Some problems and progress in symbolic dynamics (over the last 30 years)

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ETDS is thirty

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September 2010

This will be a thin slice.

For more, see on my website
"Open problems in symbolic dynamics".

1. Some definitions.

A square matrix A with nonnegative integer entries defines a shift of finite type (SFT). This dynamical system is the shift map S_A on a compact metrizable space of bisequences representing itineraries through a directed graph.

When A is nondegenerate, S_A is mixing iff A is primitive. The mixing case is the fundamental case.

DEFN. Square matrices A, B over Z_+ are ESSE(Z_+) if there are matrices R, S over Z_+ such that $A = RS, B = SR$.

Strong shift equivalence (SSE) is the transitive closure of ESSE.

THM (Williams) SFTs S_A, S_B are topologically conjugate iff A, B are SSE(Z_+).

Williams approached strong shift equivalence with a more tractable relation, shift equivalence.

DEFN Square matrices A, B are SE(Z_+) if there exist R, S over Z_+ and a positive integer ℓ such that $A^\ell = RS, B^\ell = SR, AR = RB, BS = SA$.

As it turns out, the relation SE(Z_+) is understandable, useful and decidable.

DEFN An automorphism of an SFT S is a self conjugacy (a homeomorphism h with $hS = Sh$). Aut(S) is the group of automorphisms of S.

2. The Classification Problem for SFTs

Conjecture (1974, Williams) If A and B are SE(Z_+), then they are SSE(Z_+).

Counterexamples (Kim-Roush):
1992: A, B reducible
1999: A, B primitive

The counterexamples use $\mathrm{tr}(A) = \mathrm{tr}(A^2) = 0$. Williams' Conjecture might be true for matrices defining mixing SFTs with points of all periods... and possibly there does not exist an algorithm which decides whether two matrices are SSE-Z_+.

We seem stalled on this mystery.

With matrices over other semirings, there are SSE/SE classification theories for Markov shifts, G-shifts, sofic shifts ... they are stalled until the fundamental SFT case is resolved.

3. The Extension Problem for mixing SFTs

Extension Problem. When does an automorphism of a subshift of a mixing SFT S extend to an automorphism of S?

This grew out of a question of Williams (late 70s): if a mixing SFT S_A has two fixed points, must there be an automorphism of S_A which exchanges them?

For Williams, the Question was a test for his shift equivalence conjecture: perhaps the answer would depend on more than the shift equivalence class of A. (That now seems unlikely.)

Still, tools developed for the Extension Problem were critical to the Kim-Roush counterexamples.

If a problem is hard, that might be one reason to try to solve it.

4. Two ingredients for solutions

I. Dimension representation (Krieger)

Given A $n \times n$ over Z_+, let G_A be the direct limit group

$$G_A = Z^n \to Z^n \to Z^n \to \dots .$$

G_A has a natural positive set with which it is an ordered group (more precisely, a stationary dimension group). The automorphism of G_A induced by A makes it a module (the "dimension module").

A topological conjugacy $\phi : S_A \to S_B$ induces an isomorphism $\widehat{\phi}$ of dimension modules.

When $A = B$, the rule $\phi \mapsto \widehat{\phi}$ defines a homomorphism ρ_A from $\mathrm{Aut}(S_A)$ to the group $\mathrm{Aut}(G_A)$ of automorphisms of the dimension module. ρ_A is called the dimension representation of $\mathrm{Aut}(S_A)$.

$\mathrm{Aut}(G_A)$ can be analyzed and often the image of ρ_A in $\mathrm{Aut}(G_A)$ is easily understood.

$\mathrm{Ker}(\rho_A)$ is a large, complicated, still mysterious countable group.

II. Sign-gyration homomorphism (B-Krieger, Kim-Roush-Wagoner)

Given n, pick points x_i, one from each S-orbit of cardinality n. For some permutation π, $U(x_i) = S^{n(i)}x_{\pi(i)}$. Define

sign$_n(U)$:= 1 if the permutation is even,
sign$_n(U)$:= -1 if it is odd.
gy$_n(U)$:= $\sum n(i) \in Z/n$.

