Lectures III, IV

Strong shift equivalence space and its applications

SSE(Λ) (Wagner)

Λ = subset of a ring containing $\{0,1\}$.

SSE(Λ) is a CW complex formed as a quotient of oriented simplices.

0-simplex $\leftrightarrow$ square matrix A over Λ

1-simplex $\leftrightarrow$ ordered pair (A_0, A_1)
("edge") with ordered pair (R,S)
$\ni A_0 = RS$, $A_1 = SR$

2-simplex $\leftrightarrow$ 3-tuple (A_0, A_1, A_2) with
("triangle") " $((R_1,S_1),(R_2,S_2),(R_3,S_3))$

such that for

$$\begin{tikzcd} & A_1 \arrow[dr, "{(R_2,S_2)}"] & \\ A_0 \arrow[ur, "{(R_1,S_1)}"] \arrow[rr, "{(R_3,S_3)}"'] & & A_2 \end{tikzcd}$$

- each edge is an edge
- the Triangle Identities hold:

$$R_1 R_2 = R_3$$
$$R_2 S_3 = S_1$$
$$S_3 R_1 = S_2$$

<u>$n>2$</u>;

n-simplex $\leftrightarrow (A_0, A_1, \dots, A_n)$,

with edges $A_i \to A_j$ $\quad i<j$

giving triangles

$$\begin{tikzcd} & A_j \arrow[dr] & \\ A_i \arrow[ur] \arrow[rr] & & A_k \end{tikzcd} \qquad i<j<k$$

(So far $n>2$ has no application. But this defines the "right" top. space — more later.)

A <u>simple</u> graph automorphism is one which fixes all vertices.

A <u>simple</u> automorphism of S_A is a composition of automorphisms of the form $\varphi \psi \varphi^{-1}$ where

- $\varphi : S_A \xrightarrow{\cong} S_B$
- ψ is a 1-block automorphism of S_B defined by a simple graph automorphism.

The simple automorphisms of S_A form a normal subgroup of $\mathrm{Aut}(S_A)$.

Suppose ρ is a path in $SSE(\Lambda)$ from A to B along edges.

E.g.

$$\rho = \underset{A}{\bullet} \xrightarrow{(R_1,S_1)} \bullet \xleftarrow{(R_2,S_2)} \bullet \xrightarrow{(R_3,S_3)} \underset{B}{\bullet}$$

$\boxed{\text{For } \Lambda \in \mathbb{Z}_+}$: associate to ρ a conjugacy $\varphi_\rho : S_A \to S_B$

e.g. above $\varphi_\rho = c(R_1,S_1)\big(c(R_2,S_2)\big)^{-1} c(R_3,S_3)$

(φ_ρ is well defined modulo simple automorphisms)

$\boxed{\text{For } \Lambda = \mathbb{Z}}$: associate to ρ an isomorphism of dim. modules

$$\hat\rho : (G_A, \hat A) \longrightarrow (G_B, \hat B)$$

e.g. above $\hat\rho = \hat R_1 (\hat R_2)^{-1} R_3$.

Write $\boxed{\rho \underset{\Lambda}{\sim} \rho'}$ if ρ deforms to ρ' thru triangles (leaving endpoints of ρ, ρ' fixed).

THEOREM (Wagoner)

(1) $\rho \underset{\{0,1\}}{\sim} \rho' \iff \varphi_\rho = \varphi_{\rho'}$

(2) $\rho \underset{\mathbb{Z}_+}{\sim} \rho' \iff \varphi_\rho = \varphi_{\rho'}$
(mod simple automorphisms)

(3) $\rho \underset{\mathbb{Z}}{\sim} \rho' \iff \hat{\rho} = \hat{\rho}'$

Of course the directions $\Leftarrow$ are hard. Easier: check $\Rightarrow$
Suffices to check a triangle.
~~(3) $\Leftarrow$~~

$$\begin{tikzcd} & \bullet \arrow[dr, "{(R_2, S_2)}"] & \\ \bullet \arrow[ur, "{(R_1, S_1)}"] \arrow[rr, "{(R_3, S_3)}"'] & & \bullet \end{tikzcd}$$

$R_1 R_2 = R_3$
$R_2 S_3 = S_1$
$S_3 R_1 = S_2$

(3) $\Rightarrow$ immediate! $R_1 R_2 = R_3$
(1) $\Rightarrow$ Contemplate the following makes sense:

