Path methods for strong shift equivalence

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PATH METHODS FOR SSE

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1. Introduction

Over four papers in 1990-91, Kim and Roush created a structure for studying shift equivalence and strong shift equivalence of positive matrices over dense subrings of $\mathbb{R}$. Especially $\mathbb{R}$ and $\mathbb{Q}$.

This talk is based on a 50 page joint work with Kim and Roush, on the Arxiv. It contains a complete and generalized version of the theory, along with full, detailed proofs accessible to newcomers. It is all about matrices, not using dynamics.

Before we consider the topic, we'll review strong shift equivalence and its context. All rings and semirings are assumed to contain 1. A matrix over a semiring $\mathcal{S}$ is a matrix with entries in $\mathcal{S}$.

2. Strong shift equivalence

Let $\mathcal{S}$ be a semiring with $\{0, 1\}$ (e.g. $\mathbb{R}, \mathbb{Z}, \mathbb{Z}_+, \ldots$). Williams [1973] defined :

Matrices A, B over $\mathcal{S}$ are elementary strong shift equivalent over $\mathcal{S}$ (ESSE-$\mathcal{S}$)
if they are square and there exist matrices U, V over $\mathcal{S}$ such that

$$A = UV \quad \text{and} \quad B = VU \ .$$

A, B are strong shift equivalent over $\mathcal{S}$ (SSE-$\mathcal{S}$) if there exists a chain

$$A = A_0, A_1, \ldots, A_\ell = B$$

with A_{i-1} and A_i ESSE-$\mathcal{S}$ for $0 < i \leq \ell$.

Why did Williams define SSE?

- Up to topological conjugacy, every shift of finite type (SFT) is an “edge SFT” σ_A, defined by a square matrix A over $\mathbb{Z}_+$.

- SFTs are fundamental for symbolic dynamics.

- σ_A and σ_B are isomorphic (topologically conjugate) iff A, B are SSE-$\mathbb{Z}_+$.

The trouble with SSE-$\mathbb{Z}_+$.

However, SSE for $\mathbb{Z}_+$ (or even $\mathbb{R}_+$) is a very hard relation to understand.

Given A and B: looking for the ESSE chain

$$A = A_0, A_1, \dots, A_\ell = B$$

we know NO a priori bound on that length ℓ (the "lag") or the sizes of the matrices A_i.

It is still not known if there exists an algorithm which takes matrices A, B and decides whether they are SSE-$\mathbb{Z}_+$. (Even if A, B are 2×2 !)

So, Williams introduced a more tractable relation, shift equivalence.

2. Shift equivalence

DEFN Square matrices A, B are shift equivalent over $\mathcal{S}$ (SE-$\mathcal{S}$) if $\exists$ matrices U, V over $\mathcal{S}$ and $\ell \in \mathbb{N}$ such that

$$A^\ell = UV \qquad B^\ell = VU$$
$$AU = UB \qquad BV = VA$$

Always: SSE-$\mathcal{S}$ implies SE-$\mathcal{S}$. Also

- SE-$\mathbb{Z}_+$ is decidable (Kim-Roush).

- SE-$\mathbb{Z}_+$ turns out to be reasonably tractable, and closely related to significant applications in symbolic dynamics

3. Algebraic meaning of SE-$\mathcal{S}$

Suppose A, B are matrices over a ring $\mathcal{S}$.

- If $\mathcal{S}$ is a field, then A, B are SE-$\mathcal{S}$ iff the nonnilpotent parts of their Jordan forms agree.

- If $S = \mathbb{Z}$, there are additional invariants (but they are managable).

- SE-$\mathcal{S}$ of A, B is equivalent to the isomorphism of certain associated $\mathcal{S}[t, t^{-1}]$ modules.
  (There is a "conceptual" algebraic object classifed by SE-$\mathcal{S}$.)

Example: if $\mathcal{S}$ is a field or PID (e.g. $\mathbb{Z}$), and A, B over $\mathcal{S}$ each have just one nonzero eigenvalue, with algebraic multiplicity one, then they are SE-$\mathcal{S}$ iff those eigenvalues are equal.

Let us see how relations simplify in the main cases.

DEFN A real matrix is *primitive* if it is square nonnegative and and some power is positive.

For SSE over $\mathbb{Z}_+$ (or over $\mathcal{S}_+$, for a subring $\mathcal{S}$ of $\mathbb{R}$) the fundamental case to understand is SSE of primitive matrices.
(This is quite analogous to the Perron-Frobenius theory of nonnegative matrices.)