Define the sign-gyration homomorphism
SG$_n$: Aut$(S) \to Z/n$ by
SG$_n$ = gy$_n$ + $(n/2)\sum_k$ sign$_{n/2^k}$
where the summation is over the positive integers k such that $n/2^k$ is an integer. E.g.,

SG$_{12}$ = gy$_{12}$ + (12/2)[sign$_6$ + sign$_3$]
SG$_3$ = gy$_3$.

SG$_n$ is a group homomorphism.

The product of these is SG, the sign-gyration homomorphism.

5. Solution of the heart of the extension problem

Theorem (KRW, 1992) SG(U) is determined by $\rho_A(U)$. If U is in Ker(ρ_A), then SG(U)=0.

Theorem By constructions, vanishing of SG is the only obstruction to extending an automorphism U' of a subsystem of S_A to an element of Ker(ρ_A).

(For a given U', the obstruction can be checked by the action of U' on finitely many periodic points.)

The constructions involved a number of people over the years. The final and hardest construction was due to KRW (2000).

6. Range of the dimension representation

For a mixing SFT S_A, which automorphisms of its dimension module are induced by automorphisms of S_A?

Solving this problem, and the classification problem for mixing SFTs, is equivalent to solving the classification problem for general SFTs. (Kim-Roush)

Solving this problem (which is understood in many examples) would also suffice to finish off the Extension Problem.

Kim, Roush and Wagoner showed that in some cases ρ_A is not surjective.

7. Cartoon outline of the counterexample scheme

a. Extend the definition of SG to topological conjugacies (not just automorphisms) defined by a SSE-Z_+ from A to B.

For this, to each A assign an explicit S_A. For each S_A and n, choose explicit ordered sets of representatives of periodic orbits of length n (with the ordering, regard the k orbits of length n as simply $\{1, ..., k\}$). To each elementary SSE(Z_+) associate an explicit conjugacy, so the map of length n orbits can be regarded as a map from $\{1, .., k\}$ to itself, i.e. a permutation. Now compute with the same formulas as before.

b. Show for an SSE-Z_+ from A to B that the induced isomorphism of dimension modules $G_A \to G_B$ determines the SG for the SSE (!!), by explicit formulas.

c. Arrange an example for which S_A and S_B have no points of period 1 or 2, but the only SG_2 possible (by considering possible dimension module isomorphisms) is nonzero. Contradiction.

The nonsurjectivity examples for the dimension representation have the same pattern – a certain action on dimension could only be induced by an automorphism with a nontrivial SG_2 (but the example S_A has no points of period 1 or 2).

8. Wagoner's Strong Shift Equivalence Spaces

Wagoner topologized the strong shift equivalence relation (1987-1990).

Given a subset Λ of a semiring, containing $\{0, 1\}$, he built a certain oriented CW complex, visualized as an infinite simplex.

A vertex (0-cell) is a square matrix over Λ.

An edge (1-cell) from vertex A to matrix B is a pair of matrices (R, S) over Λ such that $A = RS, B = SR$.

A path along edges from A to B corresponds to a strong shift equivalence from A to B.

A 2-cell is a triangle whose edges satisfy certain matrix equations (the "Triangle Identities").

Two paths along edges in SSE($\{0,1\}$) are homotopic (fixing endpoints) iff their associated SSE's define the same topological conjugacy.

Two paths along edges in SSE(Z_+) from A to B are homotopic (fixing endppoints) iff (modulo "simple" automorphisms of the shift) they define the same topological conjugacy from S_A to S_B,

Two paths in SSE(Z) from A to B are homotopic iff they define the same map $G_A \to G_B$.

The Kim-Roush-Wagoner obstructions are proved in the setting of Wagoner's SSE spaces.

9. Positive K-theory

(Kim Roush Wagoner; B-Wagoner)

To show SG(U') = 0 is the only obstruction to extending an automorphism U' of a subsystem of S_A to an element of Ker(ρ_A), it was necessary to produce constructions. These developed in papers by several authors. The hardest construction (KRW,2000) introduced "positive K-theory". Here (skipping a technical detail):

(i) an SFT S_A now is represented by a matrix A with entries in $tZ_+[t]$. A has infinitely many rows and columns, but only finitely many entries can be nonzero.

