SOME OPEN PROBLEMS IN SYMBOLIC DYNAMICS COPENHAGEN JUNE 2008 LECTURE 1: BACKGROUND

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Contents

1. Introduction

This is the first of 8 lectures presenting some open problems in symbolic dynamics. They are a small selection from the open problems from the long paper [4]. Most of our time will be spent on statements and context. There won't be many proofs.

Our main aim in the first four lectures will be get an idea of the state of the art for the classification problem for $\mathbb{Z}$ shifts of finite type ($\mathbb{Z}$ SFTs).

Lecture I covers elementary background, definitions and statement of algebraic invariants. This is mostly a truncated version of parts of Lectures I and II in [2], which contains more detail and proofs. Other related short topical surveys of mine are [1, 3]. All of these except [1] are on my website. A thorough introduction to the symbolic dynamics around SFT's is the book of Lind and Marcus [5]. This book is very clear and gives attribution and history. The references [3, 5] have extensive bibliographies.

2. General Subshifts

2.1. **Dynamical Systems.** For the purposes of these lectures, a topological dynamical system will be a continuous map T from a compact metric space X into itself. We can represent this as (X, T) or just T. Apart from occasional remarks, T will be a homeomorphism. I may use a letter like T to denote the map or its domain, by context.

2.2. **Full Shifts.** The system which is the full shift on n symbols (also called the n-shift) is defined as follows. We give a finite set of n elements — say, $\{0, 1, ..., n-1\}$ — the discrete topology. (This finite set is often called the *alphabet*.) We let X be the product of countably many copies of this set, with the copies indexed by $\mathbb{Z}$. We think of an element x of X as a doubly infinite sequence

$$x = ...x_{-1}x_0x_1...$$

where each x_i is one of the n elements. X is given the product topology and thus becomes a compact metrizable space. A metric compatible with the topology is given by defining, when x is not equal to y,

$$\text{dist}(x, y) = 2^{-k}, \qquad \text{where } k = \min\{|i| : x_i \neq y_i\}.$$

That is, two sequences are close if they agree in a large stretch of coordinates around the zero coordinate.

A finite sequence of elements of the alphabet is called a word. If W is a word of length $j - i + 1$, then the set of sequences x such that $x_i...x_j = W$ is called a cylinder set. The cylinder sets are closed and open, and they give a basis for the product topology on X. Thus X is zero dimensional.

There is a natural shift map S sending X into X, defined by shifting the index set by one: $(Sx)_i = x_{i+1}$. (This is the "dynamics" in symbolic dynamics.) It is easy to see that S is bijective, S sends cylinders to cylinders, and thus S is a homeomorphism. The full shift on n symbols is the system (X, S).

2.3. **Subshifts.** A subshift (or just shift) is a subsystem of some full shift (X, T) on n symbols. This means that it is a homeomorphism obtained by restriction of T to some compact subset Y invariant under the shift and its inverse. The complement of Y is open and is thus a union of cylinder sets. Because Y is shift invariant, it follows that there is a (countable) list of words such that Y is precisely the set of all sequences y such that for every word W on the list, for every $i \leq j$, W is not equal to $y_i ... y_j$. That is, Y is the subset of all sequences which avoid the forbidden words.

Concisely: any subshift may defined by excluding a countable collection of words.

If Y is a set which may be obtained by forbidding a finite list of words, then the subshift is called a subshift of finite type, or just a shift of finite type (SFT).

2.4. **Examples.** Let S be the subshift defined by restricting the two-shift to the set Y of sequences in which the word 00 never occurs. (That is: S is the subshift of the 2-shift defined by excluding the word 00.) Then S is SFT.

Let T be the subshift defined by excluding all words 100...01 where the number of zeros is odd. Then T is a subshift, and it is not SFT. (T is the "even system" of B. Weiss.)

3. Block Codes

3.1. **Homomorphisms of subshifts.** Suppose (X, S) and (Y, T) are subshifts. A map f from X to Y is a homomorphism (of topological dynamical systems) if it is continuous and intertwines the shifts, i.e. $fS = Tf$. If the homomorphism f is surjective, then it is called a quotient map or factor map or epimorphism of subshifts. If it is injective, then it is called an embedding or monomorphism of subshifts. If it is injective and surjective, then it is an isomorphism or conjugacy of subshifts. This notion of isomorphism is our fundamental equivalence relation.

