II. Automorphisms of the Shift

$\mathrm{Aut}(S_A)$ = group of homeomorphisms U such that $US_A = S_A U$

Investigate S_A via $\mathrm{Aut}(S_A)$. Suppose A is irreducible, $\lambda_A > 1$.

As an abstract group:

(1) $\mathrm{Aut}(S_A)$ is countably infinite

(2) " " residually finite

(homomorphisms into finite groups separate points)

(3) Center of $\mathrm{Aut}(S_A) = \{(S_A)^n : n \in \mathbb{Z}\}$ [Ryan]

(4) $\mathrm{Aut}(S_A)$ contains

- many complicated subgroups
- no fin. gen. subgroup with unsolvable word problem

Ryan's Theorem (3) is our only (!) tool for distinguishing different $\mathrm{Aut}(S_A)$.

~~Open A~~ Ex. The groups $\mathrm{Aut}(S_A)$ are pairwise not isomorphic for $A = [2], [4], [8]$.

Open. $\mathrm{Aut}(S_{[2]}) \cong \mathrm{Aut}(S_{[3]})$?

$\cdot\ \mathrm{Aut}(S_{[2]}) \cong \mathrm{Aut}\left(S_{\left[\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right]}\right)$?

We know only two ways to construct useful representations of $\mathrm{Aut}\, S_A$:

- via dimension module
- " periodic points.

Dimension Representation

ESSE

Suppose $A = UV$, $B = VU$ over $\mathbb{Z}_+$.

$$\begin{tikzcd} \mathbb{Q}^m \arrow[r, "A"] \arrow[d, "U"'] & \mathbb{Q}^m \arrow[d] \\ \mathbb{Q}^n \arrow[r, "B"'] \arrow[ur, "V"] & \mathbb{Q}^n \end{tikzcd}$$

$AU = UVU = UB.$

Can check: $G_A \xrightarrow[U]{\cong} G_B$.

Define $\hat{U} = U|_{G_A}$

$$\hat{U} : (G_A, \hat{A}) \xrightarrow{\cong} (G_B, \hat{B})$$

SSE

Given conjugation $\varphi : S_A \longrightarrow S_B$

φ is a composition $c(R_1, S_1)^{\varepsilon_1} \cdots c(R_K, S_K)^{\varepsilon_K}$ (*)

Define $\hat{\varphi} : (G_A, \hat{A}) \longrightarrow (G_B, \hat{B})$

as $\hat{\varphi} = \hat{R}_1^{\varepsilon_1} \cdots \hat{R}_K^{\varepsilon_K}$.

Nontrivial: indeed $\hat{\varphi}$ is independent of choice of SSE for decomposition. (*)

~~one proof: Krieger's~~

Defn The *dimension representation* of $\mathrm{Aut}(S_A)$ is the map

$$\rho_A : \mathrm{Aut}(S_A) \longrightarrow \mathrm{Aut}_+(\hat{A})$$
$$\varphi \longmapsto \hat{\varphi}$$

$\mathrm{Aut}_+(\hat{A})$ = the automorphisms of (G_A, G_A^+) which commute with $\hat{A}$.

$\mathrm{Aut}(\hat{A}) \cong \mathrm{Aut}_+(\hat{A}) \oplus \mathbb{Z}/2$

Ex. $A = [2]$

$U \in \mathrm{Aut}(\hat{A})$ is restriction of a vector space map $\mathbb{Q} \longrightarrow \mathbb{Q}$

$$x \longmapsto mx$$

G_A = dyadic rationals

$m = \pm 2^{-k}, \quad k \in \mathbb{Z}$.

$\mathrm{Aut}(\hat{A}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$

$\mathrm{Aut}_+(\hat{A}) \cong \mathbb{Z}$

Ex $A = [6] = [2 \cdot 3]$

$\mathrm{Aut}_+(\hat{A}) \cong \mathbb{Z}^2$

Ex $A = [24]$

$\mathrm{Aut}_+(\hat{A}) \cong \mathbb{Z}^2$

Ex $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$\mathrm{Aut}_+(\hat{A}) \cong \mathbb{Z}$

~~Always: $\mathrm{Aut}_+(\hat{A}) = \mathrm{Aut}_+(\widehat{A^k})$, $k \geq 1$.~~

Periodic Points

$S = S_A$

$P_n^o(S_A)$ = points of least S_A-period n

$|P_n^o(S_A)| < \infty$

Define $U_n = U|P_n^o(S_A)$

$$\mathrm{Aut}(S) \to \mathrm{Aut}(S_n)$$
$$U \mapsto U_n$$

is a group homomorphism.