(ii) If A, B are such matrices and E is a basic elementary matrix over $tZ_+[t]$ and $E(I-A) = I-B$, or $(I-A)E = B$, then there is an associated topological conjugacy $S_A \to S_B$

(iii) All conjugacies $S_A \to S_B$ are compositions of such elementary conjugacies,
$S_A \to S_{A_1} \to S_{A_2} \to \cdots \to S_B$

This gives another approach to construction, obstruction and classfication. It is more functorial than the SSE setting. Like the SSE setup, it generalizes to other kinds of symbolic systems. Positive K-theory has been especially useful for studying flow equivalence of SFTs, G-SFTs and sofic shifts.

10. Why "Positive K-Theory"?

For a ring R, GL(R) is the set of infinite invertible matrices over R which are equal to I in all but finitely many entries. The map from GL(R) to $K_1(R)$ has kernel generated by the basic elementary matrices. So, two elements of GL(R) represent the same class in $K_1(R)$ iff they are in the same double coset under the action by basic elementary matrices.

Similarly, for the ring $R = Z[t, t^{-1}]$, if matrices $I - A$ and $I - B$ over R present SFTs as before, then conjugacy of the SFTs requires that $I - A$ and $I - B$ be in the same double coset under the action by basic elementary matrices.

Two features distinguish the positive-K equivalence relation on the $I-A$ and the K_1 equivalence relation. K_1.

1. $I-A$ is not invertible.
2. Let M be the set of matrices $I-A$ defining SFTs. Elements $I-A$, $I-B$ define conjugate SFTs iff they are in the same double coset under multiplication by basic elementary matrices, where each INDIVIDUAL multiplication must take a matrix in M to a matrix in M.

The matrices $I-A$ and $I-B$ are equivalent by products of elementary matrices over R

$$U(I-A)V = I-B,$$

iff integer matrices presenting the SFTs as edge SFTs are shift equivalent over Z, which in the mixing SFT case is equivalent to shift equivalence over Z_+.

11. Aut(S), S a mixing SFT

Study of Aut(S) started with Hedlund (1969). Aut(S) is countably infinite and residually finite, contains copies of free groups, all locally finite countable groups and many others.

Its finitely generated subgroups have solvable word problem.

Ryan's Theorem: the center of Aut(S) is the set of powers of S.

Aut(S) has many normal subgroups. But apart from Ryan's Theorem, we know only two ways to compute normal subgroups: through the dimension represenation, or by actions on finite subsystems.

And that's what we know about the algebraic structure of Aut(S).

How much algebraic structure are we missing? It is another longstanding mystery.

Open: are the automorphism groups of the 2-shift and 3-shift isomorphic?

On the Math ArXiv (24 June 2010):
"The restricted Weyl group of the Cuntz algebra and shift endomorphisms",
by Conti, Hong and Szymanski.

For S a full shift on a prime number of symbols, they show Aut(S) mod its center is isomorphic to a certain subgroup of interest in the group of outer automorphisms of the associated Cuntz C^*-algebra.

Could this possibly help?

11. When is a mixing SFT S_B a factor of a mixing SFT S_A ?

Unequal entropy case (B, 1983 ETDS): iff
(*) for all n, $\mathrm{tr}(A^n) > 0 \implies \mathrm{tr}(B^n) > 0$.

Equal entropy case:
last big result was (Ashley, 1991 ETDS):
For mixing SFTs S_A,S_B of equal entropy, TFAE.
1. There is a right closing factor map from S_A onto S_B
2. (*) holds, and the dimension module G_B is a quotient of G_A.

Conjectured (1993):
For mixing SFTs S_A,S_B of equal entropy, TFAE.
1. There is a factor map from S_A onto S_B.
2. (*) holds, and the dimension module G_B is a quotient of a closed (pure) submodule of G_A.

The condition (2) in the conjecture is known to be a necessary condition for the factor map, by Kitchens-Marcus-Trow (1991).