3.2. **Block Codes.** Now suppose F is a function from words of length $2n+1$ which occur in S-sequences into some finite set A. This function F gives a rule for taking an input sequence x of S and producing an output sequence fx. The sequence fx is determined by defining each of its coordinate symbols $(fx)_i$ by the rule

$$(fx)_i = F(x_{i-n} \cdots x_{i+n}).$$

It is easy to see that the map f produced in this way will be continuous and commute with the shift. That is, such a map f defines a homomorphism (called a block code, or sliding block code, or just code) from S into the full shift on the alphabet A (or into any subshift which contains the image of f).
Example. The shift map itself is given by the code $(fx)_i = x_{i+1}$. Here $F(x_{i-n} \cdots x_{i+n}) = F(x_{i-1}x_ix_{i+1}) = x_{i+1}$. Example. Let S be the twoshift and $F(ijk) = j + k \pmod 2$, that is $(fx)_i = x_i + x_{i+1} \pmod 2$. Then at a sample point x we see

$$\begin{array}{lll} x & = \dots x_0x_1 \dots & = \dots 01101011\dots \\ fx & = \dots (fx)_0(fx)_1 \dots & = \dots 1011110\ \dots \end{array}$$

It is often convenient to abuse notation and use the same symbol for the maps we call f and F above.

The "Curtis-Hedlund-Lyndon Theorem" asserts that every homomorphism of subshifts is a block code.

3.3. **Higher Block Presentations.** A subshift S is isomorphic to many different subshifts. For example S is isomorphic to its "n-block presentation" $S^{[n]}$ in which symbols are grouped in blocks of size n. A word of length n for S becomes a symbol in the alphabet for the n-block presentation. For example with $n = 3$ a point x in S corresponds to a point y in its 3-block presentation as follows:

$$\begin{array}{lll} x & = \dots x_0x_1 \dots & = \dots 01230\dots \\ y & = \dots y_0y_1 \cdots = \dots [x_0x_1][x_1x_2] \dots & = \dots [012][123][230]\ \dots \end{array}$$

4. Edge Shifts of Finite Type

4.1. **Edge Shifts.** Let A be an adjacency matrix for a directed graph (if A is $n \times n$, then the graph has n vertices, with $A(i, j)$ from vertex i to vertex j. Let the set of edges be the alphabet. Let Y be the set of sequences y such that for all k, the terminal vertex of y_k is the initial vertex of y_{k+1}. We can think of Y as the space of doubly infinite walks through the graph, presented by the edges traversed. The shift map restricted to Y is an *edge SFT*; the map (and by context its domain) is denoted S_A. Any SFT is isomorphic to an edge shift.

Remark: the edge SFT S_{A^n} is conjugate to $(S_A)^n$ (where S^n denotes the homeomorphism obtained by iterating S n times).

4.2. **Matrices.** Because any SFT is isomorphic to an edge SFT S_A, all information about the dynamics of S_A is determined by the matrix A.

Note: matrices of arbitrarily large size can define isomorphic SFTs. For example, the n-block presentation of an edge shift S_A will be an edge shift S_B; if S_A contains infinitely many points, then the size of B goes to infinity as n goes to infinity.

5. Matrix Invariants for SFTS

There is a dictionary between various dynamical properties of and edge SFT S_A and properties (mostly algebraic) of the matrix A. To avoid trivialities, A below is assumed to be nondegenerate (no zero row and no zero column).