Suppose A, B are primitive matrices over $\mathcal{S}$, a subring of $\mathbb{R}$.

- T.F.A.E. for A, B: SE over S, SE over $\mathcal{S}_+$

- If also $\mathcal{S}$ is a field, or PID (e.g. $\mathbb{R}, \mathbb{Z}$), or Dedekind domain, then SE over $\mathcal{S}$ and SSE over $\mathcal{S}$ are equivalent.

So in the most important cases the only issue is how SSE-$\mathcal{S}_+$ fits.

4. Algebraic meaning of SSE-$\mathcal{S}$

For any ring S (we always assume $1 \in \mathcal{S}$), the equivalence relation SSE-$\mathcal{S}$ is generated by the following relations on square matrices over $\mathcal{S}$:

- conjugacy over $\mathcal{S}$ ($A \sim UAU^{-1}$), and
- "nilpotent extensions":

$$\begin{pmatrix} A & X \\ 0 & 0 \end{pmatrix} \sim A \sim \begin{pmatrix} 0 & X \\ 0 & A \end{pmatrix}$$

For example, if $A = XY$ and $B = YX$, we can find nilpotent extensions of A, B which are conjugate:

$$\begin{pmatrix} I & 0 \\ Y & I \end{pmatrix} \begin{pmatrix} A & X \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & X \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix}$$

For a general ring (even in $\mathbb{R}$) which is not a Dedekind domain, I know nothing more about how SSE over $\mathcal{S}$ refines SE over $\mathcal{S}$. I can't provide an example of matrices SE over $\mathcal{S}$ but not SSE over $\mathcal{S}$.

5. Classifying shifts of finite type.

Williams gave us:

- Theorem (Annals of Math 1973)
  SE-$\mathbb{Z}_+ \implies$ SSE-$\mathbb{Z}_+$.

- Conjecture (Annals of Math 1974)
  SE-$\mathbb{Z}_+ \implies$ SSE-$\mathbb{Z}_+$.

Eventually counterexamples were constructed (Kim Roush 1992,1999), based on a lovely algebraic topological structure created by Wagoner ("strong shift equivalence space"). The counterexamples used special conditions, especially, zero trace.

What we thoroughly lack are general sufficient conditions for SSE-$\mathbb{Z}_+$.

6. SSE over subsemirings of $\mathbb{R}$

To probe the $\mathbb{Z}_+$ problem, we attack a related problem.

From here: $\mathbb{S}$ is a nondiscrete subring of $\mathbb{R}$, and $\mathbb{S}_+ = \mathbb{S} \cap [0, \infty)$.

PROBLEM When are positive matrices over $\mathbb{S}$ SSE over $\mathbb{S}_+$?

How entangled are the algebraic and positivity obstructions? There are other motivations for the problem. We have no counterexample to the possibility that SE-$\mathbb{S}$ always implies SSE-$\mathbb{S}_+$ for positive matrices.

Theorem (Kim-Roush) A primitive matrix over $\mathbb{S}$ with positive trace is SSE-$\mathbb{S}_+$ to a positive matrix.

7. Three part approach to SSE-$\mathbb{S}_+$ of positive matrices.

(I) Understand SSE over $\mathbb{R}_+$ via paths of positive real matrices.

(II) Use (I) to get conditions in which
SSE-$\mathbb{S} \implies$ SSE-$\mathbb{S}_+$.

(III) Understand SSE-$\mathbb{S}$, especially, when do we have SE-$\mathbb{S} \implies$ SSE-$\mathbb{S}$? (Always?)

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For example, suppose consider positive real matrices with exactly one nonzero eigenvalue.
(I) [KR, 90s] done.
(II) [BKR] True for all $\mathbb{S}$.
(III) Done iff $\mathbb{S}$ is Dedekind.

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I have nothing more for (III).
Let's consider in order (I), (II).

8. (I), over $\mathbb{R}_+$: The Central Result

The Central Result (90s, KR)

Suppose (A_t), $0 \leq t \leq 1$, is a path of positive conjugate $n \times n$ real matrices.

Then A_0 and A_1 are SSE over $\mathbb{R}_+$.

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Sketch of a proof:

1. Given $t < 1$ and $\epsilon > 0$ small enough,
$A_{t+\epsilon} = V^{-1}A_tV$ with V close to I.

2. That conjugacy $A_t \to V^{-1}A_tV$ is a composition of conjugacies implemented by basic elementary matrices E close to I.

