The entropy theory of symbolic extensions

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joint work with
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This talk primarily concerns the joint work

[BD] M. Boyle and T. Downarowicz, *The entropy theory of symbolic extensions*, Inventiones Math. (2004)

and also refers a little to

[BFF] M. Boyle, D. Fiebig, U. Fiebig. *Residual entropy, conditional entropy and subshift covers*, Forum Math. (2002)

[D1] T. Downarowicz, *Entropy of a symbolic extension of a totally disconnected dynamical system*, ETDS (2001)

[D2] T. Downarowicz, *Entropy Structure*, J. d'Analyse to appear.

[DN] T. Downarowicz and S. Newhouse, *Symbolic extensions in smooth dynamical systems*, preprint (2002).

[DS] T. Downarowicz and J. Serafin, *Possible entropy functions* Israel J. Math (2003)

In this lecture:

- All spaces are compact metrizable.

- (X, T) denotes a homeomorphism, $T : X \to X$, with $\mathbf{h}_{\text{top}}(T) < \infty$.

- $\mathcal{M}_T$ is the space of T-invariant Borel probabilities.

- A subshift (Y, S) is the restriction of the full shift on a finite alphabet to a closed invariant subsystem.

- A *symbolic extension* of (X, T) is a subshift (Y, S) with a continuous surjection $\varphi : Y \to X$ such that $T\varphi = \varphi S$.

EXAMPLES: (X, T) has a symbolic extension if T admits Markov partitions (Adler-Weiss, Sinai, Bowen, Fried) or more generally if T is expansive (Reddy).

Symbolic extensions can be a tool for studying (X, T) (e.g. via Markov partitions).

Symbolic extensions will emerge as a general tool for exploring the complexity of systems (X, T).

QUESTION [Auslander 89] If $\mathbf{h}_{\text{top}}(T) < \infty$, must (X, T) have a symbolic extension?

DEFN The (topological) residual entropy of T is $\mathbf{h}_{\text{res}}(T) = \inf\{\mathbf{h}_{\text{top}}(S)\} - \mathbf{h}_{\text{top}}(T)$, where the inf is over the symbolic extensions of T.

THM [BFF, D1] For $0 < \alpha < \infty$, $0 \leq \beta \leq \infty$, there exist T with $\mathbf{h}_{\text{top}}(T) = \alpha$, $\mathbf{h}_{\text{res}}(T) = \beta$.

"Intuitively": $\mathbf{h}_{\text{res}}(T) > 0$ reflects nonuniform emergence of entropy on refining scales.

EXAMPLE 1 (extreme) $\mathbf{h}_{\text{top}} = \log 2$ with $\mathbf{h}_{\text{res}}(T) = \infty$ follows from

- At a scale ϵ, T looks like (has n, ϵ orbits like) the 2-shift;

- there is a constant $c > 0$ such that for every scale ϵ, for every periodic orbit $\mathcal{O}$ of T, there is some scale $\delta < \epsilon$ (depending on $\mathcal{O}$) at which the orbit resolves into a system with entropy $\geq c$.

There are more results in [BFF, D1], e.g.
[BFF] T is $C^\infty \implies \mathbf{h}_{\text{res}}(T) = 0$.

Questions left open included

1. For $1 \leq k < \infty$,
   if T is C^k, must $\mathbf{h}_{\text{res}}(T) = 0$?

2. Given T with $\mathbf{h}_{\text{res}}(T) < \infty$, must there exist a symbolic extension (Y, S) of (X, T) such that $\mathbf{h}_{\text{top}}(S) = \mathbf{h}_{\text{top}}(T) + \mathbf{h}_{\text{res}}(T)$?

In [BD] we investigate entropy obstructions to symbolic extensions at the level of measures. This leads to

- The answer to Question 1 is NO. [DN]
- The answer to Question 2 is NO. [BD]
- Clarification of "intuitively". [BD]
- A master entropy invariant. [D2]

Extension entropy. Consider a homeomorphism T of a compact metric space X. Given a symbolic extension $\varphi : (Y, S) \to (X, T)$ define its extension entropy function

$$h_{\text{ext}}^{\varphi} : \mathcal{M}_T \to [0, \infty)$$
$$\mu \mapsto \max\{h(S, \nu) : \varphi\nu = \mu\} \; .$$

Symbolic extension entropy. Given (X, T), we define its symbolic extension entropy function to be the function $h_{\text{sex}}^{T} : \mathcal{M}_T \to [0, \infty)$ which is the infimum of all h_{ext}^{φ} arising from symbolic extensions φ of (X, T). (So, either h_{sex}^{T} is bounded or is identically ∞.)