	Dynamics of S_A	**Matrix A**
2.	Transitivity	[irreducible]
3.	Mixing	[primitive]
4.	Entropy	[spectral radius]
5.	$\lvert \mathrm{Fix}(S_A)^n \rvert)$	[trace of A^n]
6.	Zeta function	[nonzero spectrum]
7.	Isomorphism	[SSE]
8.	Eventual isomorphism	[SE]
9.	Flow equivalence	[cok$(I - A)$ + sign]

5.1. **Nonnegative Matrices.** The matrix A is *irreducible* if for every (i, j) there exists $n > 0$ such that $A(i, j) > 0$. (This n can depend on (i, j) — consider a cyclic permutation matrix.) The matrix A is *primitive* if there exists $n > 0$ such that A^n has all entries strictly positive. The matrix A is *reducible* if it is not irreducible.

Most problems about SFTs S_A can be reduced easily to problems for the case that A is primitive (i.e. S_A is mixing). This is analogous to the situation with nonnegative matrices, which one understands by first understanding the primitive case.

5.2. **Entropy.** The premier numerical invariant of a dynamical system S is its (topological) entropy, $h(S)$. For a subshift S,

$$h(S) = \limsup_n \frac{1}{n} \log(\#W_n(S))$$

where $W_n(S)$ is the set of words of length n occuring in sequences of S. That is, the entropy is the exponential growth rate of the S-words.

For a full shift on n symbols, the entropy is $\log(n)$. For an SFT S_A,

$$h(S) = \log(\lambda_A)$$

where λ_A is the spectral radius of A. This follows from the spectral radius theorem because the number of words of length n is the sum of the entries of A^n, which is the L^1 norm of A^n.

5.3. **Isomorphism.** The basic, fundamental result, due to Williams, is that S_A and S_B are isomorphic if and only if the matrices A and B are strong shift equivalent over $\mathbb{Z}_+$. This is completely general, nondegeneracy of A and B need not be assumed.

Let $\mathcal{S}$ be a subset of a semiring containing 0 and 1. (E.g. $\mathcal{S}$ could be $\mathbb{Z}, \mathbb{Z}_+, \mathbb{Q}, \dots$; usually $\mathcal{S}$ will be $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. Matrices A and B are elementary strong shift equivalent over $\mathcal{S}$, i.e.

$$A \ \sim_{\mathcal{S}}^{\text{ESSE}} \ B \quad ,$$

if there are matrices U, V with entries from $\mathbb{Z}_+$ such that $A = UV$ and $B = VU$. For example, we have

$$\begin{pmatrix} 2 \end{pmatrix} \ \sim_{\mathbb{Z}_+}^{\text{ESSE}} \ \begin{pmatrix} 1, 1 \\ 1, 1 \end{pmatrix} \ \sim_{\mathbb{Z}_+}^{\text{ESSE}} \ \begin{pmatrix} 1, 1, 0 \\ 0, 0, 1 \\ 1, 1, 1 \end{pmatrix}$$

because

$$\begin{pmatrix} 2 \end{pmatrix} = \begin{pmatrix} 1, 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1, 1 \\ 1, 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1, 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1, 1 \\ 1, 1 \end{pmatrix} = \begin{pmatrix} 1, 1, 0 \\ 0, 0, 1 \end{pmatrix} \begin{pmatrix} 1, 0 \\ 0, 1 \\ 1, 1 \end{pmatrix}$$

$$\begin{pmatrix} 1, 1, 0 \\ 0, 0, 1 \\ 1, 1, 1 \end{pmatrix} = \begin{pmatrix} 1, 0 \\ 0, 1 \\ 1, 1 \end{pmatrix} \begin{pmatrix} 1, 1, 0 \\ 0, 0, 1 \end{pmatrix} \quad .$$

NOTE: the relation ESSE is not transitive (for example the matrix $\begin{pmatrix} 2 \end{pmatrix}$ cannot be be ESSE over $\mathbb{Z}_+$, or even over $\mathbb{Q}$) to a matrix of rank greater than 1). So we define matrices A and B to be *strong shift equivalent over* $\mathcal{S}$,

$$A \ \sim_{\mathcal{S}}^{\text{SSE}} \ B \quad ,$$

if they are linked by a finite chain of elementary strong shift equivalences. Thus

$$(2) \sim_{\mathbb{Z}_+}^{\mathrm{SSE}} \begin{pmatrix} 1,1,0 \\ 0,0,1 \\ 1,1,1 \end{pmatrix}$$

