Multidimensional shifts of finite type and sofic shifts II

Mike Boyle
University of Maryland

Outline

1. Periodic points of onto cellular automata: density?

2. How many spatially periodic points are jointly periodic?

3. Subsystems

4. Quotients

The sections on jointly periodic points are largely taken from a paper with Bryant Lee in Experimental Mathematics, 2007. The programs used for that paper are on my website.

The sections on subsystems and quotients are taken from B-Pavlov-Schraudner, “Multidimensional sofic shifts without separation and their factors” Transactions AMS to appear, on my website.

I. Jointly periodic points of onto cellular automata: density?

Let f denote a surjective endomorphism of (X_N, σ_N), i.e., an onto one-dimensional cellular automaton. A point is jointly periodic if it is periodic under both f and the shift.

A periodic point of the shift is at least preperiodic for f, but is not necessarily periodic for f. For example, for σ_2 on alphabet $\{0, 1\}$, let f be defined by $(fx)_i = x_i + x_{i+1}$ (mod 2). There are two fixed points for σ_2, and f maps both to the fixed point ...00000... . The fixed point ...111111... is not periodic for f.

In the case f is injective, then f is bijective, an automorphism of σ_N. Thus for every k, the finite set of points in shift orbits of size k (i.e. having least period k under the shift) is mapped to itself by f. In this case, the shift-periodic points are the same as the f-periodic points.

Given that f is a surjective 1-dimensional c.a. map, f is uniformly finite to one. Let $M = \max_x |f^{-1}x| < \infty$.

If f is any shift-commuting map, and maps a point of least period n to a point of least period k, then k divides n, and on the orbit of x, f is n/k-to-1.

So, if n is not divisible by a prime less than or equal to M, then f maps the finite set of points of least shift-period n into itself. So, at the very least, there will be infinitely many n for which a point of shift-period n will be also periodic under f. But how rich is this set of jointly periodic points?

A topological measure of largeness for a set is density. The collection of jointly periodic points x is dense iff for every word on N symbols there is a jointly periodic point x in which that word occurs.

Question

Are the jointly periodic points of f dense?

Since the shift-periodic points are dense, the answer is yes if f is injective. The answer is still yes if f is right or left closing (B-Kitchens) or if f has a point of equicontinuity (Blanchard-Tisseur).

Otherwise nothing is known (!). In particular we do not know whether the possibly larger set of f-periodic points must be dense.

I also do not know a counterexample for a higher-dimesional surjective cellular automaton. But perhaps someone at this conference ...

There are some experimental results on the question [B-Lee]. First I'll indicate the proof of joint density when f is closing.

Closing maps An endomorphism of an SFT is right closing if it is injective on unstable sets (never collapses left asymptotic points). For left closing, replace unstable,left with stable, right.

This is the class of factor maps about which we have some real theorems and know a lot. Conversely, we have very little understanding of factor maps which are not closing, especially when they are iterated as c.a.

Proof sketch Suppose f is a right closing c.a. map For large enough k, the map ψ which is a composition of f with the kth power of the shift is now defined as a block code on non-negative coordinates. So, it can be viewed as an endomorphism of the one sided full shift. We so view it.

Because ψ is right closing, the map ψ is an open map on the onesided shift space. Because it is open and n is large, ψ is forwardly expansive. Therefore it is a onesided SFT. It is also mixing because the full shift with which it commutes is. (We use results of Parry, Nasu, Kurka, B-Fiebigs.)

Now consider the map $\phi = \psi\sigma$. It is again a onesided MSFT (e.g., by the LR textile picture, or directly). Check: if a point is periodic for ϕ, then it is periodic for σ, hence is jointly periodic. But the periodic points of the onesided mixing SFT ϕ are dense. QED

Experimental issues

1. The number of maps grows superexponentially with the span.

The number of endomorphisms of σ_N of span at most k is N^{N^k}. Let inj(k, N) and surj(k, N) denote the number of injective and surjective endomorphisms of span k. Kim and Roush showed

$$\lim_k \frac{1}{k} \log \log \text{inj}(N, k) = \log N$$

and therefore the same superexponential growth rate holds for surj(k, n).

At the same time, the surjective c.a. of span k are a very tiny fraction of all c.a. of span k.

Problem. Find a sharper statement about the numbers inj(k, N) and surj(k, N), and their ratio as $k \to \infty$.

In part, this problem is "because it's there". I know no direct application to another problem.

However, the counts at $k+1$ are more or less unaffected by those at k. So, saying something sharper seems to require a deeper understanding of the structure of injective and surjective c.a.