Given n and U:

Let $\mathcal{O}_1, \dots, \mathcal{O}_k$ be the S-orbits of size n

Pick $x_i \in \mathcal{O}_i$, $1 \le i \le k$

$$U_n : x_i \longmapsto S^{m_i} x_{\pi(i)}$$

(π a permutation on $\{1, \dots, k\}$).

Define

$$\mathrm{sign}_n(U) = 0 \in \mathbb{Z}/2 \text{ if } \pi \text{ is even}$$
$$= 1 \in \mathbb{Z}/2 \text{ " " " odd}$$

$$gy_n(U) = \sum_{i=1}^{k} m_i \in \mathbb{Z}/n.$$

("gy" for "gyration").

Then $\mathrm{sign}_n : \mathrm{Aut}(S) \to \mathbb{Z}/2$ ~~and gy_n are~~

$gy_n : \mathrm{Aut}(S) \to \mathbb{Z}/n$

are homomorphisms (easy to check).

Remark: a longer exercise:
if $h : \prod_{n=1}^{N} \mathrm{Aut}(S_n) \longrightarrow \underbrace{A}_{\text{abelian group}}$

is a homomorphism, then h factors through
$\prod_{n=1}^{N} (gy_n \times sign_n)$.

~~Define $gy = \prod_n gy_n$, $sign = \prod_n sign_n$~~

<u>Defn</u> (sign-gyration-compatibility condition homomorphism)

$$SGCC_n : \mathrm{Aut}(S) \longrightarrow \mathbb{Z}/n$$
$$: U \mapsto gy_n(U) + \frac{n}{2} \sum_{\substack{1 \le k < n, \\ \frac{n}{k} \in \{2^j : j \in \mathbb{N}\}}} sign_k(U)$$

<u>Ex</u> $SGCC_{36} = gy_{36} + 18 \left(sign_{18} + sign_9 \right) \in \mathbb{Z}/36$

<u>Ex</u> $S = S_{[2]}$ the shift on $\{0,1\}^{\mathbb{Z}}$

S-orbits	size 1	0^∞ 1^∞
"	size 2	$(01)^\infty$
"	size 3	$(001)^\infty$ $(011)^\infty$
"	size 4	$(0111)^\infty$ $(0001)^\infty$ $(0011)^\infty$

$SGCC_2(S) = 1 \in \mathbb{Z}/2$

For flip U $((Ux)_0 = x_0 + 1)$:

$SGCC_n(U) = 0$, $n = 1, 2, 3, 4, \ldots$

Now a miraculous connection of periodic point and dimension representations:

<u>Factorization THM</u> (Kim-Roush-Wagoner, following ...)
For A irreducible over $\mathbb{Z}_+$ and $U \in \mathrm{Aut}(S_A)$:

- $\rho_A(U) = 0 \implies SGCC_n(U) = 0 \;\; \forall n$

$$\begin{tikzcd} \mathrm{Aut}(S_A) \arrow[dr, "\rho_A"] \arrow[dd, "SGCC_n"'] & \\ & \mathrm{Aut}_+(\hat{A}) \arrow[dl, "sgc_n"] \\ \mathbb{Z}/n & \end{tikzcd}$$

- Moreover there is an explicit (nasty) formula for sgc_n (more later).

~~I. For example: if $U \in \mathrm{Aut}(S_A)$ and $Ux = x$ for $x \in$~~

I. Where did SGCC come from?

II. Of what use is this?

I. What is action of $\mathrm{Aut}(S_A)$ on finite subsystems $F_N(S_A) := \bigcup_{n \le N} P^0_n(S_A)$?

(Can SE-$\mathbb{Z}_+$ mixing S_A have nonisomorphic actions?) [essentially Williams ~80]

I

[B-Krieger 85]: $\exists$ collection $\mathcal{I}$ of involutions of full shifts S such that $\forall N$, for $V \in \mathrm{Aut}(S|_{F_N})$ TFAE:

(1) $V = U|_{F_N}$ for some $U \in \langle \mathcal{I} \rangle$

(2) $SGCC_n(V) = 0$, $n \le N$.