Theorem 1. *(Williams 1973) Let A, B be square matrices over $\mathbb{Z}_+$. The following are equivalent.*

(1) *The edge SFTs S_A and S_B are conjugate.*
(2) *A and B are SSE over $\mathbb{Z}_+$.*

We will prove (at the board) the easy implication, (2) $\implies$ (1). To every SSE over $\mathbb{Z}_+$,

$$A = UV \quad , \quad B = VU$$

we will associate a topological conjugacy $c(U,V)$ from S_A to S_B. This conjugacy $c(U,V)$ will be uniquely determined by the matrices U,V when U,V have all entries in $\{0,1\}$. In general, $c(U,V)$ will be uniquely determined up to composition by a one-block map defined by a graph automorphism of leaving all vertices fixed.

Note: the inverse of the homeomorphism $c(U,V)$ is not $c(V,U)$. The composition $c(U,V)$ followed by $c(V,U)$ equals S_A.

Theorem 2 (Decomposition Theorem). *Every conjugacy of edge SFTs is a composition of conjugacies of the form $c(U,V)$ and $(c(U,V))^{-1}$.*

There is a very clear proof of the Decomposition Theorem in Lind and Marcus [5].

5.4. **Difficulties with strong shift equiuvalence.** Strong shift equivalence over $\mathbb{Z}_+$ is easy to define, but it is very difficult to understand completely.

Open Problem 1. *Does there exist an algorithm which, given square matrices A, B over $\mathbb{Z}_+$, decides whether their edge SFTs are topologically conjugate?*

So, Williams introduced another matrix relation, shift equivalence.

5.5. **Shift equivalence.** Matrices A and B are *shift equivalent over $\mathcal{S}$* (SE-$\mathcal{S}$) if there are matrices U,V over $\mathcal{S}$ and a positive integer ℓ (called the lag) such that the following equations hold.

$$\begin{aligned} A^\ell &= UV & \qquad B^\ell &= VU \\ AU &= UB & \qquad BV &= VA. \end{aligned}$$

We will understand below the goodness and tractability of shift equivalence.

Conjecture 1. *(Williams 1974) Shift equivalence over $\mathbb{Z}_+$ implies strong shift equivalence over $\mathbb{Z}_+$.*

This conjecture was finally disproved in the 90's. We will see how. Shift equivalence remains fundamental to attacking the classification problem. The question of when shift equivalence implies strong shift equivalence is still VERY poorly understood. For example:

Open Problem 2. *Suppose A is a square matrix over $\mathbb{Z}_+$ with just one eigenvalue, n, and this eigenvalue is a simple root of the characteristic polynomial. Then A is SE over $\mathbb{Z}_+$ to the one-by-one matrix $\big(n\big)$. Must A be SSE over $\mathbb{Z}_+$ to the one-by-one matrix $\big(n\big)$?*

Open Problem 3. *Let A be "Ashley's eight-by-eight", the 8×8 matrix which is the sum of the permutation matrices for the permutations which in cycle notation are (12345678) and (1)(2)(374865). Then A is SE-$\mathbb{Z}_+$ to $[2]$. Is A SSE-$\mathbb{Z}_+$ to $[2]$?*

Open Problem 4. *Suppose A and B are 2×2 and SE over $\mathbb{Z}_+$ with* $\det A = \det B < -1$. *Must A and B be SSE over $\mathbb{Z}_+$?*

Open Problem 5. *Given $k \geq 2 \in \mathbb{N}$, define the matrices*

$$A_k = \begin{pmatrix} 1 & k \\ k-1 & 1 \end{pmatrix} \qquad \text{and} \qquad B_k = \begin{pmatrix} 1 & k(k-1) \\ 1 & 1 \end{pmatrix} .$$

Then A_k is SE-$\mathbb{Z}$+ to B_k, but for $k \geq 4$ it is not known if A_k is SE-$\mathbb{Z}$+ to B_k [5, Example 7.3.13]

6. Shift equivalence is good

Immediate remarks:

- SE (unlike ESSE) is an equivalence relation.
- There is an algorithm known (Kim and Roush) which decides whether two matrices are shift equivalent over $\mathbb{Z}_+$.
- There is a dynamical characterization of shift equivalence ("eventual isomorphism").
- Shift equivalence is meaningful, tractable and useful, particularly in its "dimension" manifestation.