(Abbreviating, we call h_{sex}^{T} the sex entropy function of T.) When h_{sex}^{T} is bounded, $h_{\text{sex}}^{T}(\mu)$ gives a quantitative measure of the emergence of complexity on finer scales "near" the support of μ.

For every (X, T), we will give a functional analytic characterization of the functions on $\mathcal{M}_T$ which can arise as h^{φ}_{ext} for symbolic extension φ of (X, T), and also a functional analytic characterization of h^T_{sex}. This will reveal a remarkably rich and subtle structure.

Entropy structure. An entropy structure for (X, T) is an allowed nondecreasing sequence of nonnegative functions h_n on $\mathcal{M}_T$, converging to the entropy function h.

The sequence (h_n) will describe the emergence of entropy on refining scales. The general determination of "allowed" is achieved in [D2] (Tomasz's talk). In [BD] it is important to work with (h_n) which also has the property that the functions h_n and $h_{n+1} - h_n$ are uppersemicontinuous (u.s.c.). Here is one example of an allowed (h_n) which gives the right intuition and suffices in many cases.

Suppose the system (X, T) admits a refining sequence of partitions P_n with *small boundaries* (the boundary of the closure of each partition element has μ-measure zero for every μ in $\mathcal{M}_T$), and with the maximum diameter of elements of P_n going to zero as $n \to \infty$. Define $h_n(\mu) = h(\mu, P_n)$. Then the sequence (h_n) is an entropy structure for (X, T).

Not every system (X, T) admits such a sequence P_n. However, if the periodic point set of T is zero-dimensional (e.g. countable) and X is finite-dimensional, then there is such a sequence [Kulesza]. We will give one general construction later.

The uppersemicontinuity of h_n and of $h_{n+1} - h_n$ follows here from the small boundaries because the inf of continuous functions is u.s.c., e.g.

$$h_n(\mu) = \inf_k \frac{1}{k} H_\mu\Big(\bigvee_{i=0}^{k-1} T^{-i} P_n \Big) \ .$$

Superenvelopes. Suppose (h_n) is an entropy structure with all $h_n - h_{n-1}$ u.s.c. A bounded function E on $\mathcal{M}_T$ such that every $E - h_n$ is nonnegative u.s.c. is called a *superenvelope* of the entropy structure. (For notational reasons, we also allow the constant function $E \equiv \infty$ as a superenvelope.)

The main result of [BD] is the
Sex Entropy Theorem: a bounded function on $\mathcal{M}_T$ is the extension entropy function of a symbolic extension of (X, T) if and only if it is affine and a superenvelope of the entropy structure. (This does not depend on the choice of entropy structure.)

Corollary h^T_{sex} is the minimum superenvelope of the entropy structure (h_n).

Corollary If h^T_{sex} is bounded, then for a residual subset of $\mathcal{M}_T$, $h^T_{\text{sex}}(\mu) = h(\mu)$.

The Sex Entropy Theorem is one of two ingredients which move many questions about sex entropy into the realm of pure functional analysis. The other ingredient is a realization theorem:

THEOREM [DS] Let (h_n) be a sequence of affine nonnegative u.s.c. functions on a metrizable Choquet simplex, with nonnegative u.s.c. differences, converging to a bounded function h. Then (h_n) is an entropy structure for a dynamical system.

The entropy structure characterization of h^T_{sex} leads to a illuminating recursive characterization of h^T_{sex}.

Inductive Characterization of h_{sex}
Let $\tilde{g}$ denote the u.s.c. envelope of a function g (the inf of the continuous functions larger than g). Convention: $\tilde{g} \equiv \infty$ if $\sup g = \infty$.