For an involution U of a full shift S:

- US is top. conjugate to an SFT
- $|P_n^o(US)| = |P_n^o(S)|$ $\forall n$ $(*)$

Exploring the limits of action of $\langle \mathcal{I} \rangle$ and the constraint $(*)$ leads to SGCC.

II

<u>THM</u> (Kim-Roush-Wagoner, following BK, BLR, Fiebig, B-Fiebig, W.): for irreducible SFT S_A and $V \in \mathrm{Aut}(S_A | F_N(S_A))$ (N sufficiently large), TFAE:

(1) $SGCC_n(V) = 0$, $n \le N$

(2) $V = U|_{F_N}$ for some $U \in \mathrm{Ker}(\rho)$.

<u>THM</u> (B-Krieger) Likewise $SGCC = 0$ is the only constraint on extending automorphisms of subshifts of S_A to automorphisms of S_A.

To complete our understanding of action of $\mathrm{Aut}(S_A)$ on subsystems, we need to solve (and for other reasons)

$\boxed{\text{Open Problem (RD) Given } A \text{ primitive over } \mathbb{Z}_+, \text{ what is image of the dimension representation of } S_A \text{ in } \mathrm{Aut}_+(\hat{A})\,?}$

Some facts:

(1) The classification of SFT's up to topological conjugacy reduces to the solution of (RD) and the solution of the conjugacy problem for mixing SFT's. (Kim-Roush)

(2) $\rho_A : \mathrm{Aut}(S_A) \longrightarrow \mathrm{Aut}_+(\hat{A})$ can be nonsurjective. (Later.)

~~(3) Modulo multiplications by $\hat{A}$,~~

(3) Given A primitive over $\mathbb{Z}_+$ and $\varphi \in \mathrm{Aut}_+(\hat{A})$ Let R be ~~a rank +~~ matrix over $\mathbb{Q}$ such that $G_A \xrightarrow{R} G_A$ defines $\hat{R} = \varphi \; \mathrm{Aut}_+(\hat{A})$. Likewise let $G_A \xrightarrow{S} G_A$ have $\hat{S} = \hat{R}^{-1}$.

Let $\mathrm{rank}\, S = \mathrm{rank}\, R = \dim V_A$.

Then for all large n, m: $A^{m+n} = (RA^m)(SA^n) = (SA^n)(RA^m)$

with RA^m, SA^n over $\mathbb{Z}_+$. So,

~~$\varphi \in \mathrm{Aut}($~~

$\varphi \in \mathrm{image}(\rho_{A^\ell})$ for all large ℓ.

(3) If char. poly. (A) has no repeated nonzero root, then $\mathrm{Aut}_+(A)$ is finitely generated (and abelian), so by (2), ρ_{A^ℓ} is surjective for all large ℓ.

(4) "(N. Long) Problem (RD) is not elementary":

- if A is primitive with
  $$A = UV = VU \quad \text{over } \mathbb{Z}_+$$
  and $\lambda_A = \text{prime } p \in \mathbb{N}$
  and A has no repeated nonzero root
  then $\exists k \ni$
  $$\hat{U}^k = \mathrm{id} \text{ or } \hat{A}^{-1}\hat{U}^k = \mathrm{id}. \quad (*)$$

- $\exists$ examples with ρ_A surjective, but not using only such (*).

For A primitive and $\lambda_A > 1$:
understand $\mathrm{Aut}(S_A)$ via

- Kernel (ρ_A)
  - big, complicated, combinatorial
  - known actions are rather homogeneous,(*) modulo low-order-periodic points
- Image (ρ_A)
  - typically finitely generated abelian
  - fine structure, algebraic

(*) But : • $\mathrm{Aut}(S_A) \hookrightarrow \mathrm{Aut}(S_{[n]})$ if $n > 1$

Open Problem

For B primitive and $\lambda_B > 1$ and every A, does

$$\mathrm{Aut}(S_A) \hookrightarrow \mathrm{Aut}(S_B) \ ?$$