Two systems S and T are *eventually isomorphic* if S^n and T^n are isomorphic for all but finitely many n.

Theorem 3. *For matrices A and B over $\mathbb{Z}_+$, the following are equivalent.*

(1) *The edge SFTs S_A and S_B are eventually isomoprhic.*
(2) *A and B are SE over $\mathbb{Z}_+$.*

We'll prove the easy implication, (2) $\implies$ (1). Given the equations above, we see that A^ℓ and B^ℓ are strong shift equivalent. Moreover, so are higher powers, since for any positive integer n,

$$U(VA^n) = A^{\ell+n} \qquad \text{and} \qquad (VA^n)U = (B^nV)U = B^{\ell+n} \ .$$

Let us appreciate the tractability of shift equivalence.:

(1) For primitive matrices, SE-$\mathbb{Z}$ implies SE-$\mathbb{Z}_+$.
(2) SE-$\mathbb{Z}$ implies SSE-$\mathbb{Z}$.
(3) A square matrix over $\mathbb{Z}$ is SSE-$\mathbb{Z}$ to a nonsingular matrix.

What is particularly important is that for primitive matrices the relations SE-ZZ_+, SE-ZZ, SSE-ZZ_+ are all the same.

Now some finer points for the algebraically inclined. In the list of implications above:

- in (1), $\mathbb{Z}$ can be replaced by any unital subring of the reals.
- in (2), $\mathbb{Z}$ can be replaced by any Dedekind domain.
- in (3), $\mathbb{Z}$ can be replaced by any principal ideal domain.

Open Problem 6. *For what rings $\mathcal{S}$ does SE over $\mathcal{S}$ imply SSE over $\mathcal{S}$? Does this implication hold for every integral domain with finite cohomological dimension?*

7. The meaning of SSE over a ring

Let $\mathcal{S}$ be a ring (by which we always mean a ring with 1). Square matrices A, B are similar over $\mathcal{S}$ (SIM-$\mathcal{S}$) if there is an invertible U over $\mathcal{S}$ such that $A = UBU^{-1}$.

The equivalence relation on square matrices given by SSE- $\mathcal{S}$ is a kind of stabilized version of the relation SSE-$\mathcal{S}$.

Proposition 4. *Let $\mathcal{S}$ be a ring (by definition, containing 1). The equivalence relation SSE- $\mathcal{S}$ is the equivalence relation $\sim$ on square matrices over $\mathcal{S}$ which is generated by SIM-$\mathcal{S}$ and the following relations among square matrices with the given block forms:*

$$A \sim \begin{pmatrix} A, P \\ 0, 0 \end{pmatrix} \sim \begin{pmatrix} 0, Q \\ 0, A \end{pmatrix}$$

Above, the zero blocks and the blocks P, Q need not be square. The Proposition's easy-to-check proof is spelled out in [3]. We remark that $\sim$ above also satisfies

$$A \sim \begin{pmatrix} A, 0 \\ R, 0 \end{pmatrix} \sim \begin{pmatrix} 0, 0 \\ S, A \end{pmatrix} \ .$$

Every singular square matrix over a PID $\mathcal{S}$ is similar over $\mathcal{S}$ to a matrix in block triangular form with bottom row zero, and thus is

SSE–$\mathcal{S}$ to a smaller matrix. Thus over a PID (such as $\mathbb{Z}$ or any field), every square matrix A is SSE–$\mathcal{S}$ to some nonsingular matrix B, and the SSE–$\mathcal{S}$ of A can be identified with the similarity class of B.

