

**\mathbb{Z}^d Shifts of Finite Type
and \mathbb{Z}^d Sofic Shifts:
Subsystems and Quotients**

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***these slides will be online, via
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Outline of the talk

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III. Entropy

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V. Quotients (factors)

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I. Background on shifts

\mathbb{Z}^d Subshifts

Notations: given positive integers d, N :

- $\mathcal{A} := \{0, 1, \dots, N - 1\}$, a finite alphabet
- An element of $\mathcal{A}^{\mathbb{Z}^d}$ is pictured as a way of filling the \mathbb{Z}^d lattice with symbols
- For $d = 1$, x in $\mathcal{A}^{\mathbb{Z}^d}$ is $x = \dots x_{-1}x_0x_1\dots$, with all x_i in \mathcal{A} and $x_i = x(i)$
- For $d = 2$, x in $\mathcal{A}^{\mathbb{Z}^d}$ is a planar array of symbols, etc.

- $\mathcal{A}^{\mathbb{Z}^d}$ is a metric space, $\text{dist}(x, y) = \frac{1}{k+1}$,
where $k = \min\{\|v\| : x(v) \neq y(v)\}$
- There is a shift action σ of \mathbb{Z}^d on $\mathcal{A}^{\mathbb{Z}^d}$ by homeomorphisms. For $v \in \mathbb{Z}^d$, the shift homeomorphism σ^v is defined by
 $(\sigma^v x)(u) = x(u + v)$, $u \in \mathbb{Z}^d$
- For example, for $d = 1$ and $y = \sigma^1 x$:
if $\dots x_{-1}x_0x_1 \dots = \dots 003 \dots$, then
 $\dots y_{-2}y_{-1}y_0 \dots = \dots 003 \dots$
- If Y is a closed σ -invariant subset of $\mathcal{A}^{\mathbb{Z}^d}$, then $(Y, \sigma|_Y)$ is a *subshift* (or just *shift*). I may use the same letter for the shift or its domain.

- For any subshift Y , there exists a set \mathcal{L} of finite configurations such that $Y = X_{\mathcal{L}}$: the set of all x in $\mathcal{A}^{\mathbb{Z}^d}$ satisfying:
for every finite subset C of \mathbb{Z}^d and $u \in \mathbb{Z}^d$,

$$x|_{(u+C)} \notin \mathcal{L}.$$

If it is possible to choose \mathcal{L} to be some finite set \mathcal{F} then Y is a *shift of finite type* (SFT).

- Example: $d = 1$, $\mathcal{A} = \{0, 1\}$, $\mathcal{F} = \{00\}$.

Block codes

Let Y be a subshift on alphabet \mathcal{A} .

Let Y' be a subshift on alphabet \mathcal{A}' .

Let $U_n = \{v \in \mathbb{Z}^d : \|v\| \leq n\}$.

- A block code is a function $\phi : Y \rightarrow Y'$ for which there exists $\Phi : \mathcal{A}^{U_n} \rightarrow \mathcal{A}'$ with $(\phi x)(v) = \Phi(x|_{v+U_n})$ for all $v \in \mathbb{Z}^d$.
- Example: $Y = Y' = \{0, 1\}^{\mathbb{Z}}$ and $(\phi x)_i = x_i + x_{i+1} \pmod{2}$
- ϕ is a 1-block code if $(\phi x)(v) = \Phi(x|_v)$
- The continuous shift-commuting maps between subshifts are exactly the block codes.

Sofic Shifts

- A sofic shift is a subshift which is a quotient of an SFT.
- For example: SFT = $X_{\mathcal{F}}$, on alphabet $\{a, b, c\}$ with $\mathcal{F} = \{ba, bb, cc\}$. (I.e. a point in X is an arbitrary concatenation of the words a and bc .) Define Y as the image under the one-block code $a \mapsto 1, b \mapsto b, c \mapsto b$. If $ab^n a$ occurs in Y , then n must be even. $Y = X_{\mathcal{L}}$ where

$$\mathcal{L} = \{ab^{2n+1}a : n \in \mathbb{N}\} .$$

Y (the “even system”) is sofic and not SFT.

II. Overview: \mathbb{Z}^d SFTs and sofic shifts, $d = 1$ vs. $d \geq 2$

For \mathbb{Z} SFTs:

- There are computable, fine invariants.
- Invariants and structure have an algebraic quality.
(Any SFT is topologically isomorphic to X_B for some matrix B . Various algebraic invariants of B give invariants of topological isomorphism for X_B .)
- Qualitatively, mixing \mathbb{Z} SFTs have a homogeneous structure, rich in subsystems and quotients.
- Generally problems of SFTs reduce easily to problems of mixing SFTs.

For \mathbb{Z}^d SFTs, $d \geq 2$:

- \mathbb{Z}^d SFTs, including mixing \mathbb{Z}^d SFTs, are qualitatively heterogeneous.
- \mathbb{Z}^d SFT problems do not reduce to problems of mixing \mathbb{Z}^d SFTs.
- Essentially nothing can be computed for a general arbitrary \mathbb{Z}^d SFT if $d \geq 2$.
(Berger, Robinson, Kari, ...)
- The landscape of general possibilities is recursion-theoretic.
Hochman-Meyerovitch and Hochman give constructive results and techniques of generality unprecedented in this area.