3. With $E \geq 0$, $C > 0$ and E close to I, there is an ESSE over $\mathbb{R}_+$:

$$(E)\,(E^{-1}C) \;=\; C$$
$$(E^{-1}C)\,(E) \;=\; E^{-1}CE\ .$$

4. So for small $\epsilon = \epsilon_t > 0$,
$t \leq s < t + \epsilon \implies A_s$ and A_t are SSE-$\mathbb{R}_+$.
Likewise for $t - \epsilon < s \leq t$.

5. By compactness, there's a finite subcover of $[0,1]$ by such neighborhoods $(t - \epsilon_t, t + \epsilon_t)$; therefore A_0 and A_1 SSE-$\mathbb{R}_+$.

9. (I): More results

COR. of Central Result (Chuysurichay, 2011) Suppose A is positive $n \times n$ over $\mathbb{R}$. Let $\mathcal{C}$ be the set of positive real $n \times n$ matrices which are conjugate to A.

Then $\mathcal{C}$ intersects only finitely many SSE-$\mathbb{R}_+$ classes.

PROOF. Use a result from semialgebraic geometry. Since $\mathcal{C}$ is the solution set of finitely many polynomial inequalities and equalities in finitely many real variables (the matrix entries), $\mathcal{C}$ has only finitely many connected components. Apply the Central Result. QED

Example(C. 2011) For some A, the minimal lag required for SSE between matrices in $\mathcal{C}$ is unbounded.

The finiteness result, despite the unbounded lags, indicates some power of the path method.

More from (Kim Roush, 90s):

THM Suppose $A > 0$ and conjugate to a matrix $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M \end{pmatrix}$ with M nilpotent. Let $\mathcal{C}$ be the set of positive matrices conjugate conjugate to A.
Then $\mathcal{C}$ is path connected.

LEMMA 1 If A, B are positive and SSE-$\mathbb{R}$, then A, B are resp. SSE-$\mathbb{R}_+$ to positive A', B' which are conjugate over $\mathbb{R}$.

THM 2 If A, B are positive and SSE over $\mathbb{R}$, and each has just one nonzero eigenvalue, then A, B are SSE over $\mathbb{R}_+$.

(I) Some new results [BKR].

• (Ia) Suppose (A_t), $0 \le t \le 1$, is a path of positive, shift equivalent $n \times n$ real matrices. Then A_0 and A_1 are SSE over $\mathbb{R}_+$.

• (Ib) Suppose A, B are positive matrices and SSE-$\mathbb{R}_+$. Then A, B are SSE-$\mathbb{R}_+$ through positive matrices.

"SSE-$\mathbb{R}_+$ through positive matrices" means we have $A_0, A_1, ..., A_\ell$ all positive with $A = A_0$, $B = A_\ell$, and A_i, A_{i+1} ESSE-$\mathbb{R}_+$ for $0 \le i < \ell$.

One of the crucial ingredients for these two new results is a pure linear algebra result:

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Suppose A is a nilpotent $n \times n$ matrix. Let $\mathcal{C}$ be a conjugacy class of $n \times n$ nilpotent matrices satisfying certain conditions.

Then given a neighborhood $\mathcal{V}$ of A, whenever B, C in $\mathcal{C}$ are sufficiently close to A, there is a path of conjugate matrices from B to C which stays in $\mathcal{V}$.

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This applicability of linear algebra is an example of (I) being more accessible to standard math (than the problem of SSE-$\mathbb{Z}_+$).

10. (II), from $\mathbb{R}_+$ to $\mathbb{S}_+$: Results. From [BKR]:

LEMMA 1' If A, B are positive and SSE-$\mathbb{S}$, then A, B are resp. SSE-$\mathbb{S}_+$ to positive A', B' which are conjugate over $\mathbb{S}$.

THM 2' If A, B are positive and SSE over $\mathbb{S}$, and each has just one nonzero eigenvalue, then A, B are SSE over $\mathbb{S}_+$.

THM If A, B are positive matrices over a subfield $\mathbb{S}$ of $\mathbb{R}$, and A, B are SSE over $\mathbb{R}_+$, then A, B are SSE over $\mathbb{S}_+$.

The last result uses the new results Ia, Ib over $\mathbb{R}$. Sow in $\mathbb{R}_+$, reap in $\mathbb{S}_+$.

11. A hint of ideas behind (II)

The hardest part is LEMMA 1'. This requires one to develop analogues of state splitting and fiber product techniques for decomposing SSE over $\{0,1\}$. (These are also used in proving the new $\mathbb{R}_+$ results.) There are new features. I'll skip all of that.