$\varphi_1 \varphi_2$:

$$\begin{tikzcd}
\cdot \arrow[r, "a"] \arrow[d, "r_1"'] & \cdot \arrow[d, "r_1"] \\
\cdot \arrow[ur, "s_1"] \arrow[r] \arrow[d, "r_2"'] & \cdot \arrow[d, "r_2"] \\
\cdot \arrow[ur, "s_2"] \arrow[r, "b"'] & \cdot
\end{tikzcd}$$

φ_3

$$\begin{tikzcd}
\cdot \arrow[r, "a"] \arrow[d, "r_3"'] & \cdot \arrow[d, "r_3"] \\
\cdot \arrow[ur, "s_3"] \arrow[r, "b"'] & \cdot
\end{tikzcd}$$

We check correspondence of edges is same. 0-1 matrices guarantee uniqueness.

$$a \overset{\varphi_3}{\longleftrightarrow} r_3 s_3 \longleftrightarrow r_1 r_2 s_3 \longleftrightarrow r_1 s_1 \overset{\varphi_1 \varphi_2}{\longleftrightarrow} a$$

$$b \overset{\varphi_3}{\longleftrightarrow} s_3 r_3 \longleftrightarrow s_3 r_1 r_2 \longleftrightarrow s_2 r_2 \overset{\varphi_1 \varphi_2}{\longleftrightarrow} b$$

(2) $\Longrightarrow$ Trickier.

Given over $\mathbb{Z}_+$:

$$\begin{tikzcd} & \bullet \arrow[dr, "{(R_2, S_2)}"] & \\ \bullet \arrow[ur, "{(R_1, S_1)}"] \arrow[rr, "{(R_3, S_3)}"'] & & \bullet \end{tikzcd}$$

show $\exists$ choices for $c(R_i, S_i) = c_i$ such that

$$c_3 = c_2 c_2 \qquad (*)$$

$$\cancel{c(R_3, S_3) = c(R_1, S_1)\, c(R_2, S_2)} \quad (*)$$

Caveat! It can happen that e.g.

- $(R_1, S_1) = (R_2, S_2)$
- $(*)$ is impossible with $c_1 = c_2$.

But we only claim the existence of the c_i mod simple autos.

$SGCC_n$ for $SSE(\mathbb{Z}_+)$ Lp. 7

- To each vertex A we associate explicitly S_A an edge SFT over graph with
  - vertices $1, \dots, n$ (if A is $n \times n$)
  - edges (i, j, k), $1 \le k \le A(i,j)$

- Use a lexico order to pick from each periodic orbit $\mathcal{O}$ of S_A the lexico minimal $x_{\mathcal{O}} = \dots x_0 x_1 \dots x_{p-1} x_p \dots$ with $\underbrace{x_0 x_1 \dots x_{p-1}}_{\text{min.}}$

- ~~Define for~~
- For

$$A \xrightarrow{(R,S)} B$$

also order rs paths
choose the specific $c(R,S)$ which respects (a lexico.) order on edges where there is a choice

- order orbits $\mathcal{O}$ of given size by laxico order on $x_{\mathcal{O}}$. For each period n:

in S_A: $\mathcal{O}_1, \dots, \mathcal{O}_K$ with $x_1 \in \mathcal{O}_1, \dots, x_K \in \mathcal{O}_K$

in S_B: $\mathcal{O}'_1, \dots, \mathcal{O}'_K$ with $x'_1 \in \mathcal{O}'_1, \dots, x'_K \in \mathcal{O}'_K$

Now $c(R,S): x_i \mapsto (S_B)^{m_i} x'_{\pi(i)}$

Define $\mathrm{sign}_n(R,S) = 0$ if π is even
$= 1$ if π is odd

$gy_n(R,S) = \sum m_i \in \mathbb{Z}/n$

$$SGCC_n(R,S) = gy_n + \frac{n}{2}\left(\sum_{\dots} \mathrm{sign}_K\right) \in \mathbb{Z}/n$$