2. The number of periodic points of σ_N grows exponentially.

To find the points of period k for the shift which are f-periodic, we track the forward orbit of a point and see when there is a repetition. This involves keeping a tagged list of words of length k in memory and changing tags as f acts. There are 2^k words of length k.

So, we encounter serious limits on the k we can explore, due to memory constraints, even if $N = 2$. Increasing k by 1 roughly doubles the memory requirement.

The library of maps from Hedlund et al

In 1963, Hedlund, Appel and Welch produced a list of all onto endomorphisms f of σ_2 of span at most 5. (For which publication is still not allowed ...)

Let us say that f is m-dense at period k if every word of length m occurs in some jointly periodic point with shift period k; and f is m-dense if every word of length m occurs in some jointly periodic point.

We know left or right permutative c.a. are onto and closing and thus have jointly periodic points dense. Modulo symmetries (w.r.t. our periodic point questions), there were 64 span 4 c.a. not permutative in an end variable.

There are 141,792 surjective c.a. of span 5. Hedlund et al grouped them into classes of some regularity and a remaining set of 200 maps, generated by 26 "sporadic" maps and some operations.

Experimental results for the density conjecture

All the span 4 surjective c.a. on the 2-shift are 13-dense, and are 10-dense at some $k \leq 24$.

All the sporadic span 5, surjective c.a. of σ_2 are are 10-dense at $k = 24$.

There were similar results for small samples of types of map. Given this and the previous work, we elevated the question to a conjecture:

Conjecture [B-Lee]
The jointly periodic points of any one-dimensional cellular automaton are dense.

II. How many spatially periodic points are jointly periodic?

We continue with an onto one-dim. c.a. map.

Definition.
$\nu_k(f, \sigma_N) = |\{x \in \mathsf{Fix}(\sigma_N)^k : x \text{ is } f - \text{periodic}\}|$
and
$\nu(f, \sigma_N) = \overline{\lim}_k \, \nu_k(f, \sigma_N)^{1/k}$.

$\nu_k(f, \sigma_N)$ counts the points of period k for σ_N which are jointly periodic.
$\nu(f, \sigma_N)$ captures the growth rate.

For example, if f is injective, then $\nu(f, \sigma_N) = N$.

Note, $\nu_k(f, \sigma_N)$ does not change if f is replaced by $f^i(\sigma_N)^j$, for any $i > 0$ and $j \geq 0$.

Looking at a fairly large sample (including all span 4 onto endomorphisms of the 2-shift), out to shift orbit periods of 19 to 26, we see no obvious difference between maps which are closing or not, or permutative or not. There are some rigorous arguments in certain classes to show $\nu(f,\sigma_N) > 1$, or $\nu(f,\sigma_N) = N$.

Question.
Is $\nu(f,\sigma_N) > 1$ for every onto c.a. f?

Question.
Is $\nu(f,\sigma_N) \geq \sqrt{N}$ for every onto c.a. f?

The last question reflects a random maps heuristic (if some structure doesn't force more periodicity, then we see at infinitely many periods at least about the periodicity we'd expect of a random map). An answer yes is consistent with our data, which are suggestive but not compelling. (But if we could see $k = 50$ ...)

We investigated $\nu(f,N)$ with two programs.

The first program (among other things) computes $\nu_k(f,N)$. We looked out to k about 27 for all those sporadic span 5, all those non-permutative span 4, and a sampling of other types of c.a. The data are consistent with an answer yes to the last (“$\sqrt{N}$”) question. For the k the root of $\nu_k(f,N)$, looking at $N=2$ (the 2-shift), we see numbers generally in the range $[1.4,2]$. It would be much more telling if we could see to $k=50$. Running the program at $k=26$ used about 1.8 gigabytes of memory.

To look further, we developed a program which randomly picks a word of length k, and then computes the preperiod and eventual period of the corresponding periodic point. Typically we could for a given f investigate points to period about $k = 37$ without crashing. But the crash-in-practice k varied with the map, from 33 to 50.

We saw data consistent with the random maps heuristic: roughly, a high likelihood that the point would fall into a cycle of size something like $2^{\sqrt{k}}$, with significant dropoff in the size of unlikely eventual periods. We ran size 10 samples for many maps.

Again, this data is suggestive and consistent with the random maps heuristic, but not compelling.

The program is available at my website or the Experimental Mathematics website. Brendan Berg is updating it to the current version of C.

More ambitiously, what does the distribution of $\nu(f, \sigma_N)$ over surjective c.a. f of span k look like, as $k \to \infty$? Can we say at least $\nu(f, \sigma_N) \geq \sqrt{N}$ with asymptotic (in k) probability 1?

