Symbolic extensions of intermediate smoothness

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This talk primarily reports

[BD2] M. Boyle and T. Downarowicz, *Symbolic extension entropy: C^r examples, products and flows*, Discrete and Continuous Dynamical Systems (2006)

and also refers to the following

[A] M. Asaoka, *A simple example exhibiting C^1-persistent homoclinic tangency for higher dimensions*, preprint (2006)

[BD1] M. Boyle and T. Downarowicz, *The entropy theory of symbolic extensions*, Inventiones Math. (2004)

[BFF] M. Boyle, D. Fiebig, U. Fiebig. *Residual entropy, conditional entropy and subshift covers*, Forum Math. (2002)

[D1] T. Downarowicz, *Entropy of a symbolic extension of a dynamical system*, Erg. Th. Dyn. Syst. (2001)

[D2] T. Downarowicz, *Entropy Structure*, J. d'Analyse (2005)

[DN] T. Downarowicz and S. Newhouse, *Symbolic extensions in smooth dynamical systems*, Inventiones Math. (2005)

[M1] M. Misiurewicz, *On non-continuity of topological entropy*, Bull. Acad. Polon. Sci. (1971)

[M2] M. Misiurewicz, *Diffeomorphism without any measure with maximal entropy*, Bull. Acad. Polon. Sci. (1973)

I. Background: symbolic extensions and entropy.

- All spaces are compact metrizable.

- (X, T) denotes a homeomorphism, $T : X \to X$, with $\mathbf{h}_{\text{top}}(T) < \infty$.

- $\mathcal{M}_T$ is the space of T-invariant Borel probabilities.

- A subshift (Y, S) is the restriction of the full shift on a finite alphabet to a closed invariant subsystem.

- A *symbolic extension* of (X, T) is a subshift (Y, S) with a continuous surjection $\varphi : Y \to X$ such that $T\varphi = \varphi S$.

Definition.
The (topological) residual entropy of T is

$$\mathbf{h}_{\mathsf{res}}(T) = \inf\{\mathbf{h}_{\mathsf{top}}(S)\} - \mathbf{h}_{\mathsf{top}}(T)$$

where the inf is over the symbolic extensions of T.

Theorem. [BFF, D1]
Given $0 < \alpha < \infty$ and $0 \le \beta \le \infty$, there exists T with $\mathbf{h}_{\mathsf{top}}(T) = \alpha$, $\mathbf{h}_{\mathsf{res}}(T) = \beta$.

The intuition: $\mathbf{h}_{\mathsf{res}}(T) > 0$ reflects nonuniform emergence of entropy on refining scales.

To understand this it is essential to consider symbolic extensions in terms of invariant measures.

Extension entropy. Consider a homeomorphism T of a compact metric space X. Given a symbolic extension $\varphi : (Y, S) \to (X, T)$ define its extension entropy function

$$h^{\varphi}_{\mathrm{ext}} : \mathcal{M}_T \to [0, \infty)$$
$$\mu \mapsto \max\{h(S, \nu) : \varphi\nu = \mu\} \ .$$

Symbolic extension entropy. Given (X, T), we define its symbolic extension entropy function to be the function $h^{T}_{\mathrm{sex}} : \mathcal{M}_T \to [0, \infty]$ which is the infimum of all $h^{\varphi}_{\mathrm{ext}}$ arising from symbolic extensions φ of (X, T).
($h^{T}_{\mathrm{sex}} \equiv \infty$ if no symbolic extension exists.)

Abbreviate:
symbolic extension entropy = sex entropy.

When some symbolic extension exists, h^{T}_{sex} is a bounded function, and $h^{T}_{\mathrm{sex}}(\mu)$ gives a quantitative measure of the emergence of complexity on finer scales "near" the support of μ.

Entropy structure. An entropy structure for (X,T) is an allowed nondecreasing sequence of nonnegative functions h_n on $\mathcal{M}_T$, converging to the entropy function h.

Example of an entropy structure.
Suppose the system (X,T) admits a refining sequence of partitions P_n with *small boundaries* (the boundary of the closure of each partition element has μ-measure zero for every μ in $\mathcal{M}_T$), and with the maximum diameter of elements of P_n going to zero as $n \to \infty$. Define $h_n(\mu) = h(\mu, P_n)$. The sequence (h_n) is an entropy structure for (X,T).

- (h_n) reflects emergency of complexity on refining scales.

- The meaning of "allowed" is part of a deeper theory of entropy [D2].

- Every system has an entropy structure [BD1].

Superenvelopes. Below: (h_n) is an entropy structure with $h_0 \equiv 0$ and all $h_n - h_{n-1}$ u.s.c. A bounded function E on $\mathcal{M}_T$ such that every $E - h_n$ is nonnegative u.s.c. is called a *superenvelope* of the entropy structure. (Also allow the constant function $E \equiv \infty$ as a superenvelope.)

