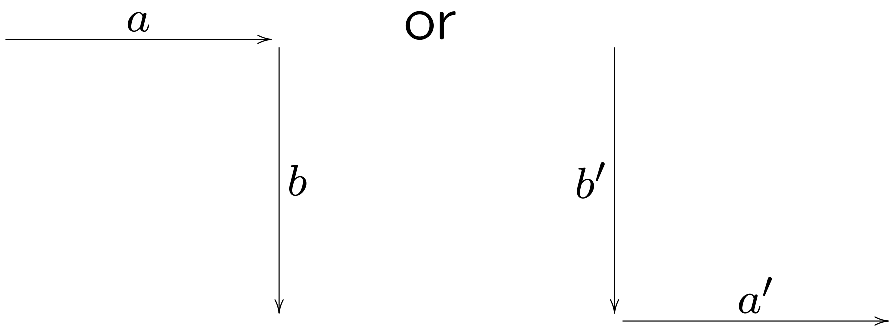


such that each ab path is the top/right of exactly one tile and each ba path is the left/bottom of exactly one tile. (In the tile pictured, a, a' are A-edges and b, b' are B-edges.)

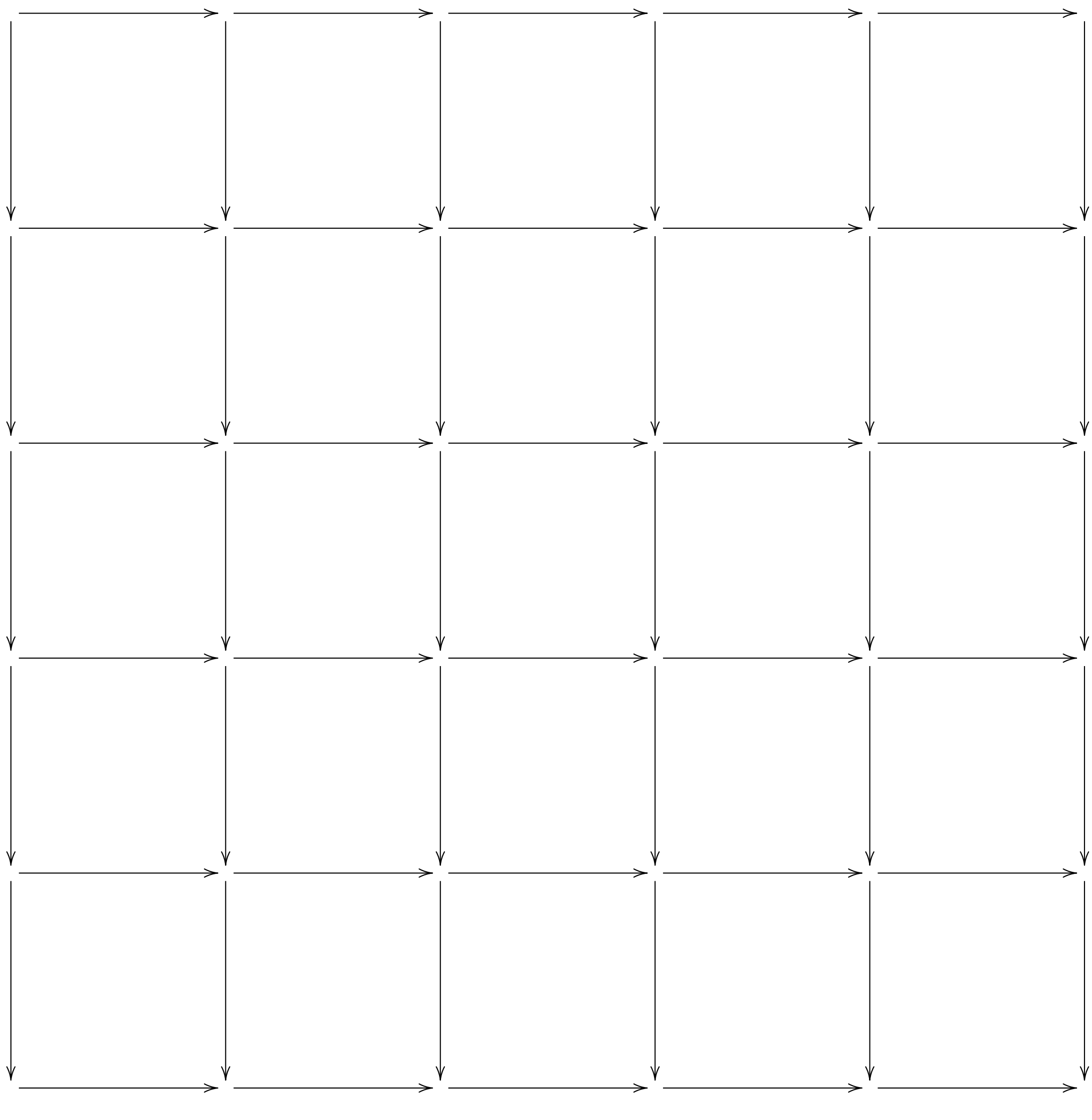
Thus each Wang tile



is determined by either of the paths



