# Borel dynamics and countable state Markov shifts

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joint work with
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## Outline of the talk

**I. Countable state Markov shifts, loop shifts.**

**II. Classes of Markov shift, defined by their loop shifts.**

**III. Entropy Conjugacy (EC) and Almost Isomorphism (AI).**

**IV. Hochman and Universality.**

**V. Almost Borel isomorphism.**

**VI. Exploring AI (and SDE).**

**VII. Bowen's entropy conjugacy definition and conjecture.**

## I. Countable state Markov shifts. Loop shifts.

Let $A$ be an $\mathbb{N} \times \mathbb{N}$ matrix over $\mathbb{Z}_+$. $A$ is the adjacency matrix of a directed graph, $G_A$; $A(i,j)$ is the number of edges from $i$ to $j$.

$X_A$ denotes the space of doubly infinite sequences $x$ of edges, such that for all $n$ the initial vertex of $x_n$ equals the terminal vertex of $x_{n-1}$. The cylinders
$C(x,n) := \{y \in X_A : y_i = x_i, -n \leq i \leq n\}$
are a basis of open sets for the topology on $X_A$.

In this talk: $A$ is irreducible (for each pair $i,j$ there is a path from $i$ to $j$) and aperiodic (the g.c.d. of periods of loops is one). So, $(X_A, \sigma_A)$ is mixing. We allow $A$ to be finite (then, with index set some integer $N$ rather than $\mathbb{N}$).

Now "Markov shift" means countable (possibly finite) state, finite entropy, mixing Markov shift.

The Markov shifts are tools for studying other dynamics, and are also topological dynamical systems themselves.

Given a power series $f(z) = \sum_{n=1}^{\infty} f_n z^n$, its loop shift $\sigma_f$ is the Markov shift whose graph has a base vertex $v$ such that the number of first return paths of length $n$ for $v$ is $f_n$, and every vertex except $v$ has a unique incoming edge and a unique outgoing edge.

Say $\sigma_f$ is a loop shift of $(X_A, \sigma_A)$ if there is a vertex $v$ in $G_A$ such that the number of first return paths to $v$ is $f_n$. Then the loop shift embeds naturally as a subsystem of $(X_A, \sigma_A)$.

This embedded loop shift supports every ergodic $\sigma_A$-invariant measure which assigns positive measure to an edge adjoining $v$.

## II. Classes of Markov shift, defined by their loop shifts

Let $\sigma_f$ be any loop shift of a Markov shift $\sigma_A$ with entropy $h(S) = \log(\lambda) < \infty$. Recall the definitions: $\sigma_A$ is

- Transient (T) if $f(1/\lambda) < 1$

- Recurrent (R) if $f(1/\lambda) = 1$

- Positive Recurrent (PR) if $f(1/\lambda) = 1$ and $f'(1/\lambda) < \infty$

- Strong Positive Recurrent (SPR) if $\limsup (f_n)^{1/n} < \lambda$.

Then $T \cap R = \emptyset$ and $R \subset PR \subset SPR$. The recurrent, not positive recurrent, shifts are Null Recurrent (NR).

The PR Markov shifts are the finite entropy Markov shifts with a measure of maximal entropy.

But it is the SPR Markov shifts which are the real "countable state shifts of finite type", with several characterizations:

- There is a meromorphic extension of the zeta function of $\sigma_f$ to a disc of radius $> 1/\lambda$.

- A measure of maximal entropy exists and is exponentially recurrent

- Every proper closed subsystem has strictly smaller entropy.

## III. Entropy Conjugacy (EC) and Almost Isomorphism (AI)

For a self isomorphism $S$ of a Borel space $X$, define its entropy as the sup of $h(S,\mu)$ over the $S$-invariant Borel probabilities $\mu$. Say a set $E$ is entropy-negligible (for $S$) if $\exists \epsilon > 0$ such that $\mu(X \setminus E) = 1$ for every $S$-invariant ergodic Borel probability $\mu$ with $h(S,\mu) > h(S) - \epsilon$.

Say a set is entropy-full if its complement is entropy negligible.

DEFINITION (Buzzi) Systems $S$ and $T$ are entropy conjugate if there is a Borel isomorphism of their actions on entropy full sets.

Buzzi showed certain interesting classes of smooth or piecewise smooth maps are entropy conjugate to SPR shifts.

Rufus Bowen gave a different (not equivalent) definition for "entropy conjugacy" (see end of this talk).