Sex Entropy Theorem [BD1].
Let E be a bounded function on $\mathcal{M}_T$. T.F.A.E.

1. E is the extension entropy function of a symbolic extension of (X, T).

2. E is affine and a superenvelope of the entropy structure.

(The statement does not depend on the choice of entropy structure.)

Functional analytic characterization of h_{sex}.
h_{sex} is the minimum superenvelope of the entropy structure (h_n).

Inductive Characterization of h_{sex}.
Let $\tilde{g}$ denote the u.s.c. envelope of a function g (the inf of the continuous functions larger than g). Convention: $\tilde{g} \equiv \infty$ if $\sup g = \infty$.

Let $\mathcal{H} = (h_n)$ be an entropy structure, $h_n \to h$. Begin with the tail sequence $\tau_n = (h - h_n)$, which decreases to zero. We will define by transfinite induction a transfinite sequence $u^{\mathcal{H}}$ of functions u_α on $\mathcal{M}_T$. Set

- $u_0 \equiv 0$
- $u_{\alpha+1} = \lim_k \widetilde{(u_\alpha + \tau_k)}$
- $u_\beta =$ the u.s.c. envelope of $\sup\{u_\alpha : \alpha < \beta\}$, if β is a limit ordinal.

THEOREM $u_\alpha = u_{\alpha+1} \iff u_\alpha + h = h_{\text{sex}}$, and such an α exists among countable ordinals (even if $h_{\text{sex}} \equiv \infty$).

The convergence above can be transfinite, and this indicates the subtlety of the emergence of complexity on ever smaller scales.

Sex entropy and smoothness

If (X, T) is C^∞, then [Buzzi following Yomdin] T is asymptotically h-expansive, and [BFF] therefore $h_{\mathsf{sex}} = h$.

Theorem [DN] A generic C^1 non-hyperbolic (i.e. non-Anosov) area preserving diffeomorphism of a compact surface has no symbolic extension (i.e. residual entropy $= \infty$).

Theorem [DN] For $r > 1$ and any compact Riemannian manifold of dimension > 1, there is a C^r-open set of C^r diffeomorphisms in which the diffeomorphisms with positive topological residual entropy are a residual set.

Theorem [A] For a smooth compact manifold M with $\dim(M) \geq 3$, there is an open subset of $\mathrm{Diff}^1(M)$ in which generic diffeomorphisms have no symbolic extension.

The DN/A proofs involve complicated iterated constructions using genericity arguments and persistent homoclinic tangencies. We'll give concrete C^r examples ($1 \le r < \infty$) a little later.

The main open problem. For a C^r diffeomorphism T, $1 < r < \infty$, is it possible that T has infinite residual entropy?

Conjecture [DN]. Suppose $2 \le r < \infty$ and T is a C^r diffeomorphism. Then

$$\mathbf{h}_{\mathsf{sex}}(T) \le \Big[R(f)\mathrm{dim}(X)\Big] \; \frac{r}{r-1} \; ,$$

where $R(f) := \lim_n (1/n) \log \max ||(T^n)'||$.

II. Functoriality of sex entropy.[BD2]

Powers. For $0 \neq n \in \mathbb{Z}$,
(1) The restriction of $h_{\mathsf{sex}}^{T^n}$ to $\mathcal{M}_T$ equals $|n| h_{\mathsf{sex}}^{T}$.
(2) $\mathbf{h}_{\mathsf{sex}}(T^n) = |n| \mathbf{h}_{\mathsf{sex}}^{T}$.

Flows. For T a flow and a, b nonzero in $\mathbb{R}$,
(1) $\mathbf{h}_{\mathsf{sex}}(T^a, \mu) = |a/b| \mathbf{h}_{\mathsf{sex}}(T^b, \mu)$,
. for all $\mu \in \mathcal{M}_{T^a} \cap \mathcal{M}_{T^b}$.
(2) $\mathbf{h}_{\mathsf{sex}}(T^a) = |a/b| \mathbf{h}_{\mathsf{sex}}(T^b)$.

Products. Suppose (X, T) is the product of finitely or countably many systems (X_k, T_k) such that $\sum_k \mathbf{h}_{\mathsf{sex}}(T_k) < \infty$, and $\mu \in \mathcal{M}_T$. Let μ_k be the coordinate projection of μ. Then
(1) $h_{\mathsf{sex}}(T, \mu) \leq \sum_k h_{\mathsf{sex}}(T, \mu_k)$.
(2) If μ is the product measure $\prod_k \mu_k$, then
. $h_{\mathsf{sex}}(T, \mu) = \sum_k h_{\mathsf{sex}}(T, \mu_k)$.
(3) $\mathbf{h}_{\mathsf{sex}}(T) = \sum_k \mathbf{h}_{\mathsf{sex}}(T_k)$.