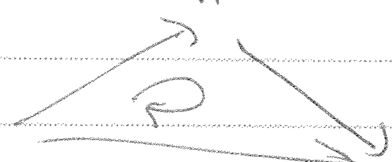
For a path $\rho = (R_1,S_1)^{\varepsilon_1} \cdots (R_K,S_K)^{\varepsilon_K}$,

$$SGCC_n(\rho) = \sum_i \varepsilon_i \, SGCC_n(R_i,S_i)$$

THEOREM (Kim-Roush-Wagoner)

$$\rho \underset{\mathbb{Z}_+}{\sim} \rho' \Rightarrow SGCC_n(\rho) = SGCC_n(\rho')$$

Proof It suffices to see $SGCC_n$ is zero around any triangle. But



- mod simple autos,
  we have $c(R_1,S_1)\,c(R_2,S_2)\,(c(R_3,S_3))^{-1} = id$
- $SGCC\,(id) = 0$
- Lemma: $SGCC$ (simple auto) $= 0$.

Also! KRW: there is an explicit (nasty) polynomial formula $sgc_n(R,S)$, which for an edge (R,S) in $SSE(\mathbb{Z}_+)$ computes $SGCC_n(R,S)$. The coefficients of $sgc_n(R,S)$ are rational numbers in

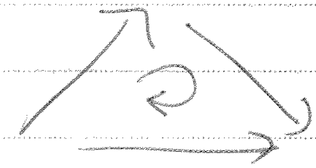
$$\frac{1}{(2n)!}\mathbb{Z}$$

Then for an SSE-$\mathbb{Z}$ path $\rho = (R_1,S_1)^{\varepsilon_1} \cdots (R_k,S_k)^{\varepsilon_k}$ define

$$sgc_n(\rho) = \sum_i \varepsilon_i \, sgc_n(R_i,S_i) \in \mathbb{Z}/n$$

<u>Theorem</u> $\rho \underset{\mathbb{Z}}{\sim} \rho' \Rightarrow sgc_n(\rho) = sgc_n(\rho')$

<u>Proof</u> To show sgc_n vanishes around a $SSE(\mathbb{Z})$ triangle:



replace all matrices with congruent (mod $n(2n!)$) matrices over $\mathbb{Z}_+$, ~~For these~~ still satisfying triangle identities. For these $sgc_n = SGCC_n$ and so vanishes around a triangle. The computation is the same as for the original triangle.

The Kim-Roush Counterexample to Williams' Conjecture (Annals Math.)

Find primitive A, B over $\mathbb{Z}_+$ and R, S over $\mathbb{Z}$ such that

- $A = RS$, $B = SR$
  (hence $A \underset{\mathbb{Z}_+}{\overset{SE}{\sim}} B$)
- $sgc_2(R,S) \neq 0$
- $tr(A) = tr(A^2) = 0$
- $sgc_2(\rho)$ vanishes for paths $A \xrightarrow{\rho} A$
  (i.e., $sgc_2 = 0$ on $Aut(\hat{A})$)

Now there can be no SSE-$\mathbb{Z}_+$ path ρ from A to B, as it would require

$$sgc_2(\rho) = sgc_2(R,S) \neq 0$$

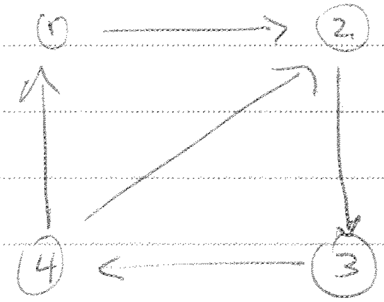
$$sgc_2(\rho) = \underbrace{SGCC_2(\rho)}_{(\#\text{ points of period} \leq 2)} = 0$$

Remarks

- A, B are 7×7 unimodular
  $\lambda_A = \lambda$ is degree 7 algebraic unit
- aut $(\hat{\lambda}) \leftrightarrow$ a group of units in $\mathbb{Q}[\lambda]$
- find basis with PARI
- evaluate sgc_2 with Maple
- this contradiction using missing low order periodic points is the only one we know

<u>KRW Example</u>: ρ_A not surjective

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$



graph

S_A has no point of period ≤ 2.

$\det(A) = \det(A - I) = \underline{1}$

Let $(R, S) = (A - I,\ A(A-I)^{-1})$

$A = RS = SR$

$sgc_2(R, S) \neq 0$

But $SGCC_2(U) = 0$ if $U \in Aut(S_A)$.

$\widehat{(A-I)} \in Aut_+(A)$

$\notin$ image (ρ_A).