Now let the tile sides be unit length and let $X_{A,B}$ be the space of infinite Wang tilings of the plane with $\mathcal{W}$, with tile corners on $\mathbb{Z}^2$. The space depends on A, B and the bijection of ab paths and ba paths used to define the tiles.

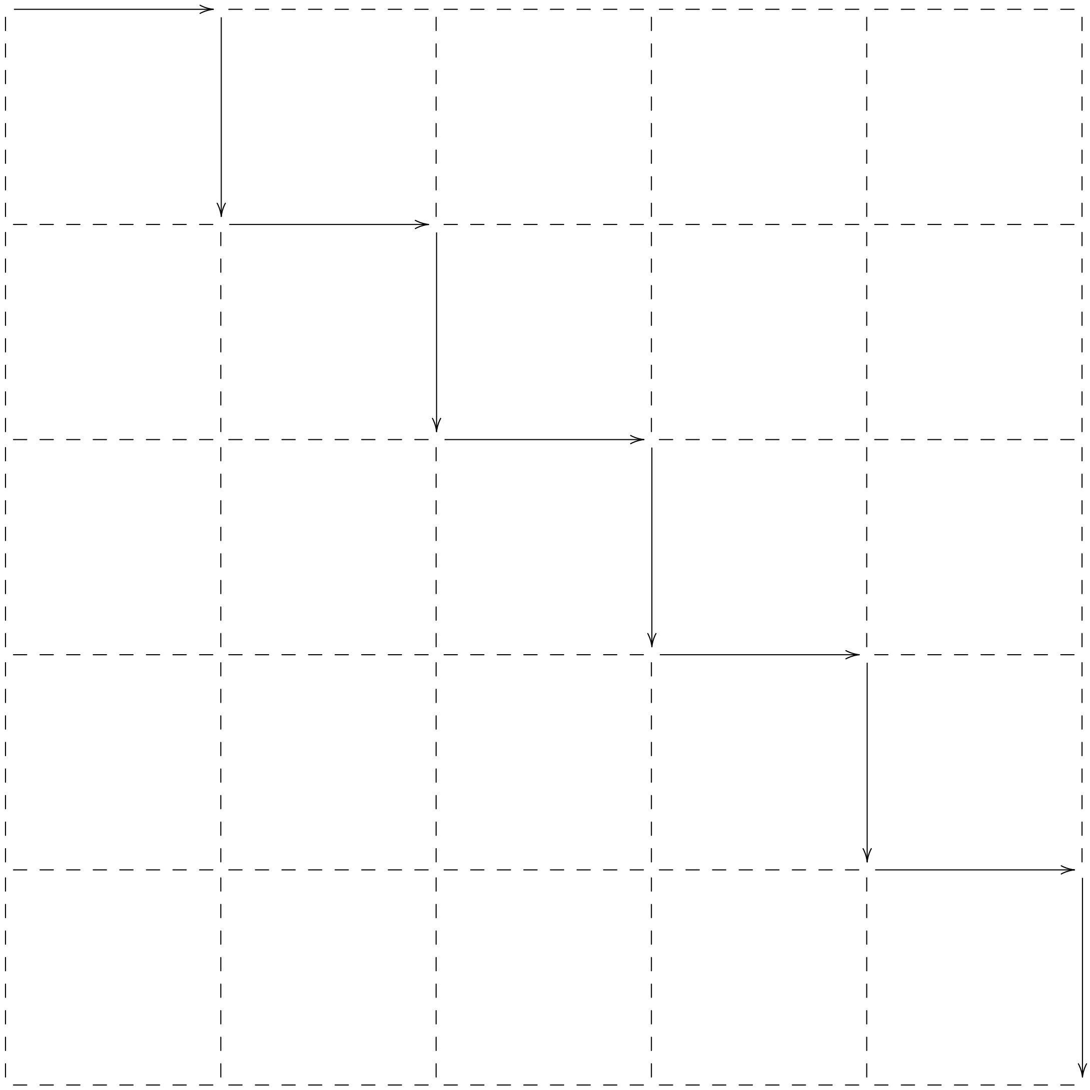


E.g., above is a finite piece of a point in $X_{A,B}$, with edge-name labels suppressed.