We know four ways to demonstrate $\nu(f, S_N)$ is large:

1. find a large shift fixed by f (or more generally by a power of f)

2. let f be a group endomorphism (e.g. [Martin, Odlyzko and Wolfram 1984]

3. use the algebra of a polynomial presenting f in very special cases [F.Rhodes, 1988]

4. finding equicontinuity points.

In all but the first case we force $\nu(f, \sigma_N) = N$.

Conjecture.
There exist f such that $\nu(f, \sigma_N) < N$.

From our data, it seems obvious that the conjectured inequality is typical. (Equality holds in the algebraic case and some other classes.) And it looks unthinkable that the conjecture could be wrong.

So, this is not a bold conjecture, but rather a proclamation of ignorance, that we cannot give a proof for any example.

But surely one of the distinguished professors or ready-for-prime-time young minds of this conference ...

IV. Subsystems of mixing Z^d SFTs and sofic shifts

Mixing

- A Z^d subshift (X, σ) is *mixing* if any two legal finite configurations can occur at all but finitely many separations.

- That is, for all x, y in X, for all n we have for all but finitely many $u \in Z^d$ there exists $w \in X$ such that
  $w|_{B_n} = x|_{B_n}$ and $w|_{u+B_n} = y|_{B_n}$.

- Generally problems of Z SFTs reduce easily to problems of mixing Z SFTs.

- For $d \geq 2$, the mixing condition splinters into a host of different conditions. Homogeneity is lost. Mixing alone is much less meaningful.

Subsystems of $\mathbb{Z}$ SFTs

Nontrivial mixing $\mathbb{Z}$ SFTs have a homogeneous structure, rich in subsystems and quotients:

- Krieger Embedding Theorem $\Longrightarrow$ if (X, σ_X) is a mixing $\mathbb{Z}$ SFT and (Y, σ_Y) is a $\mathbb{Z}$ subshift with no periodic points and $h(\sigma_Y) < h(\sigma_X)$, then (Y, σ_Y) is topologically conjugate to a subshift contained in (X, σ_X).

- Jewett-Krieger Theorem: every finite entropy measurable $\mathbb{Z}$-system is realized by a uniquely ergodic $\mathbb{Z}$-subshift

- Given any proper subsystem of a mixing $\mathbb{Z}$ SFT, all the embeddings above can be chosen disjoint from that subsystem.

“There’s always room at the Krieger Hotel.”

Subsystems of $\mathbb{Z}$ Sofic Shifts

- A mixing $\mathbb{Z}$ sofic shift X contains an increasing union of mixing $\mathbb{Z}$ SFTs X_n with $\lim_n h(X_n) = h(X)$.

- So the subsystem results of the last page for mixing $\mathbb{Z}$ SFTs also hold for mixing $\mathbb{Z}$ sofic shifts.

So: a nontrivial mixing $\mathbb{Z}$ SFT or sofic shift contains a vast family of pairwise disjoint subsystems with entropy close to $h(\sigma_X)$. Apart from considerations involving periodic points, mixing Z SFT and sofic shifts are equally rich in subsystems.

The heart of this richness is that a mixing SFT contains a rich collection of disjoint mixing SFTs as subsystems.

Subsystems of mixing Z^d SFTs

- For a special but large subclass of the mixing $\mathbb{Z}^2$ SFTs S: any smaller entropy $\mathbb{Z}^2$ subshift without periodic points can be embedded into S. [Lightwood]

- Again for a large subclass of the mixing $\mathbb{Z}^d$ SFTs S, Robinson and Sahin have proved existence of subshifts carrying completely positive entropy or Bernoulli measures.

- There is also an analogue of the Jewett Krieger Theorem for $\mathbb{Z}^d$ shifts, $d \geq 2$. [Rosenthal]

Moreover there is one result on richness of subsystems which applies to ALL $\mathbb{Z}^d$ SFTs:

THEOREM (Desai) For $d \geq 2$ and a given $\mathbb{Z}^d$ shift S:

- If S is SFT, then S contains SFTs with entropies dense in [0,$h(S)$].

Let's sketch the simple proof, for a Z^2 SFT. Clearly it is enough to show those entropies are ϵ-dense for every $\epsilon > 0$.

Proof. Let X be a Z^2 SFT with entropy $h(X) > 0$. Let Y be the SFT $X \times L$, where L is the set of translates ℓ of the lattice NZ^2 inside Z^2. So, L contains N^2 elements. A point in Y can be pictured as a point in X in which symbols in the associated ℓ are covered red. The shift action on L by Z^2 is such as to respect the coloring. So, we can view Y as having two types of symbols: copies of symbols of X; ad symbols of X colored red.

