Matrix problems arising from symbolic dynamics

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I.1. Shifts of finite type

- Given: A an $n \times n$ matrix over $\mathbb{Z}_+$,

- view A as adjacency matrix of directed graph G_A on vertices $1, 2, \ldots, n$
  $A(i,j) =$ number of edges from i to j

- Let X_A be the space of doubly infinite sequences $x = \ldots, x(-1), x(0), x(1), \ldots$ such that for all n, $x(n)$ is an edge of G_A, and $x(n+1)$ follows $x(n)$ in G_A.

- X_A is naturally a compact metrizable space

- $\sigma_A : X_A \to X_A$ is the shift homeomorphism, $(\sigma_A(x))(n) = x(n+1)$.

The topological dynamical system σ_A is a shift of finite type (SFT). It is a mixing SFT if the matrix A is primitive (nonnegative, with some A^n strictly positive). The mixing SFTs (analogous to primitive among nonnegative square matrices) are the basic building blocks and the most important case of SFT.

Two top. dyn. systems S and T are topologically conjugate, or isomorphic,

$$S \cong T$$

if there is some homeomorphism h such that $hS = Th$. Every SFT is isomorphic to some σ_A. SFTs play a significant role in dynamical systems.

To study topological conjugacy of SFTs σ_A in terms of the defining matrices A, we must define some matrix relations.

I.2. Strong shift equivalence

$\mathcal{S} :=$ a subset of a ring, containing 0 and 1.

Matrices A, B are elementary strong shift equivalent over $\mathcal{S}$ (ESSE-$\mathcal{S}$) if there exist matrices U, V over $\mathcal{S}$ such that $A = UV$ and $B = VU$.

The relation strong shift equivalence over $\mathcal{S}$ (SSE-$\mathcal{S}$) is the transitive closure of (ESSE-$\mathcal{S}$).

Example.

$$A = \begin{pmatrix} 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = U_1 V_1$$

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = V_1 U_1$$

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = U_2 V_2$$

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = V_2 U_2$$

So, A and C are SSE-$\mathbb{Z}_+$.

A and C are not ESSE-$\mathbb{Z}_+$ (or even ESSE-$\mathbb{R}_+$): it is easy to see that a one-by-one real matrix can only be ESSE to a rank one matrix.

As another example, if A is a real matrix such that $A^k \neq 0$ and $A^{k+1} = 0$, then A is SSE-$\mathbb{R}$ to (0) in k elementary steps but not fewer.

SSE is clearly a natural relation to consider for matrices. But what does it have to do with symbolic dynamics?

I.3. SSE and SFTs

THEOREM (Williams, 1973)
$\sigma_A \cong \sigma_B \iff A, B$ are SSE-$\mathbb{Z}_+$

BUT it is still unknown if SSE-$\mathbb{Z}_+$ is decidable, i.e., whether there exists an algorithm which will take two matrices and decide whether they are SSE-$\mathbb{Z}_+$.

We have no bounds on the sizes of matrices which might be involved in a chain of ESSE.

So, Williams introduced a more tractable relation, shift equivalence.

I.4. Shift equivalence

DEFN Square matrices A, B are shift equivalent over $\mathcal{S}$ (SE-$\mathcal{S}$) if $\exists$ matrices U, V over $\mathcal{S}$ and $\ell \in \mathbb{N}$ such that

$$A^\ell = UV \qquad B^\ell = VU$$
$$AU = UB \qquad BV = VA$$

- SE-$\mathbb{Z}_+$ is decidable (Kim-Roush)

- SE-$\mathbb{Z}_+$ is conceptual – equivalent to isomorphism of certain associated ordered modules (Krieger).

- A, B are SE-$\mathbb{Z}_+$ iff $(\sigma_A)^n \cong (\sigma_B)^n$ for all large n

SHIFT EQUIVALENCE CONJECTURE (Williams 1974): SE-$\mathbb{Z}_+$ implies SSE-$\mathbb{Z}_+$.

Counterexamples (Kim Roush 1992,1999) are few and so far require quite special conditions on the matrices. Explaining these is beyond the scope of the talk. Now we want to know, with what extra assumptions does SE-$\mathbb{Z}_+$ imply SSE-$\mathbb{Z}_+$?