Let $\mathcal{H} = (h_n)$ be an entropy structure, $h_n \to h$. Begin with the tail sequence $\tau_n = (h - h_n)$, which decreases to zero. We will define by transfinite induction a transfinite sequence $u^{\mathcal{H}}$ of functions u_α on $\mathcal{M}_T$. Set

- $u_0 \equiv 0$
- $u_{\alpha+1} = \lim_k \widetilde{(u_\alpha + \tau_k)}$
- $u_\beta =$ the u.s.c. envelope of $\sup\{u_\alpha : \alpha < \beta\}$, if β is a limit ordinal.

THEOREM $u_\alpha = u_{\alpha+1} \iff u_\alpha + h = h_{\text{sex}}$, and such an α exists among countable ordinals (even if $h_{\text{sex}} \equiv \infty$).

The convergence above can be transfinite, and this indicates the subtlety of the emergence of complexity on ever smaller scales. However the characterization is also of practical use for constructing examples.

[In the actual talk the examples to follow were sketched as blackboard pictures.]

EXAMPLE 2 Let the Choquet simplex $\mathcal{M}_T$ have as its set $\mathcal{M}_T^e$ of ergodic measures a sequence μ_n such that $\lim_n \mu_n = \mu_1$. Here $\mathcal{M}_T^e$ is a closed set, and in this case it suffices to apply the inductive construction to the restriction of h_{sex} to $\mathcal{M}_T^e$. Define h_n restricted to $\mathcal{M}_T^e$ to be

$$1_{\mu_1} + \cdots + 1_{\mu_n}$$

(the sum of indicator functions of $\mu_1, \ldots, \mu_n$). Now the tail τ_n is

$$1_{\mu_{n+1}} + 1_{\mu_{n+2}} + \cdots$$

and the u.s.c. envelope of τ_n is $\tau_n + h_1$. Then

$$u_1 := \lim_n \widetilde{\tau_n} = \lim_n (\tau_n + h_1) = h_1 \ .$$

The terms of the sequence $h_1 + \tau_n$ are already u.s.c., so

$$u_2 := \lim_n (\widetilde{u_1 + \tau_n}) = \lim_n (h_1 + \tau_n) = u_1 \ .$$

Thus on $\mathcal{M}_T^e$ we have $h_{\text{sex}} = h + h_1$. Since $h \equiv 1$ and $\max(h + h_1) = 2$, the residual entropy of T is 1.

EXAMPLE 3 In the previous example, we had $0 = u_0 \neq u_1 = u_2$. Let us indicate a modification such that $0 = u_0 \neq u_1 \neq u_2 = u_3$. Keep the sequence of measures μ_n just as before. Now for each of μ_k, $k > 1$, add a sequence (μ_{kj}), $1 \leq j < \infty$, with $\lim_j \mu_{kj} = \mu_k$. Enumerate all these measures as $\nu_1, \nu_2, \ldots$ and let $h_n = 1_{\nu_1} + \cdots + 1_{\nu_n}$. Indeed the previous induction does run one step longer and we get

- $u(\mu_{kj}) = 0$ for all those new measure μ_{kj}
- $u(\mu_k) = u_1(\mu_k) = 1$ for $k = 2, 3, \ldots$
- $u(\mu_1) = u_2(\mu_1) = 2$.

Then $h_{\text{sex}} = h + u = 1 + u$.

Similarly one can get higher orders of accumulation, and transfinite orders (with more careful control over the size of h at the different ergodic measures).

EXAMPLE 4 This will give an "easy" example of a system with finite entropy but with infinite residual entropy.

We choose a system (again by [DS]) with a sequence (μ_n) of distinct ergodic measures dense in $\mathcal{M}_T^e$, where $\mathcal{M}_T^e$ has no isolated points. On $\mathcal{M}_T^e$ take

$$h_n = 1_{\mu_1} + \cdots + 1_{\mu_n}$$

Now $\max h = \mathbf{h}_{\text{top}}(T) = 1$. However, every tail τ_n equals 1 on a dense subset of $\mathcal{M}_T^e$, so $u_1 = \lim_n \widetilde{\tau_n} \equiv 1$ on $\mathcal{M}_T^e$.