III. Entropy

Let us pause to consider two examples of the power of the new constructive techniques

Given a \mathbb{Z}^d subshift $(X, \sigma|X)$:

let $B_n = \{v \in \mathbb{Z}^d : 0 \leq v_i \leq n - 1, 1 \leq i \leq d\}$.

The *entropy* of this shift (as a \mathbb{Z}^d action) is

$$h(\sigma|X) = \lim_n \frac{1}{n^d} \log \text{card}\{x|_{B_n} : x \in X\} .$$

This is the main numerical measure of the complexity of a subshift.

For $X = \{0, \dots, N - 1\}^{\mathbb{Z}^d}$, the entropy is simply $\log N$.

It is easy to compute the entropy of a \mathbb{Z} SFT. For $d \geq 2$: there are not many exact computations of entropy for systems “in nature”; and no Turing machine can compute $h(\sigma_{\mathcal{F}})$ for arbitrary finite \mathcal{F} .

DEFN: A recursive sequence is the output of a Turing machine – intuitively, a sequence produced by any kind of algorithm you could imagine a computer implementing.

The classes of possible entropies:

- For \mathbb{Z} SFTs: the logs of a well understood class of algebraic integers. (Lind)
- For \mathbb{Z}^d SFTs, $d \geq 2$: the numbers $\alpha = \lim_n \alpha_n$, where $(\alpha_n)_{n \in \mathbb{N}}$ is a decreasing recursive sequence of rational numbers. (Hochman-Meyerovitch)
- For d -dimensional cellular automata maps: the numbers $\alpha = \liminf_n \alpha_n$, where $(\alpha_n)_{n \in \mathbb{N}}$ is a recursive sequence of rational numbers. (Hochman)

IV. Subsystems

Mixing

- A \mathbb{Z}^d subshift (X, σ) is *mixing* if any two legal finite configurations can occur at all but finitely many separations.
- That is, for all x, y in X , for all n we have for all but finitely many $u \in \mathbb{Z}^d$ there exists $w \in X$ such that $w|_{B_n} = x|_{B_n}$ and $w|_{u+B_n} = y|_{B_n}$.
- Generally problems of \mathbb{Z} SFTs reduce easily to problems of mixing \mathbb{Z} SFTs.
- For $d \geq 2$, the mixing condition splinters into a host of different conditions.

Subsystems of \mathbb{Z} SFTs

Nontrivial mixing \mathbb{Z} SFTs have a homogeneous structure, rich in subsystems and quotients:

- Krieger Embedding Theorem \implies if (X, σ_X) is a mixing \mathbb{Z} SFT and (Y, σ_Y) is a \mathbb{Z} subshift with no periodic points and $h(\sigma_Y) < h(\sigma_X)$, then (Y, σ_Y) is topologically conjugate to a subshift contained in (X, σ_X) .
- Jewett-Krieger Theorem: every finite entropy measurable \mathbb{Z} -system is realized by a uniquely ergodic \mathbb{Z} -subshift
- Given a proper subsystem of a mixing \mathbb{Z} SFT, all the embeddings above can be chosen with images missing that subsystem. “There’s always room at the Krieger Hotel.”

Subsystems of \mathbb{Z} Sofic Shifts

- A mixing \mathbb{Z} sofic shift X contains an increasing union of mixing \mathbb{Z} SFTs X_n with $\lim_n h(X_n) = h(X)$.
- So the subsystem results of the last page for mixing \mathbb{Z} SFTs also hold for mixing \mathbb{Z} sofic shifts.

So: a nontrivial mixing \mathbb{Z} SFT or sofic shift contains a vast family of pairwise disjoint minimal subsystems with entropy close to $h(\sigma_X)$.

Subsystems of \mathbb{Z}^d Shifts, $d \geq 2$

- For a large subclass of the mixing \mathbb{Z}^2 SFTs S : any smaller entropy \mathbb{Z}^2 subshift without periodic points can be embedded into S .
- Again for a large subclass of the mixing \mathbb{Z}^d SFTs S , Robinson and Sahin have proved existence of subshifts carrying completely positive entropy or Bernoulli measures.
- There is also an analogue of the Jewett Krieger Theorem for \mathbb{Z}^d shifts, $d \geq 2$.

Moreover there is a result on richness of subsystems which applies to ALL \mathbb{Z}^d SFT/sofic shifts:

THEOREM (Desai) For $d \geq 2$ and a given \mathbb{Z}^d shift S :

- If S is SFT, then S contains SFTs with entropies dense in $[0, h(S)]$.
- If S is sofic, then S contains SFTs with entropies dense in $[0, h(S)]$.
- In either case every number in $[0, h(S)]$ is the entropy of a subsystem.

So there are a lot of subsystems.