Fiber Products. Let (X, T) be the fiber product of (X', T') and (X'', T'') over their common factor (X, T'''). Then

(1) $h_{\mathsf{sex}}(T, \mu) \leq h_{\mathsf{sex}}(T', \mu') + h_{\mathsf{sex}}(T'', \mu'') - h(T''', \mu''')$ where $\mu \in \mathcal{M}_T$ and the other measures are its projections.

(2) If above μ is the relatively independent joining of μ' and μ'', and T'' is asymptotically h-expansive, then

$h_{\mathsf{sex}}(T, \mu) \geq h_{\mathsf{sex}}(T', \mu') + h_{\mathsf{sex}}(T'', \mu'') - h_{\mathsf{sex}}(T''', \mu''')$

(3) If above $\mathbf{h}(T''') = 0$ and T'' is asymptotically h-expansive, then

$h_{\mathsf{sex}}(T, \mu) = h_{\mathsf{sex}}(T', \mu') + h_{\mathsf{sex}}(T'', \mu'')$.

We need (3) for our explicit examples.

The proofs for products and fiber products use the (transfinite) inductive characterization and also the Downarowicz entropy structure defined from continuous functions [D2].

III. Examples.

Given $1 \le r < \infty$, Misiurewicz (1973) manipulated several vector fields to construct a C^r system $D : V \times S^1 \to V \times S^1$ with no measure of maximal entropy (the first smooth examples with no such measure). (Dim(V)=3.) Features of the example, given r:

- Each $V \times \{t\}$ is D-invariant. Let
  $V_t = V \times \{t\}$
  $D_t = D|V_t$
  $S^1 = (-1/2, 1/2]$.

- $\mathbf{h}_{\text{top}}(\mathbf{D_0}) = \mathbf{0}$.

- Restriction of D to $\cup_{t \ge \epsilon} V_t$ is C^∞ with entropy $< h(D)$.

- $\limsup_{t \to 0} h(D_t) = h(D) > 0$.

It turns out that the sex entropy function h^D_{sex} is simply the u.s.c. envelope $\widetilde{h}$ of the entropy function h on $\mathcal{M}_D$.

The proof of this [BD2] uses the functional analytic characterization of the sex entropy function, and a study of the lift of h_{sex} from $\mathcal{M}_D$ to a function on the Bauer simplex whose boundary is the closure of the ergodic measures in $\mathcal{M}_D$.

Sex Entropy Variational Principle [BD1]. The topological sex entropy is the max of its sex entropy function.

So for D, the topological sex entropy equals its topological entropy.

Another Misiurewicz example.

Another (much easier) Misiurewicz example (1971): a smooth system $(W \times S_1, R)$ with the entropy function on $\mathcal{M}_R$ not lower semicontinuous:

- R is C^∞

- Each $W \times \{t\} := W_t$ is R-invariant
  $R_t : W_t \to W_t$

- $h(R_t) = 0$ if $t \neq 0$

- $h(R_0) > 0$.

Because W is C^∞, it is asymptotically h-expansive. The sex entropy function on $\mathcal{M}_W$ is simply the entropy function, and the residual entropy is zero.

We will combine the two Misiurewicz examples in a fiber product to get an explicit example of a C^r diffeo with positive topological sex entropy.

Smooth examples with positive residual entropy.

- Set $X = V \times W \times S^1$.

- Define $T : X \to X$,
  $T : (v, w, t) \mapsto (D_t(v), R_t(w), t)$.

- $h_{\text{top}}(R_t) = 0$ if $t \neq 0$, and
  $h_{\text{top}}(D_0) = 0$.

- Thus $h_{\text{top}}(T) = \max\{h_{\text{top}}(D), h_{\text{top}}(R)\}$.

- To prove T has positive topological residual entropy: by the Sex Entropy Variational Principle, it suffices to show the sup of h^T_{sex} is larger than the max above.

- $T : (v, w, t) \mapsto (D_t(v), R_t(w), t)$.

- T is a fiber product of V and W over S^1. Apply the functorial fiber product result (3) to $\mu \in \mathcal{M}_T$ with projections μ_D, μ_R:

$$\begin{aligned} h_{\text{sex}}(T, \mu) &= h_{\text{sex}}(D, \mu_D) + h_{\text{sex}}(R, \mu_R) \\ &= \widetilde{h}(\mu_D) + h(\mu_R) \end{aligned}$$

where we used $h^R_{\text{sex}}(\mu_R) = h(\mu_R)$, which holds because R is asymptotically h-expansive, which holds because R is C^∞.

- Now choose a μ_D and μ_R on V_0 and W_0 to maximize the $\widetilde{h}(\mu_D)$ and $h(\mu_R)$ above, respectively at $h_{\text{top}}(D)$ and $h_{\text{top}}(R)$, and let μ be their product measure on $V \times W \times \{0\}$. We get

$$\begin{aligned} h^T_{\text{sex}}(\mu) &= h_{\text{top}}(D) + h_{\text{top}}(R) \\ &> \max\{h_{\text{top}}(D), h_{\text{top}}(R)\} \ . \end{aligned}$$

This finishes the proof.