Almost isomorphism of countable state Markov shifts

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joint work with
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Outline of the talk

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1. Markov shifts and shifts of finite type

Notations:

- $\mathcal{G}$ is a graph (:= directed graph)

- $\mathcal{G}$ has at most countably many vertices

- $\mathcal{G}$ has at most finitely many edges between any two vertices

- $A =$ adjacency matrix of $\mathcal{G}$

- $\mathcal{G}_A :=$ graph with adjacency matrix A

- $\mathcal{E}_A :=$ edges of $\mathcal{G}_A$

- X_A is the set of bisequences $x = \dots x_{-1}x_0x_1\dots$ in $(\mathcal{E}_A)^{\mathbb{Z}}$ coming from walks through $\mathcal{G}_A$, topologized with subset top. of product top. of discrete top.

- σ_A is the shift homeomorphism, $X_A \to X_A$, where $(\sigma_A x)_n = x_{n+1}$

By a *Markov shift* we mean such a top. dyn. system (X_A, σ_A), where in addition σ_A is *irreducible*: there is a path from any vertex of $\mathcal{G}_A$ to any other vertex of $\mathcal{G}_A$.
I use σ_A for (X_A, σ_A), σ_A or X_A.

By a *shift of finite type (SFT)* we mean a Markov shift σ_A where $\mathcal{G}_A$ has only finitely many vertices and edges (i.e. an SFT is a compact Markov shift).

SFTs and other Markov shifts have various uses, in particular for studying invariant measures of some smooth or piecewise smooth systems.

2. **Classes of Markov shifts**

Given a vertex v in $\mathcal{G}_A$:

- $f_n :=$ number of first-return loops of length n from v to v

- $r_n :=$ number of loops of length n from v to v

- $\lambda := \overline{\lim}_n \ |r_n|^{1/n}$

** λ and (f_n) can be used to define some classes of Markov shifts (next slide).

** Those classes, and the number λ, don't depend on the particular vertex chosen.

** Define the topological entropy $h(\sigma_A)$ as the sup of the measure entropies over invariant Borel probabilities. Then $h(\sigma_A) = \log \lambda$. From here we always assume $\lambda < \infty$.

The classes:

- (T) σ_A is transient if $\sum f_n/\lambda^n < 1$

- (R) σ_A is recurrent if $\sum f_n/\lambda^n = 1$

- (PR) σ_A is positive recurrent if $\sum f_n/\lambda^n = 1$ and $\sum n f_n/\lambda^n < \infty$

- (SPR) σ_A is strongly positive recurrent if $\overline{\lim} |f_n|^{1/n} < \lambda$

Above, R contains PR contains SPR. The classes T/R/PR come from Vere-Jones' work in the 60's on infinite nonnegative matrices.
The class SPR was defined independently by U. Fiebig and B. Gurevic [1996], following work of I. Salama [1988,1992]. (Gurevic used the term stable recurrent. We like the progression of initials R, PR, SPR and also think of SPR as stable positive recurrent.)

Theorem [Gurevich 1969] σ_A has a measure of maximal entropy iff σ_A is PR. In this case there is a unique measure of maximal entropy.

We will be interested only in PR shifts, and primarily in SPR shifts.

Fix a vertex v and define f_n as before (number of first return loops to v) and then define

$$f(z) := f_1 z + f_2 z^2 + f_3 z^3 + \cdots$$

Given f, define the Markov shift σ_f whose defining graph $\mathcal{G}$ has a distinguished base vertex V and for each n f_n loops of length n, with distinct loops intersecting at V and nowhere else. Then σ_f is topologically conjugate to $\sigma_{A,v}$, via an edge labeling on $\mathcal{G}$.

The Artin-Mazur zeta function of σ_f is

$$\zeta(z) := \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \ |\mathrm{Fix}(\sigma_f)^n| \ z^n\right] = \frac{1}{1 - f(z)} \ .$$

The following are some of the equivalent conditions on σ_A characterizing SPR.

1. σ_A is SPR (i.e. $\overline{\lim} |f_n|^{1/n} < \lambda$).

