Symbolic extensions and entropy structure

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Positive K-theory

This is the "recent progress" topic I almost chose. Positive K-theory is a systematic tool for constructions and classifications in symbolic dynamics. See

- M. Boyle and J.B.Wagoner, *Positive Algebraic K-theory and shifts of finite type*, Katok Festschrift.

and its references. Two recent developments:

- M. Boyle and M. Sullivan, *Equivariant flow equivalence for shifts of finite type*.
- M. Boyle, J. Buzzi and R. Gomez, *Almost isomorphism for countable state Markov shifts*.

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On to our actual topic.

This talk refers to the following
(mostly [BD1] and [D2]):

[BD1] M. Boyle and T. Downarowicz, *The entropy theory of symbolic extensions*, Inventiones Math. (2004)

[BFF] M. Boyle, D. Fiebig, U. Fiebig. *Residual entropy, conditional entropy and subshift covers*, Forum Math. (2002)

[D1] T. Downarowicz, *Entropy of a symbolic extension of a totally disconnected dynamical system*, ETDS (2001)

[D2] T. Downarowicz, *Entropy Structure*, J. d'Analyse to appear.

[DN] T. Downarowicz and S. Newhouse, *Symbolic extensions in smooth dynamical systems*, preprint (2002).

[DS] T. Downarowicz and J. Serafin, *Possible entropy functions* Israel J. Math (2003)

In this lecture:

- All spaces are compact metrizable.

- (X, T) denotes a homeomorphism, $T : X \to X$, with $\mathbf{h}_{\mathrm{top}}(T) < \infty$.

- $\mathcal{M}_T$ is the space of T-invariant Borel probabilities.

- A subshift (Y, S) is the restriction of the full shift on a finite alphabet to a closed invariant subsystem.

- A *symbolic extension* of (X, T) is a subshift (Y, S) with a continuous surjection $\varphi : Y \to X$ such that $T\varphi = \varphi S$.

The main messages

1. We will get a good entropy theory for the possible symbolic extensions of a system (X, T).

2. This leads to the Entropy Structure theory of Downarowicz. This is a meaningful, general theory for the emergence of entropy on refining scales.

A finer general structure behind entropy theory is revealed.

DEFN The (topological) residual entropy of T is $\mathbf{h}_{\text{res}}(T) = \inf\{\mathbf{h}_{\text{top}}(S)\} - \mathbf{h}_{\text{top}}(T)$, where the inf is over the symbolic extensions of T.

THM [BFF, D1] For $0 < \alpha < \infty$, $0 \le \beta \le \infty$, there exist T with $\mathbf{h}_{\text{top}}(T) = \alpha$, $\mathbf{h}_{\text{res}}(T) = \beta$.

"Intuitively": $\mathbf{h}_{\text{res}}(T) > 0$ reflects nonuniform emergence of entropy on refining scales.

EXAMPLE 1 (extreme)
$\mathbf{h}_{\text{top}} = \log 2$ with
$\mathbf{h}_{\text{res}}(T) = \infty$ follows from

• At a scale ϵ, T looks like (has n, ϵ orbits like) the 2-shift;

• there is a constant $c > 0$ such that for every scale ϵ, for every periodic orbit $\mathcal{O}$ of T, there is some scale $\delta < \epsilon$ (depending on $\mathcal{O}$) at which the orbit resolves into a system with entropy $\geq c$.

There are more results in [BFF, D1], e.g.
[BFF] T is $C^\infty \implies \mathbf{h}_{\text{res}}(T) = 0$.

Questions left open included

1. "What about smooth examples?"
   [A. Katok, 1991]

2. Given T with $\mathbf{h}_{\text{res}}(T) < \infty$, must there exist a symbolic extension (Y, S) of (X, T) such that $\mathbf{h}_{\text{top}}(S) = \mathbf{h}_{\text{top}}(T) + \mathbf{h}_{\text{res}}(T)$?