But note: the proof requires in a fundamental way the assuption SSE-$\mathbb{S}$, not SE-$\mathbb{S}$. Likewise Wagoner's strong shift equivalence spaces are built from SSE-$\mathbb{S}$, not SE-$\mathbb{S}$.

This switch of hypothesis (after so much investment in Williams' conjecture) seems useful. We separate out as another step (III) the relation of SSE-$\mathbb{S}$ and SE-$\mathbb{S}$.

Next let $(A_t), 0 \leq t \leq 1$ be a path of positive matrices, conjugate over $\mathbb{R}$, from $A = A_0$ to $B = A_1$, with A, B over $\mathbb{S}$, and conjugate over $\mathbb{S}$ (by Lemma 1').
What do you need to conclude A, B are SSE over $\mathbb{S}_+$ (not just over $\mathbb{R}_+$)?
There are two pieces to this.

There is a path G_t in GL(n,$\mathbb{R}$) with $A_t = G_t^{-1} A G_t$ and $G_0 = I$.

Piece 1.

If G_1 is over $\mathbb{S}$, then via approximation methods, A and B are SSE over $\mathbb{S}+$.

The approximations use heavily the simple fact that an element of SL($n, \mathbb{R}$) is a product of elementary matrices, which can be approximated arbitrarily closely by elementary matrices over $\mathbb{S}$.

Piece 2.

So, beginning with a path of positive conjugate matrices $(A_t) = (G_t^{-1} A G_t)$, what do you need to pass to a new path (G'_t) with $G'_0 = I$ and G'_1 over $\mathbb{S}$?

Assume $A = V^{-1} B V$, with V in GL(n,$\mathbb{S}$).

Define Cent(A) to be
$\{M \in \mathsf{GL}(n, \mathbb{R}) : AM = MA\}$.

As it works out: you need exactly that the connected component of Cent(A) containing $U_1^{-1} V$ also contains a matrix over $\mathbb{S}$.

This is no problem if $\mathbb{S}$ is a field.

It is an actual obstruction to the method for some rings $\mathbb{S}$, e.g. some algebraic number rings.

For any $\mathbb{S}$, the number of distinct SSE-$\mathbb{S}_+$ classes which interesect a path connected set of positive, conjugate real matrices containing a matrix A cannot exceed $|\pi_0(\text{Cent}(A))|$ (which is finite and easily analyzed).

So, we generalize Chuyisurichay's result to all $\mathbb{S}$: the set of $n \times n$ positive matrices over $\mathbb{S}$ lying in a given conjugacy-over-$\mathbb{S}$ class intersects only finitely many distinct SSE-$\mathbb{S}_+$ classes.

12. The one-component proof.

Theorem (KR, 90s) Suppose $A > 0$ and conjugate to a matrix $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M \end{pmatrix}$ with M nilpotent. Let $\mathcal{C}$ be the set of positive matrices conjugate conjugate to A.

Then $\mathcal{C}$ is path connected.

Proof. We have $A = P + N$, where
P is rank one positive; N is nilpotent; and $PN = NP = 0$.

With $A' \in \mathcal{C}$, likewise $A' = P' + N'$.

We want to produce the positive path of conjugate matrices from A to A'.

We have U such that $U^{-1}AU = A'$, and also $\det(U) > 0$. Then there is a path (U_t) of invertible matrices from I to U, giving a path of matrices $(A_t) = (U_t^{-1}AU_t)$ from A to A'. We do not have $A_t \geq 0$, but (slightly technical) we can arrange that for all t, $A_t = P_t + N$ with $P_t > 0$ and N nilpotent.

By compactness: $\exists\epsilon > 0$ such that for all t, $P_t + \epsilon N_t$ is positive. Then,

$$P_t + \epsilon N_t \ , 0 \leq t \leq 1$$

is a positive path of similar matrices.

And! For N nilpotent and s a nonzero number, sN is similar to N . So

$$P_1 + sN_1 \ , 1 \geq s \geq \epsilon$$

is a path of similar matrices from A to $P_1+\epsilon N_1$, and it is a positive path.

Producing likewise a path from $P_2 + \epsilon N_2$ to B, and composing the three paths, we get the desired positive path from A to B. QED.

13. Conclusion. Progress on understanding SSE-$\mathbb{R}_+$ for positive matrices yields a good deal for SSE-$\mathbb{S}_+$ for other dense subrings $\mathbb{S}$ of $\mathbb{R}$.

The problem of understanding SSE-$\mathbb{R}_+$ for positive matrices is accessible by more standard mathematics. It may be that good progress could follow if the problem were attacked by some who could bring something extra to the problem.

More geometry?

More matrix theory?

More youth?