2. Any proper subsystem of σ_A has strictly smaller entropy.

3. There is a measure of maximal entropy, and with respect to it the shift is exponentially recurrent.

4. For some (or for any) vertex v, the series $(1 - f(z))/(1 - \lambda z)$ has radius of convergence $> 1/\lambda$.

(1) $\iff$ (2) [U. Fiebig 1996]

(1) $\iff$ (3) [BBG 2004] (routine)

(1) $\iff$ (4) [Gurevich-Savchenko 1998]

3. Almost isomorphism of Markov shifts

Markov shifts σ_A and σ_B are *almost isomorphic* if there exists another Markov shift σ_C and injective one-block codes $\alpha : \sigma_C \to \sigma_A$, $\beta : \sigma_C \to \sigma_B$, each of which has a *magic word* (a concept from SFT coding theory).

A *magic word* for α is a σ_A-word W such that for any σ_A-word WUW,

- If $WUW = (\alpha x)[i, i + |WUW| - 1]$ and also $= (\alpha y)[i, i + |WUW| - 1]$, then $x[i + |WU| - 1] = y[i + |WU| - 1]$.

- If z is a point in σ_A and W occurs in z infinitely often to the left and also to the right, then z is in the image of α.

Almost isomorphism is indeed an equivalence relation on Markov shifts.

Let $\alpha : \sigma_C \to \sigma_A$, $\beta : \sigma_C \to \sigma_B$ be an almost isomorphism, with W a magic word for α. Define $\gamma = \beta\alpha^{-1}$. So, γ is a Borel bimeasurable bijection from the image of α to the image of β. Let us see the map γ is good for measures (meaning shift invariant Borel probabilities).

First suppose μ ergodic for σ_A and $\mu W > 0$. Then the α-image of σ_C has μ-measure 1. Thus there is a unique ergodic μ' on σ_B such that γ sends μ to $\gamma\mu := \mu'$. Moreover, one can check that if μ has full support in σ_A (i.e. every nonempty open set has positive μ-measure), then the same will hold for μ' in σ_B. So:

- γ induces a bijection $\mu \leftrightarrow \mu'$ of ergodic measures with full support, and simultaneous isomorphisms of their measurable systems, $\gamma : (\sigma_A, \mu) \to (\sigma_B, \mu')$.

- Each such isomorphism $(\sigma_A, \mu) \to (\sigma_B, \mu')$ is *finitary*: in each direction, continuous on a measure 1 set.

  Here for this set we use the set of points which see the magic word infinitely often to the left and to the right. The points seeing a word WUW in certain coordinates form an open set in the relative topology.

- If σ_A has a measure of maximal entropy, so does σ_B. (So PR is invariant under AI.)

- SPR is invariant under AI (because finitary isomorphism respects exponential recurrence, and exp. rec. of the max. entropy measure characterizes SPR).

- In the case σ_A and σ_B are AI and SPR, they are *entropy conjugate*: there is a $\delta > 0$, and Borel sets $\mathcal{F}_A \subset X_A$ and $\mathcal{F}_B \subset X_B$, with measure 1 for all ergodic invariant measures of entropy within δ of $h(\sigma_A) = h(\sigma_B)$, and an isomorphism of Borel systems $(\mathcal{F}_A, \sigma_A|\mathcal{F}_A) \to (\mathcal{F}_B, \sigma_B|\mathcal{F}_B)$.

  If σ_A is SPR, then there exists $\epsilon > 0$ such that any measure supported on the subsystem which misses the magic word W has entropy at most $h(\sigma_A) - \epsilon$. Likewise there is ϵ' for σ_A and β. For $\delta = \min(\epsilon, \epsilon')$, the map γ gives an entropy-conjugacy from σ_A to σ_B.

- In the SPR case, for the maximal entropy measures the map γ has exponentially fast coding time. (I.e. the measure of the set of x where $x[-n \dots n]$ does not determine $(\gamma x)[0]$ goes to zero exponentially in n.)