LITTLE SHIFT EQUIVALENCE CONJECTURE If A over $\mathbb{Z}_+$ has a single nonzero eigenvalue n, then A and (n) are SSE-$\mathbb{Z}_+$.

THEOREM (Kim-Roush, 1990) The last conjecture is true with $\mathbb{R}$ or $\mathbb{Q}$ in place of $\mathbb{Z}$.

POSITIVE RATIONAL SHIFT EQUIVALENCE CONJECTURE If A, B are shift equivalent over $\mathbb{Q}$ and have all entries positive, then A and B are SSE-$\mathbb{Q}_+$.

I.5. More on SSE as a matrix relation

FACT (Williams) For a unital subring $\mathcal{S}$ of $\mathbb{R}$, primitive matrices are SE-$\mathcal{S}_+$ iff they are SE-$\mathcal{S}$.

FACT (Williams,Effros,B-Handelman) For $\mathcal{S}$ a PID or Dedekind domain, SE-$\mathcal{S}$ implies SSE-$\mathcal{S}$.

PROBLEM For which other unital subrings $\mathcal{S}$ of $\mathbb{R}$ does SE-$\mathcal{S}$ implies SSE-$\mathcal{S}$?

At any rate:
for primitive matrices over $\mathcal{S} = \mathbb{Z}$ or $\mathbb{Q}$,
SE-$\mathcal{S}_+$ is equivalent to SSE-$\mathcal{S}$ and to SE-$\mathcal{S}$.

For any nonnilpotent matrix A over a PID $\mathcal{S}$ (e.g. $\mathbb{Z}$ or $\mathbb{Q}$), there is an invertible U over $\mathcal{S}$ such that UAU^{-1} has block form $\begin{pmatrix} A' & X \\ 0 & N \end{pmatrix}$ where A' is nonsingular and N is nilpotent triangular (or empty). Given likewise B, B':

$$A, B \text{ SSE} - \mathbb{Q} \iff A', B' \text{ SIM} - \mathbb{Q}$$
$$A, B \text{ SSE} - \mathbb{Z} \impliedby A', B' \text{ SIM} - \mathbb{Z}$$

For any unital ring $\mathcal{S}$, we think of SSE-$\mathcal{S}$ as a stabilized version of similarity over $\mathcal{S}$. Here, the equivalence relation SSE-$\mathcal{S}$ is generated by the following relations on square matrices over $\mathcal{S}$: similarity over $\mathcal{S}$ ($A \sim UAU^{-1}$), and

$$\begin{pmatrix} A & X \\ 0 & 0 \end{pmatrix} \sim A \sim \begin{pmatrix} A & 0 \\ X & 0 \end{pmatrix}$$

(i.e., if in a matrix row n or column n is all zero, then we can delete row n and colum n).

More facts around SSE as a stabilized similarity.

The SIM-$\mathbb{Z}$ classes of matrices with a given irreducible characteristic polynomial with root λ are in bijective correspondence with the ideal classes of $\mathbb{Z}[\lambda]$ (Taussky-Todd). Their SSE-$\mathbb{Z}$ classes are in bijective correspondence with the ideal classes of $\mathbb{Z}[1/\lambda]$ (B-Marcus-Trow).

Suppose A is a square matrix over $\mathbb{Z}$ whose characteristic polynomial has no nonzero repeated root. Then the class of matrices SE-$\mathbb{Z}$ to A is a union of finitely many SIM-$\mathbb{Z}$ classes.

I.6. The Spectral Conjectures

Let Λ denote a list of complex numbers and Λ^n the list of their nth powers. Let $\text{tr}(\Lambda)$ be the sum of the entries of Λ. So, if Λ is the nonzero spectrum Λ_A of a matrix A, then $\text{tr}(\Lambda^n) = \text{trace}(A^n)$.

[Remark: $\det(I - tA)$ encodes Λ_A.]

For A over $\mathbb{Z}_+$, the trace of A^n is the number of fixed points of σ_A^n. So, Λ_A encodes the periodic point counts of σ_A. What can these counts be?

SPECTRAL CONJECTURE (B-Handelman 1991)
Suppose $\mathcal{S}$ is a subring of $\mathbb{R}$ containing 1, and $(\lambda_1, ..., \lambda_k)$ is a list of nonzero complex numbers. Then Λ is the nonzero spectrum of a primitive matrix over $\mathcal{S}$ if the following necessary conditions hold.