Again, we seem to be stalled.

12. Z^d shifts of finite type

The world of multidimensional SFTs (Z^d SFTs, $d > 1$) is vastly richer and more varied than Z-SFTs. We are still exploring the landscape.

There is a deep and satisfactory theory of Z^d SFTs given as automorphisms of compact abelian groups (Schmidt + ...). For general Z^d SFTs, the algebraic structure central to the Z case disappears.

Long appreciated: “typically”, if there is a property of interest for Z^d SFTs ($d > 1$), an algorithm to decide it in general won’t exist.

Recently, systematic constructions have led to Z^d SFT results of the following flavor: a general recursion theoretic obstruction is the only obstruction to realization of some phenomena. Quickly, two examples:

Example 1: (Hochman-Meyerovitch) TFAE for a real number α.
1. α is the Z^d entropy of a Z^d SFT.
2. There is a Turing machine which produces a decreasing sequence of positive rational numbers α_n such that $\lim \alpha_n = \alpha$.

Example 2 (Hochman) Supose $d > 2$, d an integer, α a real number. TFAE.
1. α is the entropy of a d-dimensional c.a.
2. There is a Turing machine which produces a sequence of positive rational numbers α_n such that $\liminf \alpha_n = \alpha$.

There is a more done and continuing. It is a good period for multidimensional symbolic dynamics.

9. Nasu's Textile Systems

This is a kind of calculus of Wang tilings which Nasu has developed to study dynamics of endomorphisms and automorphisms of SFTs (and more). For a quick-to-state example of results:

Theorem (Nasu)
Suppose U is a homeomorphism commuting with a onesided full shift on n symbols.
Then U is topologically conjugate to a two sided SFT which is shift equivalent to a full shift.
Moreover, if $\log k$ is the entropy of U, then k and n are divisible by the same primes; and if a prime p divides n, then p^2 divides k.

Nasu's machine also cranked out an example of two commuting mixing SFTs which do not have a common measure of maximal entropy, and whose defining matrices have no algebraic relations.

Conjecture (B). If S and T are nontrivial mixing SFTs, then given any sufficiently large k, m there exist commuting maps S', T' which are topologically conjugate respectively to S^k and T^m.

10. Symbolic extensions and entropy structure

Let T be a finite entropy homeomorphism of a compact metrizable space. When is T the factor a subshift – i.e when does T have a symbolic extension? What is the symbolic extension entropy $h_{\mathsf{sex}}(T)$ (:= the infimum of the entropies of such subshifts)?

To approach this question, Downarowicz [ETDS 2001] took the essential step of considering a sequence (h_n) of functions on the Choquet simplex of invariant Borel probablities for T. It turns out that $h_{\mathsf{sex}}(T)$ is computed in terms of a transfinite procedure applied to (h_n), when (h_n) is in a suitable (very large) equivalence class. Various subtle related problems have exact functional analytic characterizations.

The theory of symbolic extension entropy is at the heart of the Entropy Structure theory of Downarowicz, which unifies and extends the classical entropy theory of continuous maps on compact metric spaces, and represents some kind of rigorous theory of the emergence of complexity on refining scales.

There were other contributors to the general development (B, Fiebigs, Serafin, Weiss...), and then a large effort to understand the constraints intermediate smoothness places on $h_{\mathsf{sex}}(T)$, begun by Downarowicz and Newhouse and continued by Asaoka, Downarowicz-Maass, Burguet, Diaz-Fisher, Cowieson-Young, Diaz-Fisher-Pacifico-Vietez, Gang-Viana-Yang ...

In particular, there is support for the C^r Sex Entropy Conjecture of Downarowicz and Newhouse. This gives a formula for a finite upper bound to $h_{\mathsf{sex}}(T)$ when T is C^r, $1 < r < \infty$.

Downarowicz has written a book:

"Entropy in dynamical systems"
Cambridge University Press
(Spring 2011)

It covers the entropy structure theory and more.

This year he was awarded Poland's Banach Prize for his mathematics. His university announced it on its home page ...