For $\mathbf{v} \in \mathbb{Z}^2$, let $\sigma_{\mathbf{v}}$ denote the shift map on $X_{A,B}$ in direction $\mathbf{v}$. I'll draw with coordinates increasing downward and to the right.

We will see that for $i > 0$ and $j > 0$, the map $\sigma_{(i,j)}$ on $X_{A,B}$ is topologically conjugate to the edge SFT $\sigma_{A^iB^j}$.

To get the basic idea, we consider $(i,j) = (1,1)$. Edges in the graph with adjacency matrix AB can be identified with ab paths. To each point x in $X_{A,B}$, associate the bisequence y such that y_n is the ab path in x from (n,n) to $(n+1,n+1)$.



In the tiling x above, the dark arrows grouped in pairs define the associated bisequence y.

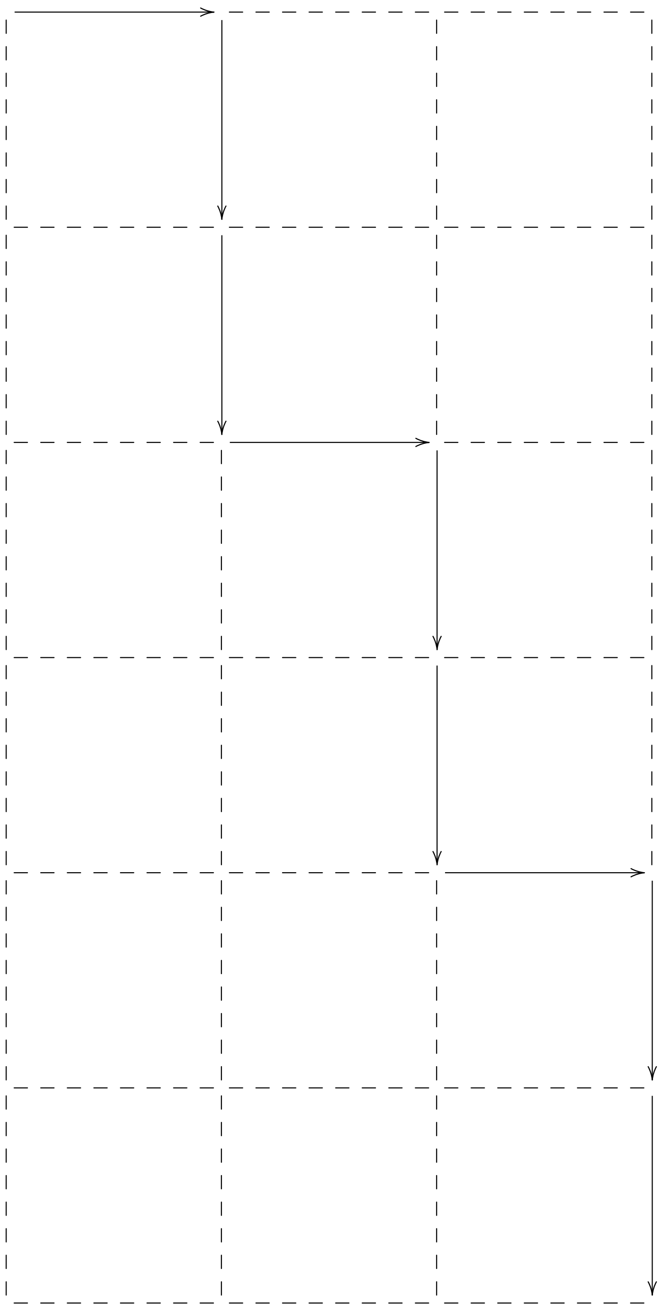
The map $\pi : x \mapsto y$ is continuous, and intertwines the homeomorphisms $\sigma_{(1,1)}$ and σ_{AB}.

From the bijections cited earlier, one sees the chosen ab bisequence y determines all of x, so π is injective.

Again from those bijections, one sees that any legal biinfinite path of ab edges occurs as y for some tiling x. So, π is surjective.

Therefore π is a topological conjugacy.