Repeating, we find
$u_2 = \lim_n \widetilde{\tau_n + u_1} \equiv 2$ on $\mathcal{M}_T^e$; then
$u_3 = \lim_n \widetilde{\tau_n} \equiv 3$ on $\mathcal{M}_T^e$; etc.
We get $u_\alpha \equiv \infty$ for $\alpha = \aleph_0$, the first infinite ordinal, so the residual entropy is infinite.

EXAMPLE 5 Here without proof is an example of a system (X, T) for which the max of $h_{\text{sex}}(T)$ exceeds the sup of $h_{\text{sex}}(T)$ over ergodic measures.

Let the ergodic measures topologically look like the following subset of the plane:
$\{(0,0),(1,0)\} \cup \{(1/2, 1/k) : k = 1, 2, 3, ...\}$
so that the measures corresponding to the points $(1/2, 1/k)$ converge to a measure μ which is the average of two ergodic measures. Enumerate all these measures as $\mu_1, \mu_2, \ldots$ and on $\mathcal{M}_T^e$ set

$$h_n = 1_{\mu_1} + \cdots + 1_{\mu_n}$$

and extend to all measures via ergodic decomposition of measure theoretic entropy.

Then the maximum of h_{sex} is 2, achieved at μ, and at every ergodic measure h_{sex} is 1. In particular there is no "ergodic decomposition" for sex entropy.

Example 6.
We sketch an example of a system (X, T) where $\mathbf{h}_{\text{top}}(T) = 1$ and $\mathbf{h}_{\text{sex}}(T) = 1$, but there is no symbolic extension $\varphi : (Y, S) \to (X, T)$ with $\mathbf{h}_{\text{top}}(S) = 1$.

More precisely, appealing to [DS], we describe an entropy structure (h_n) on a Choquet simplex K. The ergodic measures (extreme points of K) are two sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, where $\lim a_n = b_1$ and $\lim b_n = c := \sum 2^{-n} a_n$. Note c is not an extreme point. jWe determine the entropy structure by dictating that on the extreme points, h_n is the sum of the indicator functions of $b_1, ..., b_n$.

Now suppose that a symbolic extension $\varphi : (Y, S) \to (X, T)$ exists with with $\mathbf{h}_{\text{top}}(S) = 1$. We will argue to a contradiction.

Let E denote the affine superenvelope of (h_n) given by the entropy extension function h^{φ}_{ext}. Now $E \leq \mathbf{h}_{\text{top}}(S) = 1$. Since $E \geq h$, we have $E(b_n) = 1$ for all n. Since E is u.s.c., we have $E(c) = 1$. Since E is affine and $E \leq 1$ it follows that $E(a_n) = 1$ for all n. Now for any $n \geq 1$, $E - h_n$ is zero at b_1, but

$$\lim_k (E - h_n)(a_k) = 1 > 0 = (E - h_n)(b_1) \ .$$

Therefore $E - h_n$ is not u.s.c., which contradicts its being a superenvelope of (h_n).

More Consequences of the Sex Entropy Theorem.
Define $h_{\text{res}}(\mu) = h_{\text{sex}}(\mu) - h(\mu)$.
Suppose h_{sex} is bounded (i.e. not $\equiv \infty$).Then

- h_{res} and h_{sex} are u.s.c.

- The sup of h_{sex} can exceed the sup over the ergodic measures (but the max will be achieved on the closure of the ergodic measures).

- (Sex Entropy Variational Principle)
 $\mathbf{h}_{\text{sex}}(T) = \max_\mu h_{\text{sex}}(\mu)$, where
 $\mathbf{h}_{\text{sex}}(T) := \inf\{\mathbf{h}_{\text{top}(S)} : S$ is a sym. ext. of $T\}$

- The infimum of the topological entropies of symbolic extensions of (X, T) need not be realized by any symbolic extension of (X, T).

- If $\mathcal{M}_T$ is a Bauer simplex (i.e. the ergodic measures form a closed set), then h_{sex} is affine, and is realized as the extension entropy function for some symbolic extension.

Topological tail entropy

This is the term we use for $\mathbf{h}^*(T)$, the "conditional topological entropy" of Misiurewicz.

Given an entropy structure, the topological tail entropy has a quick description:

$$\mathbf{h}^*(T) \ = \ \lim_n ||h - h_n||_{\text{sup}} \ .$$

It is a difficult theorem of Downarowicz [D2] that the RHS of this equation agrees with the original definition of Misiurewicz.