Dimension Rep. & Reducible SFTs

(Kim R) (JAMS)

Let $U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

$A = \begin{pmatrix} U & O \\ I & U \end{pmatrix}$ $\quad B = \begin{pmatrix} U & O \\ X & U \end{pmatrix}$

with $X = U^{20}(U - I) > 0$

- $A \underset{\mathbb{Z}_+}{\overset{SE}{\sim}} B$ via

$R' = \begin{pmatrix} I & O \\ O & X \end{pmatrix}$

$S' = A^{20} R' = \begin{pmatrix} U^{20} & O \\ 20U^{19} & U^3 + U^2 + U \end{pmatrix}$

- If $(R_1, S_1), \dots, (R_k, S_k)$ gives $A \underset{\mathbb{Z}_+}{\overset{SSE}{\sim}} B$

then $(R, S) = (R_1 \cdots R_k, S_k \cdots S_1)$

gives $A \underset{\mathbb{Z}_+}{\overset{SE}{\sim}} B$

with $\hat{R} : G_A \to G_B$

Here $R = \begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix}$

$$S = \begin{pmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{pmatrix}$$

and $\hat{R}_{11}$, $\hat{R}_{22}$ are in image ρ.

$$AR = RB$$

$$\begin{pmatrix} U & 0 \\ I & U \end{pmatrix}\begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix}\begin{pmatrix} U & 0 \\ X & U \end{pmatrix}$$

$$R_{11} + UR_{21} = R_{21}U + R_{22}X$$

$$R_{11} - R_{22}X = R_{21}U - UR_{21}$$

For $Y \in \mathbb{Q}[U]$:

$$Y\big(\underbrace{R_{11} - R_{22}X}_{\in \mathbb{Q}[U]}\big)Y = \underbrace{Y\big(R_{21}U - UR_{21}\big)Y}_{\substack{\shortparallel \\ YR_{21}UY - YUR_{21}Y}}$$

$$\underbrace{\begin{matrix} \shortparallel \\ \cancel{YR_{21}UY - UYR_{21}Y} \\ \cancel{YR_{21}YU - YUR_{21}Y} \end{matrix}}_{\text{trace} = 0}$$

since $UY = YU$

For this U,
$W \in \mathbb{Q}[U]$
and $\mathrm{trace}(YWY) = 0$
$\forall Y \in \mathbb{Q}[U]$
$\Longrightarrow W = 0$

Then $\widehat{R_{22}X} \in \mathrm{Image}(\rho)$
and $\widehat{R_{22}} \in \mathrm{Image}(\rho) \Rightarrow \hat{X} \in \mathrm{Image}(\rho)$ $\not\#$.

$\boxed{\text{Main Problem}}$ Find a reasonable <u>sufficient</u> <u>condition</u> (C) on primitive A, B such that

$$A \underset{\mathbb{Z}}{\overset{SSE}{\sim}} B \quad + (C)$$

$$\Rightarrow \quad S_A \cong S_B$$

The SSE($\mathbb{Z}$) space seems to be a good framework.

Another candidate is the "positive K-theory" analogue (see B-Wagoner).

<u>Literature</u>: start with

- Kim & Roush Annals Math,
- Wagoner's survey article

and work back.

SSE(Λ) again

For a vertex (matrix) A in SSE(Λ) let $\mathcal{C}_{A,\Lambda}$ be the connected component of SSE(Λ) containing A. Let

$$G = \pi_1(\mathcal{C}_{A,\Lambda}; A) = \pi_1(SSE(\Lambda); A).$$

Then $\mathcal{C}_{A,\Lambda}$ is a classifying space for G (unique up to homotopy).

$\Lambda = \{0,1\} \longrightarrow G \cong \mathrm{Aut}(S_A)$

$\Lambda = \mathbb{Z}_+ \longrightarrow G \cong \mathrm{Aut}(S_A) / \text{simple autos}$

$\Lambda = \mathbb{Z} \longrightarrow G \cong \mathrm{Aut}(\hat{A})$

The map $\{0,1\} \hookrightarrow \mathbb{Z}$ induces

$$SSE(\{0,1\}) \longrightarrow SSE(\mathbb{Z})$$

and

$$\pi_1(SSE(\{0,1\}); A) \to \pi_1(SSE(\mathbb{Z}); A)$$

which is a version of the dimension representation