A magic word for a 1-block code $\phi$ between Markov shifts, $\phi : X_A \to X_B$, is a word $W$ for $X_B$ with the following properties:

(1) If $y \in X_B$ and $W$ occurs infinitely often on both negative and positive coordinates of $y$, then $x$ is in the range of $\phi$.

(2) If $y[i,j] = WUW =$ and $\phi(x) = y$, then on the coordinates of $[i,j]$ where $U$ occurs, the word in $x$ is determined by $U$.

DEFINITION Markov shifts $\sigma_A, \sigma_B$ are almost isomorphic (AI) if there exists a Markov shift $\sigma_C$, and injective 1-block codes from $X_C$ into $X_A$ and $X_B$, each with a magic word.

Every Markov shift is AI to any of its loop shifts.

For SPR shifts: AI implies Entropy Conjugacy (by a map which is continuous on an entropy full set in a very nice way).

Here is the countable-state generalization of the Adler-Marcus Theorem for finite state Markov shifts (shifts of finite type).

THEOREM (BBG 2006) SPR Markov shifts of equal entropy are AI.

COROLLARY (BBG 2006) SPR Markov shifts of equal entropy are EC.

So, what about EC for PR, we asked.

Mike Hochman answered, with much more.

## IV. Hochman and Universality

DEFINITIONS

For a Borel system $S$, a set $F$ is $t$-full if it is Borel with full measure for every ergodic nonatomic $S$ invariant probability $\mu$ such that $h(\mu, T) < t$.

A $t$-slice of a Borel system $S$ is its restriction to a $t$-full set with zero measure for every ergodic invariant probability of entropy $\geq t$.

DEFINITION A Borel system $S$ is $t$-universal if a $t$-slice of any Borel system embeds as a Borel subsystem of $S$.

Hochman [H] showed

THEOREM A mixing Markov shift of entropy $h$ is $h$-universal.

Corollaries

- If $S$ and $T$ are $h$ universal, then they have $h$-slices restricted to which they are Borel conjugate.

- If $h$ is the sup of entropies of Markov shifts contained in $S$, then $S$ is $h$-universal.

- If $S, T$ are Markov shifts of equal entropy $h$, then they have $h$-slices on which they are Borel conjugate. $S$ and $T$ are entropy conjugate if and only if [both are recurrent, or neither is recurrent].

Note the second item. $h$ universal systems are all over the place.

What is left to ask after such a decisive result?

[Hochman]:
For Markov shifts of equal entropy,

1. Can the t-slices for the Theorem be chosen such that between these slices the Borel conjugacy is a homeomorphism?

(Hochman's Borel isomorphisms, for a nonatomic ergodic probability $\mu$, are continuous AFTER restriction to a set of $\mu$ measure 1.)

2. Is there a Borel conjugacy between the restrictions of $S, T$ to the complements of their periodic points? ("almost Borel isomorphism")

(The sets to which Hochman restricts are not constructed to support the infinite invariant measures.)

## V. Almost Borel isomorphism

Theorem (BBG)
SPR Markov shifts of equal entropy are Borel isomorphic on the complement of the periodic points.

(And, the isomorphism is continuous on an entropy-full set.)

Proof ingredients:

- SPR of equal entropy are AI
- Krieger's embedding theorem
- Cantor-Bernstein type argument
- some trickery

## VI. Exploring AI (and SDE)

We would like to understand better AI and its context for Markov shifts. As usual, we reduce to the study of loop shifts.

For a loop shift $\sigma_f$ with base vertex $v$, set
$p_n =$ the number of points of period $n$;
$q_n =$ the number of points of least period $n$;
$t_n =$ the number of length $n$ paths from $v$ to $v$.
Define series $\zeta, t, p, q$ for $\sigma_f$:

- $\zeta(z) = \exp(\sum_{n=1}^{\infty} \frac{1}{n} p_n z^n) \;=\; \frac{1}{1-f(z)}$

- $t(z) = \sum_{n=1}^{\infty} t_n z^n \;=\; \frac{1}{1-f(z)} - 1$

- $p(z) = \sum_{n=1}^{\infty} p_n z^n \;=\; \frac{zf'(z)}{1-f(z)}$

- $q(z) = \sum_{n=1}^{\infty} q_n z^n$ .

For power series $f, g$ :
$f \leq g$ means $f_n \leq g_n$ for all $n$.