The next theorem is very reminiscent of the Adler-Marcus classification of irreducible shifts of finite type up to almost topological conjugacy.

The classification of (countable state) Markov shifts up to almost conjugacy is difficult and unsolved. The relation "almost isomorphism" seems much better suited to the study of ergodic measures on countable state Markov shifts.

4. Main Markov Results

THEOREM 1. For SPR Markov shifts, entropy and period are complete invariants of AI.

COROLLARY. SPR Markov shifts of equal entropy and period are entropy conjugate, and their maximal-entropy-measure systems are finitarily isomorphic with exponentially fast coding time.

KEY TECHNICAL REMARK: Let v be a vertex of σ_A, and let σ_f be the corresponding loop shift. The natural one-block code $\sigma_f \to \sigma_A$ is injective with a magic word (any edge beginning at the vertex v will be one).

So, for studying AI of Markov shifts, we can restrict to loop shifts σ_f – a vast simplification (power series instead of matrices).

THEOREM 2. Suppose σ_f and σ_g are loop shifts with equal entropy $\log \lambda$ and

(**) the power series $(1-f)/(1-g)$
has spectral radius $> 1/\lambda$.

Then σ_f and σ_g are AI (and the AI can be chosen so that the associated finitary Borel map is an entropy conjugacy).

The condition (**) always holds for SPR loop shifts of equal entropy [Gurevich-Savchenko] (elementary complex variables argument, critical fact for us.)
So, Theorem 1 is a corollary of Theorem 2.

The condition (**) means that

$$\overline{\lim_n} \Big| |\mathcal{O}_n(\sigma_f) - \mathcal{O}_n(\sigma_g)| \Big|^{1/n} < \lambda$$

where $\mathcal{O}_n$ means the set of orbits containing exactly n points.

5. Applications

We begin with a list of some systems.

1. Subshifts of quasi finite type
   (a generalization of SFT used to describe some nonuniformly hyperbolic systems)

2. Piecewise monotonic interval maps which have nonzero topological entropy.

3. The multi-dimensional β-transformations. ($[0,1]^d \to [0,1]^d$ by $x \mapsto Bx$ mod $\mathbb{Z}^d$, where B is expanding affine.)

4. C^∞ smooth entropy-expanding maps, e.g. smooth interval maps with nonzero topological entropy.

Our applications follow from work of Buzzi which reduce the entropy-conjugacy classification of certain systems, including those on the list, to the classification of SPR Markov shifts.

THEOREM 3 (from earlier work of Buzzi)
The following measurable dynamical systems have natural extensions entropy-conjugate to the disjoint union of finitely many SPR Markov shifts of equal entropy.

Moreover, the entropy conjugacy can be chosen to preserve the hyperbolic structure in some of these cases:

- (1) and (3)

- (2), when each of the monotonic branches is discontinuous at all endpoints other than 0 and 1.

COROLLARY Two systems from the list have natural extensions which are entropy conjugate if and only if they have equal entropy and for each p the same number of ergodic maximal entropy measures of period p.

COROLLARY Two topologically mixing piecewise monotonic interval maps are entropy conjugate if and only if they have the same entropy.

6. Proof.

A BASIC MOVE

Suppose $f = f_1 z + f_x z^2 + f_3 z^3 + \cdots$ is a power series for a loop shift and $f = h + k$ with h and k nonzero with coefficients in $= \mathbb{Z}_+$. Set

$$F = h + hk + hk^2 + hk^3 + \cdots = h/(1-k)$$

which by an easy computation implies

$$\frac{1}{1-F(z)} = \frac{1-k}{1-f} \ .$$

The series F defines another loop system. There is a natural one-block code (edge labeling) which gives an injection $\sigma_F \to \sigma_f$, as in the following example, where we take h to be a single term z^N.