T is *asymptotically h-expansive* if $\mathbf{h}^*(T) = 0$. (E.g. a zero-dimensional system is asymptotically h-expansive if and only if it embeds as a subsystem of some countable product S of subshifts such that $\mathbf{h}_{\text{top}}(S) < \infty$.) Misiurewicz showed the entropy function $h : M_T \to \mathbb{R}$ is u.s.c. when T is asymptotically h-expansive (so, there are measures of maximal entropy).

In the setting of residual entropy, the asymptotically h-expansive systems form the one really special and distinguished class.

THEOREM

Let (h_n) be an entropy structure for (X,T). T.F.A.E.

(1) $h_{\text{sex}} = h$

(2) h_n converges to h uniformly.

(3) (X,T) has a symbolic extension φ which is a principal extension (i.e. $h^{\varphi}_{\text{ext}} = h$)

(4) T is asymptotically h-expansive.

SEX ENTROPY AND SMOOTHNESS

How compatible is the complexity of residual entropy with smoothness?

If (X,T) is C^{∞}, then [Buzzi following Yomdin] T is asymptotically h-expansive, and therefore $h_{\text{sex}} = h$.

THEOREM [DN] A generic C^1 non-hyperbolic (i.e. non-Anosov) area preserving diffeomorphism of a compact surface has no symbolic extension (i.e. residual entropy $= \infty$).

THEOREM [DN] For $r > 1$ and any compact Riemannian manifold of dimension > 1, there is a C^r-open set of C^r diffeomorphisms in which the diffeomorphisms with positive topological residual entropy are a residual set.

In the last theorem, D & N obtain (by generic constructions of homoclinic tangencies) a specific lower bound for the residual entropy, which they expect to be actually an equality for typical nonhyperbolic C^r systems ($2 \leq r < \infty$). They also conjecture a specific finite upper bound for the topological sex entropy of any C^r map ($2 \leq r < \infty$):

$$\mathbf{h}_{\text{sex}}(T) \leq \Big[R(f)\text{dim}(X)\Big] \ \frac{r}{r-1} \ , \quad \text{where}$$
$$R(f) = \lim_n (1/n) \log \max ||(T^n)'|| \ .$$

Proof of the Sex Entropy Theorem.
[Not actually included in the talk due to time.]

The hardest part is the construction of the symbolic extension matching the given affine superenvelope. This goes by a proof in the zero dimensional case and (in the case where we must use the more general entropy structure above) a reduction to that case. The hardest part is the zero dimensional argument which splits into four parts, given the barest of sketches below.

Part 1. Define a "simplified word oracle": a relatively simple axiomatized object related to words from a presentation of (X, T) as an inverse limit of subshifts.

Part 2. From the SWO construct a closely related symbolic extension. (Not hard.)

Part 3. From an affine superenvelope construct an SWO. (The hardest part). As in [D1], we relate words to measures by interpreting a measure as approximated by a long word B, and considering B as a measure on a periodic orbit $\dots BBB \dots$. Estimates involve information theoretic lemmas (esp. as in [D1] a conditional version of a lemma of Blanchard-Glasner-Host) and separation theorems of the following sort:

Fact. Suppose h and f are functions defined on a Choquet simplex, $h < f$, h is affine u.s.c. and f is l.s.c. Then there is a continuous affine function g such that $h < g < f$.

Part 4: Check that the constructed symbolic extension has extension entropy function matching the given affine superenvelope. Uses more functional analysis and the Ledrappier-Walters Relative Variational Princple.

General Entropy Structure.

We conclude with the promised definition of an entropy structure which will work in any system.

Given (X, T), take a strictly ergodic zero entropy nonperiodic (Z, R) with unique invariant measure λ. An easy consequence of the deep work of Elon Lindenstrauss on mean dimension, following his earlier work with Benjy Weiss, is that the product $(X \times Z, T \times R)$ does have a refining sequence of partitions with small boundaries. We can then define an entropy structure (h'_n) on this product system as before. Now for our entropy structure on (X, T) we use (h_n) where $h_n(\mu) := h'_n(\mu \times \lambda)$.