DEFINITION Loop shifts $\sigma_f, \sigma_g$ are Shift Dominant Equivalent (SDE) if there exists $n > 0$ such that (coefficientwise) $z^n t_f(z) > t_g(z)$ and $z^n t_g(z) > t_f(z)$.

Always: for Markov shifts,
AI $\Longrightarrow$ SDE $\Longrightarrow$ equal entropy.

Converses depend on the class of the Markov shift:

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|---|---|---|---|---|---|---|
| $h < \infty$ | SPR | $h$ | $=$ | SDE | $=$ | AI |
| | PR | $h$ | $=$ | SDE | $\ll$ | AI |
| | N | $h$ | $\ll$ | SDE | $\Leftarrow$ | AI |
| | T | $h$ | $\ll$ | SDE | $\Leftarrow$ | AI |
| $h = \infty$ | LZF | $h$ | $\ll$ | SDE | $\Leftarrow$ | AI |

Above,

- $A = B$ means that the equivalence relations $A, B$ are the same

- $A << B$ means a B-equivalence class can contain uncountably many A-equivalence classes

- At the three implication arrows, we don't know if AI $=$ SDE, or if SDE $<<$ AI.

In progress: there may be series characterizations of AI. Given $\sigma_f$, let $e^{(f,N)}$ denote the series $z^N/(1-f(z))$. Let $q(e)$ denote the $q$-series for a series $e(z)$. To be checked:

EXPECTED T.F.A.E. for loop shifts $\sigma_f, \sigma_g$.

1. They are AI.

2. There exists $N$ such that $q(e^{(f,N)}) \leq q(g)$ and $q(e^{(g,N)}) \leq q(f)$.

3. There exists $N$ such that $p(e^{(f,N)}) \leq p(g)$ and $p(e^{(g,N)}) \leq p(f)$.

## VII. Bowen's entropy conjugacy definition and conjecture.

DEFINITION [Bowen,1977, Topological Entropy for Noncompact Sets] Let $S, T$ be continuous maps on compact metric spaces. Then $S, T$ are entropy conjugate if they have restrictions to "entropy-full" sets which are topologically conjugate.

REMARK. Bowen demands more of entropy conjugacy (than Buzzi does), in two ways.
1. Bowen requires topological (not just Borel) conjugacy after restriction
2. Bowen's entropy-negligible sets are defined as for Buzzi, but with a different definition of entropy (later). Bowen's entropy negligible sets are entropy-negligible in Buzzi's sense, but not conversely in general.

CONJECTURE [Bowen] Mixing shifts of finite type (finite state Markov shifts) of equal entropy are entropy conjugate.

This conjecture of Bowen (for EC in his sense) is to my knowledge still open.

Curiously, off the shelf symbolic dyanmics gives a strong though partial constructive result.

Associated to a mixing SFT of entropy $\log(\lambda)$ is an ideal class in the ring $\mathbb{Z}[1/\lambda]$. Given $\lambda$, there are finitely many possible ideal classes.

Using results from [Boyle-Marcus-Trow 1987]: mixing SFTs with the same entropy and ideal class are entropy conjugate in the sense of Bowen.

For some entropies, (e.g. $\lambda \in \mathbb{N}$ or $\lambda = (1+\sqrt{5})/2$), there is just one ideal class, and therefore all mixing SFTs of that entropy are entropy conjugate in the sense of Bowen.

Here is a version of Bowen's definition of the entropy of a set $G$, in the case that $G$ is a subset of a finite state subshift $T$. It is a dynamical analogue of Hausdorf dimension designed to apply to noncompact sets.

Suppose $\alpha > 0$ and $N \in \mathbb{N}$.
Say an $N, k$ cylinder is one of the form
$C = \{y : y_i = x_i, -N \leq i \leq k\}$, for some $k > N$.
Then define $d_\alpha(C) = \alpha^{-k}$.

For a cover $\mathcal{C}$ of $G$ by countably many $N, k$ cylinders, define $d(\alpha, \mathcal{C}) = \sum_{C \in \mathcal{C}} d_\alpha(C)$.

Define $d_{(N,M,\alpha)}(G)$ as the infimum of $d(\alpha, \mathcal{C})$ over covers $\mathcal{C}$ of $G$ by $N, k$ cylinders with $k > M$.

Finally, the entropy of $G$ is $\log(\alpha_0)$, where

$$\alpha_0 = \inf\{\alpha : \lim_N \lim_M d_{(N,M,\alpha)}(G) < \infty\} .$$