EXAMPLE

Let $f = z^2 + z^3$. Choose $k = z^N = z^2$. Then $h = z^3$ and F must be

$$z^3 + z^3(z^2) + z^3(z^2)(z^2) + z^3(z^2)(z^2)(z^2) + \cdots$$

There are two first return loops (to the base vertex) for σ_f, a loop $a = a_1a_2a_3$ corresponding to z^3 and a loop $b = b_1b_2$ corresponding to z^2 (the a_i and b_j are distinct edges). The display of F corresponds to certain concatenations of a and b:

$$a + ab + abb + abb + abbb + \cdots$$

This lets us choose an edge labeling ℓ (by the a_i and b_j), of the first return loops for σ_F, which gives a map from the first return loops of σ_F to certain concatenations of loops in σ_f. Points of σ_f and σ_F correspond to doubly infinite concatenations of first return loops, and ℓ induces an injection $\ell : \sigma_F \to \sigma_f$.

Note

- The injective one block code ℓ does have a magic word – any edge which does not lie on the “deleted loop” b.

- The only periodic orbit of σ_f not in the image of ℓ is $\ldots aaaaaa \ldots$, an orbit of length N. So,
  $|\mathcal{O}_n(F)| = |\mathcal{O}_n(f)|$ if $n \neq N$, and
  $|\mathcal{O}_N(F)| = |\mathcal{O}_N(f)| - 1$.

IDEA OF THE PROOF

We have series $f = \sum f_n z^n$ and $g = \sum g_n z^n$ defining loop systems with entropy $\log \lambda$. We will perform basic moves recursively on these series until in the limit we have changed each into the same series c. Then we'll get maps $\sigma_c \to \sigma_f$, $\sigma_c \to \sigma_g$ giving the AI.

The algorithm. Let $f = f^{(0)}$ and $g = g^{(0)}$. Given $f^{(k)}$ and $g^{(k)}$:

1. If $f^{(k)} = g^{(k)}$, set $c = f^{(k)}$.

2. Otherwise, choose the smallest N such that $f_N^{(k)} \neq g_N^{(k)}$. If $g_N^{(k)} < f_N^{(k)}$:

   - Set $g^{(k+1)} = g^{(k)}$.

   - Apply the basic move ("remove an N-loop") to $f^{(k)}$ to get $f^{(k+1)}$, i.e. $f^{(k+1)} = (f^{(k)} - z^N)/(1 - z^N)$.

If $f_N^{(k)} < g_N^{(k)}$ in case 2 above, of course we reverse the roles of f and g. This finishes the description of the algorithm.

As $k \to \infty$, the smallest N for which $f_N^{(k)} \neq g_N^{(k)}$ also goes to infinity. We define c as the power series produced in the limit. This c defines a loop shift. We want an injective one-block code $\sigma_c \to \sigma_f$ with a magic word.

We have our injective one-block maps with magic words,

$$\sigma_f \leftarrow \sigma_{f^{(1)}} \leftarrow \sigma_{f^{(2)}} \leftarrow \sigma_{f^{(3)}} \cdots$$

Composing finitely many such maps gives another. So in the case that for some k we get $f^{(k)} = g^{(k)}$, we are done.

In the remaining case, we can track the edge labellings up the chain of graphs $\mathcal{G}_k$ associated to the $f^{(k)}$, and produce an induced labeling on the loop graph $\mathcal{G}_\infty$ associated to σ_c. This will be an injective one-block code. We have to show our choices of deleted loops can be made such that this code has a magic word.

REDUCTION: after reducing to the mixing case and applying a construction (which we skip here), we can assume that we began with f and g such that there exists M such that

1. $f_m = g_m$ if $m < M$, and
2. $\min(f_m, g_m) > |\mathcal{O}_m(f) - \mathcal{O}_m(g)|$ if $m \geq M$.

Main step in the reduction: a technical lemma (we skip) gets the f_m, g_m with growth rate near λ. Then condition 2 holds for large m by our key assumption,
$\overline{\lim}_n \Big| |\mathcal{O}_n(\sigma_f) - \mathcal{O}_n(\sigma_g)| \Big|^{1/n} < \lambda$.

KEY OBSERVATION

In our inductive construction, we recursively "remove" terms z^N (and a corresponding orbit of length N), for $N = N_1, N_2, \dots$.