1. (Perron condition) $\lambda_1 > |\lambda_i|, \ \forall i > 1$.

2. (Coefficients condition) The polynomial $(t-\lambda_1)\cdots(t-\lambda_k)$ has all coefficients in $\mathcal{S}$.

3. (Trace condition)

- If $\mathcal{S} \neq \mathbb{Z}$, then for all $k, n \in \mathbb{N}$
  - $\mathrm{tr}(\Lambda^n) \geq 0$,
  - $\mathrm{tr}(\Lambda^n) > 0 \implies \mathrm{tr}(\Lambda^{nk}) > 0$.
- If $\mathcal{S} = \mathbb{Z}$, then for all $n \in \mathbb{N}$,

$$\sum_{k|n} \mu(n/k)\mathrm{tr}(\Lambda^k) \geq 0$$

  where μ is the Mobius function.

The Spectral Conjecture is true for $\mathcal{S} = \mathbb{R}$ (B-Handelman 1991) and $\mathcal{S} = \mathbb{Z}$ (Kim-Ormes-Roush, 2000).

What is a grand analogue realization conjecture for "all" the stable algebraic structure (not just nonzero spectrum)?

GENERALIZED SPECTRAL CONJECTURE (B-Handelman 1993) Suppose B is a square matrix over $\mathcal{S}$ and its nonzero spectrum saisfies the three conditions of the Spectral Conjecture. Then there is a primitive matrix A over $\mathcal{S}$ such that A and B are SSE-$\mathcal{S}$.

The GSC is open for every $\mathcal{S}$. It holds for B over $S = \mathbb{Z}$ if all eigenvalues of B are rational (B-Handelman 1993).

I.7. G-SFTs

The relations SSE and SE adapt well to several other classification problems in symbolic dynamics. Here is an example.

Let G be a finite group. Say a G-SFT is an SFT together with a continuous free G action on it which commutes with the shift.

Let $\mathbb{Z}G$ denote the integral group ring of G and let $\mathbb{Z}_+G$ denote the subset of elements $\sum_g n_g g$ for which every n_g is a nonnegative integer.

It turns out (Parry) that square matrices over $\mathbb{Z}_+G$ present G-SFTs; SSE-$\mathbb{Z}_+G$ of A and B is equivalent to topological conjugacy of the G-SFTs; SE-$\mathbb{Z}_+G$ of A and B is equivalent to eventual topological conjugacy of the G-SFTs; and so on (B-Sullivan 2005).

II.1 Polynomial Matrices

A square matrix B over $t\mathbb{Z}_+[t]$ presents a certain directed graph G_B. In G_B: for each term t^k in $B(i,j)$, there is a path of k edges from i to j. These paths do not intersect except as required at the beginning and end vertices.

Define σ_B to be the SFT σ_D, where D is the adjacency matrix of G_B.
(E.g., if $B = tA$ with A over $\mathbb{Z}_+$, then $D = A$.)
[Remark: $\det(I - B) = \det(I - tD)$.]

Given a finite matrix B, define the $\mathbb{N}\times\mathbb{N}$ matrix

$$B_\infty = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$$

For notational simplicity, we identify B and B_∞. From here matrices are $\mathbb{N}\times\mathbb{N}$ unless indicated. The matrix I is the $\mathbb{N}\times\mathbb{N}$ identity matrix.

II.2 Positive Equivalence

DEFN A basic elementary matrix over a ring $\mathcal{R}$ is a matrix over $\mathcal{R}$ equal to the identity except possibly in a single offdiagonal entry.

DEFN E($\mathcal{R}$) is the group of matrices generated by $\mathbb{N} \times \mathbb{N}$ basic elementary matrices.

DEFN Matrices A, B are basic positive equivalent over $\mathcal{R}$ if there is a basic elementary matrix E such that $EA = B$ or $AE = B$.

DEFN Let $\mathcal{M}$ be a set of matrices containing A and B. Then A, B are positive equivalent in $\mathcal{M}$ over a ring $\mathcal{R}$ if there are matrices $A = A_0, A_1, \ldots, A_n = B$, all in $\mathcal{M}$, such that A_i and A_{i-1} are basic positive equivalent over $\mathcal{R}$, $1 \leq i \leq n$.