Define a nested finite chain of SFTs
$Y = Y_0, Y_1, ..., Y_m$ inductively: if possible, pick in Y_k two $N \times N$ squares with the same red boundary configuration, and disallow one of them, to define Y_{k+1}. The process stops at Y_m when every red boundary can be filled in uniquely. For N large: the entropy of Y_m is small, and the entropy drops from Y_k to Y_{k+1} are small for all k define an SFT Y_1 in Y by disallowing an $N \times N$ word with a given red boundary.

Suppose $\epsilon > 0$. For N large, the entropy of Y_M is less than ϵ, and so are the entropy drops $h(Y_k) - h(Y_{k+1})$. The map from Y to X forgetting color is finite to one and thus entropy preserving. This gives a set of subshifts in X whose entropies are ϵ-dense in $[0, h(X)]$.

A subshift W in an SFT X is a decreasing limit of SFTs W_n with $h(W_n)$ converging to $h(W)$. So, for any ϵ we can find a set of SFTs in S with entropies ϵ-dense in $[0, h(X)]$. QED

Note, this simple proof of dense entropies never actually computes an entropy. This is good, because of course in general we cannot.

From the previous result, one easily constructs for any number in $[0, h(S)]$ a subshift of S with that entropy.

So, any positive entropy Z^2 SFT has many subsystems, including many SFT of large entropy. But they cannot in general be separated.

I want to show a construction of a positive entropy Z^d sofic shift, $d \geq 2$, whose only minimal subsystem is a fixed point. Thus, any two of its subsystems have nonempty intersection. (There are stronger related results, and implications for quotient maps. Michael Schraudner discussed much of this on Monday.)

The proof uses new constructive results (next slide). It is an example of the strength of the new results, for providing general constructive tools.

Effective Systems

Hochman has defined an Effective Symbolic System (ESS) to be a $\mathbb{Z}^d$ subshift $\sigma_{\mathcal{L}}$ such that the defining set $\mathcal{L}$ of forbidden finite configurations is the output of a Turing machine.

THEOREM (Hochman) For each $d \geq 3$, up to topological conjugacy the following classes of $\mathbb{Z}$ subshift are the same:

- $\mathbb{Z}$ subshifts isomorphic to σ^{e_1} for some $\mathbb{Z}^d$ sofic shift σ
- The class of ESS's.

(With SFT in place of sofic, the class of ESS's becomes just slightly more narrow.)

The sofic result was improved to $d \geq 2$ by Durand, Romashchenko and Shen, "Fixed-point tile sets and their applications". Again, there is a mildly weaker statement for SFTs.

VI. A Hint of Proof

The construction ($d = 2$).

- Construct an effective $\mathbb{Z}$ subshift W such that arbitrarily large blocks of 0's occur syndetically in all points, and every W word occurs with positive frequency in every point. [Standard ergodic theory exercise.]

- For $i = 1, 2$, pick a $\mathbb{Z}^2$ sofic shift T_i for which σ^{e_i} is a copy of W.

- Then in each coordinate of $T_1 \times T_2$, for each $M \in \mathbb{N}$, strings of M consecutive zeros occur syndetically.

- Let W be the quotient of $T_1 \times T_2$ by the map which replaces a symbol (a, b) with 0 if either of a, b is zero, and otherwise replaces (a, b) with 1.

- For every M and $w \in W$, every finite configuration in W occurs inside some large block configuration on which the boundary is covered by M-thick slabs of zeros.

- Define S by freely allowing the replacement of 1 in a configuration with symbols from $\{1_1, 1_2, ..., 1_k\}$ for k large.

- Easily:

  - $h(\sigma_X) > M$ (for large enough k)

  - σ_X has a unique minimal subsystem, 0^{Z^2}.

  - The only SFT which is in X or in a quotient of X is a fixed point.

Note how convenient the general theorem is for giving us a tool for now-simple construction. We do not have to make an ingenious our intricate argument.

The system S constructed may look too degenerate to be satisfying. For example, replacing each 1_i with 1 produces a factor map from S onto the zero entropy system W. Such defects can be repaired.

For example, if in S we replace the symbol 1_1 with 0, then we get a new sofic shift, of equal entropy, but now without any zero entropy factor other than the system consisting of a fixed point. Alternately, we could do something more complicated and produce a mixing system.

The construction, together with some information about the general method for producing the sofic system with the prescribed directional subshift, produces an example of a positive entropy Z^d SFT $d \geq 2$, all of whose subsystems must intersect a given zero entropy subshift.

It is an open question as to whether, in the SFT case, this zero entropy subshift can be a fixed point.