In [BD1] we investigate entropy obstructions to symbolic extensions at the level of measures. This leads to

- Lots of bad smooth examples! [DN]
- The answer to Question 2 is NO. [BD1]
- Clarification of "intuitively". [BD1]
- A master entropy invariant. [D2]

Extension entropy. Consider a homeomorphism T of a compact metric space X. Given a symbolic extension $\varphi : (Y,S) \to (X,T)$ define its extension entropy function

$$\begin{aligned} h^{\varphi}_{\text{ext}} : \mathcal{M}_T &\to [0,\infty) \\ \mu &\mapsto \max\{h(S,\nu) : \varphi\nu = \mu\} \ . \end{aligned}$$

Symbolic extension entropy. Given (X,T), we define its symbolic extension entropy function to be the function $h^T_{\text{sex}} : \mathcal{M}_T \to [0,\infty)$ which is the infimum of all h^{φ}_{ext} arising from symbolic extensions φ of (X,T). (So, either h^T_{sex} is bounded or is identically ∞.)

(Abbreviating, we call h^T_{sex} the sex entropy function of T.) When h^T_{sex} is bounded, $h^T_{\text{sex}}(\mu)$ gives a quantitative measure of the emergence of complexity on finer scales "near" the support of μ.

For every (X,T), we will give a functional analytic characterization of the functions on $\mathcal{M}_T$ which can arise as h^{φ}_{ext} for a symbolic extension φ of (X,T), and also a functional analytic characterization of h^T_{sex}. This will reveal a rich and subtle structure.

Entropy structure. An entropy structure for (X,T) is an allowed nondecreasing sequence of nonnegative functions h_n on $\mathcal{M}_T$, converging to the entropy function h.

The sequence (h_n) will describe the emergence of entropy on refining scales. The correct determination of "allowed" is achieved in [D2] (more later). In [BD1] it is important to work with (h_n) which also has the property that the functions h_n and $h_{n+1} - h_n$ are uppersemicontinuous (u.s.c.). Here is one example of an allowed (h_n) which gives the right intuition and suffices in many cases.

Suppose the system (X, T) admits a refining sequence of partitions P_n with *small boundaries* (the boundary of the closure of each partition element has μ-measure zero for every μ in $\mathcal{M}_T$), and with the maximum diameter of elements of P_n going to zero as $n \to \infty$. Define $h_n(\mu) = h(\mu, P_n)$. Then the sequence (h_n) is an entropy structure for (X, T).

Not every system (X, T) admits such a sequence P_n. However, if the periodic point set of T is zero-dimensional (e.g. countable) and X is finite-dimensional, then there is such a sequence [Kulesza]. We will give one general construction later.

The uppersemicontinuity of h_n and of $h_{n+1} - h_n$ follows here from the small boundaries because the inf of continuous functions is u.s.c., e.g.

$$h_n(\mu) = \inf_k \frac{1}{k} H_\mu\Big(\bigvee_{i=0}^{k-1} T^{-i} P_n\Big) .$$

Superenvelopes. Suppose (h_n) is an entropy structure with all $h_n - h_{n-1}$ u.s.c. A bounded function E on $\mathcal{M}_T$ such that every $E - h_n$ is nonnegative u.s.c. is called a *superenvelope* of the entropy structure. (For notational reasons, we also allow the constant function $E \equiv \infty$ as a superenvelope.)

The main result of [BD1] is the

Sex Entropy Theorem.
Let E be a bounded function on $\mathcal{M}_T$. T.F.A.E.

1. E is the extension entropy function of a symbolic extension of (X, T).

2. E is affine and a superenvelope of the entropy structure.

(The statement does not depend on the choice of entropy structure.)

Corollary h^T_{sex} is the minimum superenvelope of the entropy structure (h_n).

Corollary If h^T_{sex} is bounded, then for a residual subset of $\mathcal{M}_T$, $h^T_{\mathsf{sex}}(\mu) = h(\mu)$.

The Sex Entropy Theorem is one of two ingredients which move many questions about sex entropy into the realm of pure functional analysis. The other ingredient is a realization theorem:

THEOREM [DS] Let (h_n) be a sequence of affine nonnegative u.s.c. functions on a metrizable Choquet simplex, with nonnegative u.s.c. differences, converging to a bounded function h. Then (h_n) is an entropy structure for a dynamical system.

The entropy structure characterization of h^T_{sex} leads to a illuminating recursive characterization of h^T_{sex}.