THM* Let A, B be matrices in the set $\mathcal{M}$ of matrices over $t\mathbb{Z}_+[t]$. Then T.F.A.E.

1. $\sigma_A \cong \sigma_B$

2. $I - A$ and $I - B$ are positive equivalent in $\mathcal{M}$ over $\mathbb{Z}[t]$.

This gives a useful "positive K-theory" structure for studying SFTs. Just as SSE/SE adapts to several other problems, so does this setup.

Of course (2) implies there are U, V in E($\mathcal{R}$) such that

$$U(I - A)V = I - B \ .$$

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*The theorem statement is slightly wrong for simplicity. See reference 2 listed in Section V.

II.3 Positive K-theory

Why the name "positive K-theory"?

For any ring $\mathcal{R}$, its stable general linear group GL($\mathcal{R}$) is the group of $\mathbb{N} \times \mathbb{N}$ matrices of the form

$$\begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix}$$

where X is finite square invertible over $\mathcal{R}$.

The group $K_1(\mathcal{R})$ of algebraic K-theory is the abelianization of GL($\mathcal{R}$). Elements A, B of GL($\mathcal{R}$) are in the same element of $K_1(\mathcal{R})$ iff there exist U, V in E($\mathcal{R}$) such that $UAV = B$.

Our setup is similar but more difficult.
(1) Our matrices $I - A$ are not invertible.
(2) Instead of E($\mathcal{R}$) equivalence, we have the more complicated relation of composition of basic positive equivalences.

III.1 Flow equivalence

Suppose X is a compact metric space and $T : X \to X$ is a homeomorphism.

The *mapping torus* of T, Map(T), is the quotient of $X \times [0,1]$ by the map which for each x in X identifies $(x,1)$ and $(Tx,0)$.

Equivalently Map(T) is the quotient of $X \times \mathbb{R}$ under the map which identifies (x,s) and $(T^k x, s+k)$ for every k in $\mathbb{Z}$.

The *suspension flow* on Map(T) corresponds to (x,t) moving to $(x,s+t)$ at time s.

DEFN T is flow equivalent (FE) to T' if $\exists$ homeomorphism $h :$ Map(T) $\to$ Map(T') which sends flow lines to flow lines respecting the direction of flow ("orientation preserving").

III.2 Flow equivalence of mixing SFTs'

The polynomial matrices and positive K-theory setup really pay off in analyzing flow equivalence of SFTs.

Very roughly: for an SFT presented by a matrix over $\mathbb{Z}_+[t]$, "t" plays the role of time, and the flow equivalence class is unaffected by time changes (changing positive exponents in the matrix to other positive exponents).

A positive equivalence over $\mathbb{Z}[t]$ corresponds to conjugacy of $I - A(t)$ and $I - B(t)$; under modest conditions, a positive equivalence over $\mathbb{Z}[1] = \mathbb{Z}$ of $I - A(1)$ and $I - B(1)$ corresponds to flow equivalence.

If $A(t)$ is tC with C over $\mathbb{Z}_+$, then $A(1) = C$.

THEOREM Suppose A and B are nontrivial primitive matrices over $\mathbb{Z}$. TFAE.
(1) σ_A and σ_B are FE.
(2) $I-A$ and $I-B$ are pos. equivalent over $\mathbb{Z}$.
(3) $I-A$ and $I-B$ are equivalent over $\mathbb{Z}$.

Given a matrix C over $\mathbb{Z}$, there are matrices U,V in $\mathsf{E}(\mathbb{Z}) = \mathsf{SL}(\mathbb{Z})$ such that UCV is a diagonal matrix D with $D(k+1,k+1)$ dividing $D(k,k)$ and all entries nonnegative except perhaps $D(1,1)$. This list of diagonal entries is a complete invariant for equivalence over $\mathsf{E}(\mathbb{Z})$.

This is a complete invariant of dreamlike simplicity.

[It was long known (Parry, Sullivan, Bowen, Franks) that (1) is equivalent to $\det(I-A) = \det(I-B)$ and $\mathrm{cok}(I-A) \cong \mathrm{cok}(I-B)$.]