Similarly we can associate to a point in $X_{A,B}$ a bisequence of abb paths, and show that $\alpha_{(1,2)}$ is topologically conjugate to σ_{AB^2}. Etc.



Commuting one-sided SFTs arise from commuting matrices.

Suppose A, B are nonnegative integer matrices with $AB = BA$. Let $X^+_{A,B}$ by the space of tilings of $Z_+ \times Z_+$ which are restrictions of the tilings in $X_{A,B}$ constructed earlier (which depends on a chosen bijection of ab paths and ba paths). Nasu proved that if commuting maps S, T on a space Y are topologically conjugate to onesided SFTs, then there are commuting matrices A, B and a space of tilings $X^+_{A,B}$ and a homeomorphism $h : Y \to X^+_{A,B}$ which conjugates the action of S and T to the the horizontal and vertical shifts on $X^+ A, B$.

For this he constructs an associated "textile system" and shows it can be modified to be an "LR Textile System" (more later).

Mixing SFTs σ_A and σ_B (onesied or twosided) which can commute using Nasu's construction using $AB = BA$ have severely constrained joint dynamics. The dimension groups of the SFTs are isomorphic; their zeta functions are reciprocals of polynomials of equal degree; as realized in the construction, the commuting SFTs have the same measure of maximal entropy. (There is yet more structure for onesided SFTs [B-Fiebig[2]].) But ...

Commuting Z-SFTs without commuting matrices

The situation is quite different for twosided SFTs.

EXAMPLE (Nasu 95): $\sigma_A T = T\sigma_A$, $T \sim \sigma_B$,

- $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

- $\chi_B(x) = (x+1)^2(x^3 - 2x^2 + x + 1)$.

(σ_A and T cannot even have the same measure of maximal entropy.)

Nasu gave a complicated algorithm which, given an automorphism U of an irreducible SFT, will find a matrix B such that $\sigma_B \sim U$, *IF* U is SFT. The example above came from applying the algorithm to a particular automorphism.

It would be very interesting to see any systematic construction of commuting SFTs which need not be algebraically related.

CONJECTURE: Suppose S and T are mixing SFTs. Then for all large i, j, S^i and T^j can commute.

The passage to powers in the conjecture addresses low-order periodic point obstructions and follows a standard pattern in symbolic dynamics. See [B-Open problems].

Jointly invariant measures.

Here is one reason to be interested in the previous Conjecture.

There has been intense interest in certain commuting actions which have few nonatomic jointly invariant measures (e.g., the Furstenberg example $\times 2, \times 3$ mod 1 on $[0, 1)$).

We would like a larger supply of Z^2 actions for which we had some understanding of the jointly invariant measures. As is well know following Rudolph, the nonatomic jointly invariant Borel probabilities for $\times 2, \times 3$ are in a natural correspondence with the nonatomic jointly invariant measures of the 6-shift and a certain automorphism of it. So, it is natural to consider the symbolic dynamical possibilities.

For example, in Nasu's example, the distinct measures of maximal entropy of the two commuting SFTs are jointly invariant ergodic and positive entropy. In contrast, the Furstenberg example has just one, Lebesgue measure.

If two commuting MSFTs, with entropies log(α) and log(β) share the same measure of maximal entropy, then the number fields $Q(\alpha)$ and $Q(\beta)$ must be equal (a rarity).

V. Nasu's textile systems: a graphical calculus for Wang tilings

For the problems we have been considering so far, many of the sharpest results are due to Nasu and his theory of textile systems. The aim here is to give an initial idea of this theory, its consequences and its relation to the expansive subdynamics. (We do not attempt a reasonable outline of the theory.)

We will give Nasu's formal definition of a *textile system* later. A textile system will arise from a set $\mathcal{T}$ of Wang tiles of a certain form:

- there are directed graphs G_A, G_B on a common vertex set $\mathcal{V}$
- the top and bottom sides of a tile are right-pointing edges from G_A

- the left and right sides are down-pointing edges from G_B

- initial and terminal vertices of edges are (of course) required to be consistent at corners.

So, the color of a Wang tile side is an edge in a directed graph. Any Wang tile can be viewed in this way by taking $\mathcal{V}$ to be a singleton.

There is another directed graph here, $G(\mathcal{T})$. The edges of $G(\mathcal{T})$ are the tiles t in $\mathcal{T}$. The initial vertex of t is its left side and the terminal vertex is its right side.

A doubly infinite horizontal sequence z of legal (adjacent sides match) Wang tiles is now a point in an SFT, $X(\mathcal{T})$.