Inductive Characterization of h_{sex}
Let $\tilde{g}$ denote the u.s.c. envelope of a function g (the inf of the continuous functions larger than g). Convention: $\tilde{g} \equiv \infty$ if $\sup g = \infty$.

Let $\mathcal{H} = (h_n)$ be an entropy structure, $h_n \to h$. Begin with the tail sequence $\tau_n = (h - h_n)$, which decreases to zero. We will define by transfinite induction a transfinite sequence $u^{\mathcal{H}}$ of functions u_α on $\mathcal{M}_T$. Set
- $u_0 \equiv 0$ • $u_{\alpha+1} = \lim_k \widetilde{(u_\alpha + \tau_k)}$
- $u_\beta =$ the u.s.c. envelope of $\sup\{u_\alpha : \alpha < \beta\}$, if β is a limit ordinal.

DEFINITION (u_α) is the *transfinite sequence* of $\mathcal{H}$.
THEOREM $u_\alpha = u_{\alpha+1} \iff u_\alpha + h = h_{\text{sex}}$, and such an α exists among countable ordinals (even if $h_{\text{sex}} \equiv \infty$).
The convergence above can be transfinite, and this indicates the subtlety of the emergence of complexity on ever smaller scales. However the characterization is also of practical use for constructing examples.

More Consequences of the Sex Entropy Theorem.
Define $h_{\text{res}}(\mu) = h_{\text{sex}}(\mu) - h(\mu)$.
Suppose h_{sex} is bounded (i.e. not $\equiv \infty$). Then

• h_{res} and h_{sex} are u.s.c.

• The sup of h_{sex} can exceed the sup over the ergodic measures (but the max will be achieved on the closure of the ergodic measures).

• (Sex Entropy Variational Principle)
$\mathbf{h}_{\text{sex}}(T) = \max_\mu h_{\text{sex}}(\mu)$, where
$\mathbf{h}_{\text{sex}}(T) := \inf\{\mathbf{h}_{\text{top}(S)} : S \text{ is a sym. ext. of } T\}$

• The infimum of the topological entropies of symbolic extensions of (X,T) need not be realized by any symbolic extension of (X,T).

• If $\mathcal{M}_T$ is a Bauer simplex (i.e. the ergodic measures form a closed set), then h_{sex} is affine, and is realized as the extension entropy function for some symbolic extension.

Topological tail entropy

This is the term we use for $\mathbf{h}^*(T)$, the "conditional topological entropy" of Misiurewicz.

Given an entropy structure (h_n), the topological tail entropy has a quick description:

$$\mathbf{h}^*(T) \ = \ \lim_n ||h - h_n||_{\text{sup}} \ .$$

It is a difficult theorem of Downarowicz [D2] that the RHS of this equation agrees with the original definition of Misiurewicz.

The case $\mathbf{h}^*(T) = 0$ (h_n converges uniformly) is the case T is *asymptotically h-expansive*. TFAE:

(1) $h_{\text{sex}} = h$.

(2) $\mathbf{h}^*(T) = 0$.

(3) (X, T) has a symbolic extension φ which is a principal extension (i.e. $h^{\varphi}_{\text{ext}} = h$).

A reference entropy structure.

[BD1] provides a general (but complicated) method for producing one entropy structure (h_n) on any given (X,T).

Given (X,T), take a strictly ergodic zero entropy nonperiodic (Z,R) with unique invariant measure λ. An easy consequence of the deep work of Elon Lindenstrauss on mean dimension, following his earlier work with Benjy Weiss, is that the product $(X \times Z, T \times R)$ does have a refining sequence of partitions with small boundaries. We can then define an entropy structure (h'_n) on this product system as before. Now for our entropy structure on (X,T) we use (h_n) where $h_n(\mu) := h'_n(\mu \times \lambda)$. The Sex Entropy Theorem holds for this structure.

Entropy Structure [D2]

Let $\mathcal{F} = (f_k)$ and $\mathcal{G} = (g_k)$ be nondecreasing sequences of functions on a common domain. Say $\mathcal{F}$ *uniformly dominates* $\mathcal{G}$ ($\mathcal{F} > \mathcal{G}$) if

$$\forall k \ \forall \epsilon \ \exists m, \quad f_m > g_k - \epsilon \ .$$

Say $\mathcal{F} = (f_k)$ and $\mathcal{G} = (g_k)$ are *uniformly equivalent* if $\mathcal{F} > \mathcal{G}$ and $\mathcal{G} > \mathcal{F}$.