III.3 Equivariant flow equivalence for G-SFTs'

Let G be a finite group and A a matrix over $\mathbb{Z}_+G$. Recall A presents a G SFT, σ_A. The mapping torus of σ_A carries an induced free G action commuting with the suspension flow. Two G-SFTs are G-flow equivalent if there is a homeomorphism between their suspension flows, sending flow lines to flow lines respecting the direction of the flow, and intertwining the G-actions.

The classification of mixing G-SFTs up to G-flow equivalence reduces to the following theorem (in which the "weights group" is an easily computed invariant we won't discuss). It is the G-SFT/$\mathbb{Z}G$ analogue of the SFT/$\mathbb{Z}$ theorem.

THEOREM Suppose A and B are nontrivial primitive matrices over $\mathbb{Z}G$. TFAE.
(1) σ_A and σ_B are FE.
(2) $I - A$ and $I - B$ are pos. equivalent over $\mathbb{Z}G$.
(3) $I - A$ and $I - B$ are equivalent over $\mathbb{Z}G$.

Of course the hard direction is (3) $\implies$ (2).

In contrast to the $\mathbb{Z}$ case, there is nothing like a simple, general complete invariant for equivalence over $\mathbb{Z}G$.

Example. Let $G = \mathbb{Z}/2 = \{e, g\}$ and

$$A = \begin{pmatrix} e+g & e-g \\ 0 & 2e+2g \end{pmatrix} .$$

Then A is not GL($\mathbb{Z}G$) equivalent to its transpose.

Example. Let $G = \mathbb{Z}/2 = \{e, g\}$ and

$$B = \begin{pmatrix} e+g & e-g \\ e-g & 2e \end{pmatrix} .$$

Then B is not GL($\mathbb{Z}G$) equivalent to a triangular matrix.

For simple proofs, see Sec. 8 in (B-Sullivan 2005), where there are also some positive facts.

PROBLEM For a finite group G, when are two matrices over $\mathbb{Z}G$ equivalent over E($\mathbb{Z}G$)?

IV. Flow equivalence of sofic shifts

Given an edge shift of finite type $\sigma_A : X \to X$, and a map Φ from the symbols (edges) to some finite set, and $x \in X$, define a doubly infinite sequence ϕx by the rule $(\phi x)(n) = \Phi(x(n))$. Let S be the shift map on $Y = \{\phi x : x \in X\}$. Then S is a sofic shift. Every sofic shift is obtained in this way.

To a sofic shift a cover ϕ of this type can be canonically associated. We say S is n-sofic if it has a canonical cover ϕ for which no point has more than n preimages.

For sofic shifts, the flow equivalence relation is far more complicated. Soren Eilers, Toke Carlsen and I have work in progress to classify all 2-sofic shifts up to flow equivalence. (3-sofic is beyond us.)

The classification of 2-sofic shifts requires the full force of the classification of mixing G-SFTs up to FE (for the case $G = \mathbb{Z}/2$); the full force of the classification of general (reducible) SFTs up to flow equivalence by Danrun Huang (for which the general complete invariant includes isomorphism of a complicated diagram of group homomorphisms); and more.

CONCLUSION: The study of symbolic dynamical systems leads to interesting and difficult algebraic problems, especially problems involving matrices.

V. References

The papers below are on my website with references to papers of others. 1 and 2 are expository.

1. M. Boyle, Algebraic aspects of symbolic dynamics.

2. M. Boyle, Positive K-theory and symbolic dynamics.

3. M. Boyle and M. Sullivan, Equivariant flow equivalence for shifts of finite type, by matrix equivalence over group rings

(thorough paper including self contained matrix results)

4. M. Boyle, Flow equivalence of shifts of finite type via positive factorizations.

(positive K-theory approach to FE and more for SFTs. The reduction from positive equivalence to equivalence for SFTs, which in the reducible case has a block structure.)

5. M. Boyle and D. Huang, Poset block equivalence of integral matrices.

(Works out the complete invariants for the block-structured equivalence of blocked integral matrices. Together, 4 and 5 give a self-contained presentation of the Huang classification of SFTs up to FE.)

6. M. Boyle, Open problems in symbolic dynamics.