Suppose $m = N_k$ for k minimal (we will "remove" a term z^m for the first time in forming $f^{(k+1)}, g^{(k+1)}$ from $f^{(k)}, g^{(k)}$).

Then for $j \leq k$, in forming $f^{(j)}$ we only removed orbits of length smaller than m. So, $|\mathcal{O}_m(\sigma_{f^{(k)}})| = |\mathcal{O}_m(\sigma_f)|$ and $|\mathcal{O}_m(\sigma_{g^{(k)}}) = |\mathcal{O}_m(\sigma_g)|$.

Because $f_j^{(k)} = g_j^{(k)}$ for $j < m$, it follows that

$$\left|(f^{(k)})_m - (g^{(k)})_m\right| = \left||\mathcal{O}_m(\sigma_f)| - |\mathcal{O}_m(\sigma_g)|\right| .$$

So! The terms we must change along the inductive construction are controlled by the original zeta functions of σ_f and σ_g.

Now pick any first return loop W of length M for σ_f. We claim that we can make choices in the inductive construction such that W is a magic word for the map $\sigma_c \to \sigma_f$.

The maps $f^{(0)} \leftarrow f^{(1)} \leftarrow f^{(2)} \cdots$ produce for $k \geq 1$ one block codes $f^{(0)} \leftarrow f^{(k)}$ and associated labeled graphs $\mathcal{G}_{(k)}$, where for each k, all of $\mathcal{G}_{(k)}$ except the distinguished loop chosen for deletion is a subgraph of $\mathcal{G}_{(k+1)}$. The condition 2 of the reduction tells us that at every stage, going from $f^{(k)}$ to $f^{(k+1)}$, we may choose that distinguished loop to be a loop corresponding to a first return loop of $\mathcal{G}_{(0)}$ (that is, a first return loop in the graph $\mathcal{G}_{(0)}$ minus those first return loops deleted at earlier stages of the construction).

This means that at every stage of the construction, if W occurs as the label of a path in $\mathcal{G}_{(k)}$, then W begins with an edge with initial vertex the base vertex of $\mathcal{G}_{(k)}$. This property is inherited by $\mathcal{G}_{(\infty)}$ and gives the required magic word.

7. The positive K-theory aspect.

Since "algebra" is in the activity title along with "topological dynamics", I mention an algebraic aspect.

"Positive K-theory" is a framework for various classification problems in symbolic dynamics, and sometimes constructions. In the framework a symbolic system is presented by an $\mathbb{N} \times \mathbb{N}$ matrix A over a suitable ordered ring, with all entries nonnegative and only finitely many nonzero, and isomorphisms of systems are induced by multiplications of matrices of the form $I - A$ by basic elementary matrices satisfying a positivity condition (i.e., "positive paths of elementary multiplications").

There is a "positive K-theory" expository paper on my website.

Also Mike Sullivan will give a talk in which the ring is $\mathbb{Z}G$ for a finite group G and the classification is of G-SFTs up to G-equivariant flow equivalence.

The good finitary isomorphisms we've induced with magic words are *magic word isomorphisms*. Surprisingly, the classification of Markov shifts with a Markov measure up to isomorphism by magic word isomorphisms can be put in this postive K-theory framework. Here the underlying ring is the ring of power series with coefficients in $\mathbb{Z}G$, where G is the positive real numbers under multiplication. This is due to Ricardo Gomez [ETDS, 2003].

Our result then gives conditions for the existence of positive paths of elementary multiplications when the presenting matrices correspond to the measure of maximal entropy.

Some open questions.

1. If two positive recurrent Markov shifts have the same finite entropy, must they be entropy conjugate?
[The map giving the entropy conjugacy must be nasty, not even finitary.]

2. Is SPR an invariant of entropy conjugacy?

3. Classify positive recurrent Markov shifts up to almost isomorphism.

4. Consider Markov shifts together with some Markov measure. Are these methods useful for the good finitary classification of Markov shifts with Markov measures?