Let p be the map which sends a tile to its top edge. This induces a map, called ξ, which sends $z \in X(\mathcal{T})$ to a point $\xi(z)$ in the SFT X_A.

Likewise, let q be the map which sends a tile to its bottom edge, and determining $\eta(z)$ in the SFT X_A.

So: for a horizontal bisequence z of Wang tiles, $\xi(z)$ is the bisequence of its top edges and $\eta(z)$ is the bisequence of its bottom edges.

So what? Given an endomorphism ϕ of an SFT X_A, Nasu constructs tiles such that ξ is a topological conjugacy, and the map ϕ is ξ^{-1} followed by η. That is, a point x in X_A will be the top of a unique bisequence z of Wang tiles, and the bottom of z will be the image $\phi(x)$. The block code defining ϕ has been put into the Wang tiles. The subshift X_ϕ is given by the horizontal side sequences in the Wang tiling space.

Nasu's definition of a textile system.

A *textile system* is a pair of directed graphs Γ, G together with a pair p, q of graph homomorphisms from Γ to G.

Nasu's textile system gives the Wang tiles simply by setting the edges of G_B to be the vertices of Γ.

There is an obvious dual textile system obtained by interchanging the roles of top, bottom, horizontal with left, right, vertical. This is an important ingredient in Nasu's work.

The textile systems are as general as Wang tiles. To study problems of endomorphisms and automorphisms of an SFT, Nasu considers various *resolving conditions* on the graph homomorphisms p, q.

A graph homomorphism h is *right resolving* if, whenever it sends a vertex i to the initial vertex of an edge e, there exists a unique edge e' beginning with i which is sent to e. The *left resolving* maps are defined analogously using terminal vertices. These resolving maps and the block codes they define play a fundamental role in the coding theory of Z SFTs and Z sofic shifts. (Caveat: the definition of "resolving" varies a bit by author and context–this is Nasu's use for the textile systems.)

A textile system is LR if p is left resolving, q is right resolving and the map $z \mapsto \xi(z)$ is one to one. The LR textile systems are exactly those arising as $X_{A,B}$ earlier from commuting Z_+ matrices A and B. Here the resolving conditions are a translation of the requirement that the Wang tile be determined by the top+right sides and also by the left+bottom sides, as in the earlier construction with commuting matrices. The LR textile sytems are important in Nasu's theory.

An automorphism ϕ of an SFT σ_A is an LR automorphism of σ_A if there is an LR textile system, with ξ invertible, such that ϕ is ξ^{-1} followed by β.

NOTE: given the automorphism ϕ of the SFT σ_A, there is AT MOST ONE LR textile system (up to graphical isomorphism) for which ϕ can be presented as an LR automorphism of σ_A. In particular, the transition matrix B is unique. This suggests some unexpected power of the textile system formulation.

A map ψ is an essentially LR (ELR) automorphism of an SFT S if there exists an LR automorphism ϕ of an SFT σ_A and a homeomorphism h such that $h^{-1}Sh = \sigma_A$ and $h^{-1}\psi h = \phi$.

Nasu has a theory of "ELR cones" quite independent of the connected components $\mathcal{C}$ of $E_1(X)$. However, he related the viewpoints by showing the following: if in a Z^2 subshift σ_v is SFT, then the interior of the ELR cone containing v is equal to $\mathcal{C}$. So, up to conjugacy and a choice of generators of Z^2, every Z^2 SFT occurs as an LR textile system.

The foundation of the textile systems theory is Nasu's 1995 memoir. He has a number of significant results since then, for example

Theorem. An expansive automorphism of a onesided SFT must be a twosided SFT.

Theorem. If an expansive automorphism of an SFT is conjugate to an automorphism with no memory, then it is SFT.

Nasu's theory can be easily used to show that for an automorphism ϕ of a mixing SFT σ_A there is an integer k and a computable positive number γ and a matrix B determined by the (well understood and computable) action of ϕ on the dimension group of σ_A such that for all (i,j) in Z^2 with $i > 0$ and $|j/i| < \gamma$, the action of the directional shift σ_v is topologically conjugate to $\sigma_{A^i B^j}$.

So within an open cone we get well understood individual directional dynamics.

It is important to appreciate that these individual dynamics in an open cone can be put together in quite different ways, and by no means determine the joint dynamics of the Z^2 action, or the topological conjugacy classes of σ_v outside the component in E_1 containing $(1,0)$.