8. Dimension Modules

Let A be an $n \times n$ matrix over $\mathbb{Z}$. Our convention is to let matrices act on row vectors, and thus write composition of matrix maps left to right. Define

$$\begin{aligned} V_A &= \cap_{k>0}\{vA^k : v \in \mathbb{Q}^n\} \\ &= \{vA^n : v \in \mathbb{Q}^n\} \ = \mathbb{Q}^n A^n \\ G_A &= \{v \in V_A : \exists k > 0 \text{ such that } vA^k \in \mathbb{Z}^n\} \ . \end{aligned}$$

When A is an $n \times n$ matrix over $\mathbb{Z}_+$, we also define

$$G_A^+ = \{v \in V_A : \exists k > 0 \text{ such that } vA^k \geq 0\} \ .$$

• Here A maps $\mathbb{Z}^n$ to $\mathbb{Z}^n$ and G_A is simply a concrete presentation of the associated direct limit group $\mathbb{Z}^n \to \mathbb{Z}^n \to \mathbb{Z}^n \cdots$.

• For A over $\mathbb{Z}_+$, G_A is an ordered group, with G_A^+ as its positive set. An ordered group (G_A, G_A^+) of this form is called a *stationary dimension group.*

• Note: for $k > 0$, $G_A = G_{A^k}$ and $G_A^+ = G_{A^k}^+$. .

• Let $\hat{A}$ denote the isomorphism $G_A \to G_A$ given by $v \mapsto vA$. The direct limit module of A is given by the pair $(G_A, \hat{A})$.

• For square matrices A, B over $\mathbb{Z}$, the following are equivalent:

(1) A and B are SE-$\mathbb{Z}$.
(2) $(\hat{A}, G_A) \cong (\hat{B}, G_B)$, meaning there is a group isomorphism $\psi : G_A \to G_B$ such that $\hat{A}\psi = \psi\hat{B}$.

If A and B are SSE-$\mathbb{Z}$ via $A = RS, B = SR$, then $w \mapsto wR$ defines a group isomorphism $\hat{R}$ such that $\hat{R}\hat{B} = \hat{A}\hat{R}$. I.e. $\hat{R}$ is an isomorphism of modules, $(G_A, \hat{A}) \to (G_B, \hat{B})$. Every module isomorphism $(G_A, \hat{A}) \to (G_B, \hat{B})$ arises as $\hat{R}$ for some matrix R over $\mathbb{Q}$ such that $AR = RB$. Here, when A and B are primitive, $\hat{R}$ sends G_A^+ onto G_B^+ if and only if R sends a positive eigenvector for A to a positive eigenvector for B.

• For A over $\mathbb{Z}_+$, the *dimension module* of A is given by the triple

$(G_A, G_A^+, \hat{A})$. (Formally, (G_A, G_A^+) is an ordered module for the ordered Laurent ring $(\mathbb{Z}[t,t^{-1}], \mathbb{Z}_+[t,t^{-1}])$, where the action of t is by $\hat{A}^{-1}$.) Matrices are SE-$\mathbb{Z}_+$ if and only if they have isomorphic dimension modules. Matrices are SE-$\mathbb{Z}$ if and only if they have isomorphic direct limit modules.

- Examples.
  (1) If A is $n \times n$ with $|\det A| = 1$, then $G_A = \mathbb{Z}^n$.
  (2) If $A = \big(2\big)$, then $G_A = \mathbb{Z}[1/2] = \{m2^k : k \in \mathbb{Z}\}$.

References

[1] M. Boyle, *Symbolic Dynamics and Matrices*, in Combinatorial and Graph-Theoretical Problems in Linear Algebra, IMA Volumes in Math. and its Applications **50** (1993), 1-38.

[2] M. Boyle, *Algebraic aspects of symbolic dynamics*, in Topics in symbolic dynamics and applications (Temuco, 1997), London Math. Soc. Lecture Note Ser.**279** (2000), 57-88, Cambridge Univ. Press, Cambridge.

[3] M. Boyle, *Positive K-theory and symbolic dynamics. Dynamics and randomness (Santiago, 2000)*, 31–52, Nonlinear Phenom. Complex Systems **7**, Kluwer Acad. Publ., Dordrecht, 2002.

[4] M. Boyle, Open problems in symbolic dynamics, Contermporary Math., to appear.

[5] D. Lind and B. Marcus, *An introduction to Symbolic Dynamics and Coding*, Cambridge University Press (1995).