An *entropy structure* for (X, T) is any nondecreasing nonnegative sequence $\mathcal{F}$ of functions f_n on $\mathcal{M}_T$ which is uniformly equivalent to the reference structure (h_n) of the previous page.

NOTE: f_n and $f_{n+1} - f_n$ need not be u.s.c.
DEFN: a bounded function E is a superenvelope of $\mathcal{F}$ if the defect of upper semicontinuity of $E - f_n$ goes to zero with n.
(Equiv. to earlier condition in the case all $f_{n+1} - f_n$ u.s.c.)

Uniformly equivalent sequences have the same limit, superenvelopes and transfinite sequence. For any entropy structure $\mathcal{F}$:

- $\lim f_n = h$ and $\lim_n \max f_n = \mathbf{h}_{\text{top}}$.

- The superenvelopes of $\mathcal{F}$ are the symbolic extension entropy functions.

- The infimum of those superenvelopes is h^T_{sex}.

Various approaches to entropy can be adapted to give entropy structures:

• **Katok** (count Bowen ball covers)
• **Brin-Katok** (shrink rate of measure of (n, ϵ) ball
• **Newhouse** (local entropy)
• **Bowen** (entropy of a set)
• **Ornstein-Weiss/D.** (return times to (n, ϵ) ball)
• **Romagnoli** (how entropy $h_\mu(T, P)$ is forced when elements of partition P must have small diameter)
Downarowicz analyzes everything [D2].

For an arbitrary refining sequence of partitions P_n generating the Borel σ-algebra, the sequence $h_n(\mu) = h_n(\mu, P_n)$ need NOT give an entropy structure.

An entropy structure from Katok's entropy

Here is one example of adapting an existing approach to give an entropy structure.
Given μ ergodic in $\mathcal{M}_T$ and $\epsilon > 0$:
fix any σ, $0 < \sigma < 1$: define
$h^{\text{Katok}}(\mu, \epsilon)$:= growth rate of min. number of (n, ϵ) balls with union of μ measure at least σ.

For nonergodic μ, define via ergodic decomposition:
$h^{\text{Katok}}(\mu, \epsilon) := \int h^{\text{Katok}}(\mu, \epsilon)\ dt$

For a sequence ϵ_n decreasing to 0, define
$h_n^{\text{Katok}}(\mu) = h^{\text{Katok}}(\mu, \epsilon_n)$.

Then (h_n^{Katok}) is an entropy structure.

A new idea [D2] gives an especially tractable entropy structure. For $g \in C(X, [0,1])$ let $\mathcal{A}_g$ be the partition of $X \times [0,1]$ into $\{(x,t) : t \leq g(x)\}$ and its complement. For a finite family $\mathcal{G}$ in $C(X,[0,1])$ and $n \in \mathbb{N}$, let $Q_n(\mathcal{G})$ be the join of the $\mathcal{A}_{g \circ T^i}$, $g \in \mathcal{G}$, $0 \leq i < n$. Define $h(\mu, \mathcal{G}) = \lim_n \frac{1}{n} H(\mu \times \lambda, Q_n)$ where λ is Lebesgue measure on $[0,1]$.

If $\mathcal{G}_k$ is a refining sequence such that $h(\mu, \mathcal{G}_k)$ converges to the entropy function $h(\mu)$ then the sequence of functions $h(\mu, \mathcal{G}_k)$ is an entropy structure. (The sequence does not even have to separate points!)

In particular, it is now easy to see that a topological conjugacy of systems will induce a bijection of entropy structures. So, entropy structure is a topological invariant.

Given

- the uniform equivalence of most ways of computing entropy on refining scales,

- the very fine entropy invariants determined by a uniform equivalence class, and

- the accessability (at least sometimes) of these invariants

it is reasonable to regard the equivalence class of entropy structures as a master entropy invariant.

A finer general structure behind entropy theory is revealed.

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Smooth results and open problems:
go to Sheldon's talk!