But they cannot in general be separated:

THEOREM (B-Pavlov-Schraudner) Given $d \geq 2$ and $M \in \mathbb{R}$, there is a mixing \mathbb{Z}^d sofic shift S with the following properties.

- $h(S) > M$ and S is mixing
- S contains a unique minimal subsystem, which is a fixed point for the shift action.

THEOREM (B-Pavlov-Schraudner) Given $d \geq 2$ and $M \in \mathbb{R}$, there is a mixing \mathbb{Z}^d SFT S with the following properties.

- $h(S) > M$ and S is mixing
- S contains a zero entropy sofic shift which intersects every subshift in S .

V. Quotient (Factor) Maps

\mathbb{Z} SFTs and \mathbb{Z} sofic shifts also enjoy a rich supply of quotient maps.

THEOREM

1. If X is a \mathbb{Z} sofic shift, and Y is a mixing \mathbb{Z} SFT with $h(X) > h(Y)$, then Y is a quotient of X .
2. If Y is a \mathbb{Z} SFT and $h(Y) \geq \log N$, then Y has as a factor the full shift on N symbols.

Johnson and Madden asked whether (2) generalizes to \mathbb{Z}^d SFTs. Their work, as extended by Desai, proved the conclusion of (2) for a \mathbb{Z}^d SFT Y with the “corner gluing” mixing condition when $h(Y) > \log N$.

If S is a quotient of T , then disjoint subsystems of T pull back to disjoint subsystems of S . So we see that if S has poorly separated subsystems, then it cannot factor onto a T with well separated subsystems, such as a full shift. The BPS examples for subsystems thus not surprisingly have some pathological properties with regard to their possible factors.

THEOREM (BPS) Given $d \geq 2$ and $M \in \mathbb{R}$, there is a \mathbb{Z}^d sofic shift S with the following properties.

- $h(S) > M$
- S contains a unique minimal subsystem, which is a fixed point for the shift action.
- And with regard to quotients:

- Any subshift quotient Y of S satisfies the following:
 - If Y is SFT, then Y is a fixed point.
 - Y is not block-gluing (i.e. not mixing on block shapes with uniform separation).
 - Y supports no σ -invariant measure which is of completely positive entropy.
 - Y does have a subshift factor of topologically completely positive entropy.

- S can be chosen mixing, except that in this case we can't control top. c.p.e. of non-trivial factors for $d \geq 3$.

There are similar slightly weaker results for S SFT. The key to the proof for $d \geq 3$ is a theorem of Hochman.

Effective Systems

Hochman has defined an Effective Symbolic System (ESS) to be a \mathbb{Z}^d subshift $\sigma_{\mathcal{L}}$ such that the defining set \mathcal{L} of forbidden finite configurations is the output of a Turing machine.

THEOREM (Hochman) For each $d \geq 3$, up to topological conjugacy the following classes of \mathbb{Z}^d subshift are the same:

- \mathbb{Z}^d subshifts isomorphic to σ^{e_1} for some \mathbb{Z}^d sofic shift σ
- The class of ESS's.

(With SFT in place of sofic, the class of ESS's becomes just slightly more narrow.)

Heuristically: not only are we faced with many bad examples: in fact every (bad) thing we could possibly imagine happening, does happen.

VI. A Hint of Proof

We can indicate how Hochman's subdynamics theorem lets one easily construct a nonmixing sofic example S for $d = 3$.

- Construct an effective \mathbb{Z} subshift W such that arbitrarily large blocks of 0's occur syndetically in all points, and every W word occurs with positive frequency in every point.
- By Hochman: for $i = 1, 2, 3$, pick a \mathbb{Z}^3 sofic shift T_i for which σ^{e_i} is a copy of W .
- Then in each coordinate of $T_1 \times T_2 \times T_3$, for each $M \in \mathbb{N}$, strings of M consecutive zeros occur syndetically.

- Let W be the quotient of $T_1 \times T_2 \times T_3$ by the map which replaces a symbol (a, b, c) with 0 if any of a, b, c is zero, and otherwise replaces (a, b, c) with 1.
- For every M and $w \in W$, every finite configuration in W occurs inside some large block configuration on which the boundary is covered by M -thick slabs of zeros.
- Define S by freely allowing the replacement of 1 in a configuration with symbols from $\{1, 2, \dots, k\}$ for k large.
- Easily:
 - $h(\sigma_X) > M$ (for large enough k)
 - σ_X has a unique minimal subsystem, $0^{\mathbb{Z}^3}$.
 - The only SFT which is in X or in a quotient of X is a fixed point.

The nonexistence of factors with meas.th. c.p.e. measures uses the disjointness result of Glasner, Thouvenot and Weiss.

VI. The Playground

We have entered a certain period in this topic where one can imagine some wildly general recursion theoretic obstruction to \mathbb{Z}^d SFT/sofic behavior, and then try to show there is no other obstruction.

The Hochman-Meyerovitch/Hochman techniques are very concrete — given the oracle Turing machine.

However: while the very general landscape of \mathbb{Z}^d SFTs and sofic shifts seems quite recursion theoretic, this will not hold for special important classes, at least in the same generality.