Northwest Dynamics Conference
University of Victoria
August 2005

Expansive maps commuting with shifts of finite type**

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**with bonus remarks and questions on jointly periodic points of cellular automata

Definitions

- An *automorphism* of a continuous map f is a homeomorphism U commuting with f ($Uf = fU$).

- Continuous maps f, F are *topologically conjugate* ($f \sim F$) if $\exists F \sim f, G \sim g$ with $FG = GF$.

- Continuous maps f, g *can commute* if $\exists F \sim f, G \sim g$ with $FG = GF$.

- σ_A is the twosided edge shift of finite type (SFT) defined by the square $\mathbb{Z}_+$ matrix A.

- S is SFT if $S \sim \sigma_A$, for some A.

Which maps can commute with SFTs? [Nasu]

Which SFTs can commute?

CONJECTURE: Suppose S and T are mixing SFTs. Then for all large i, j, S^i and T^j can commute.

If S, T are commuting bijections and $\forall n > 0$ $|\text{Fix}(S^n)| < \infty$ and $|\text{Fix}(T^n)| < \infty$, then $\text{Per}(S) = \text{Per}(T)$. Thus low-order periodic point obstructions sometimes imply two maps cannot commute: e.g.,

- if $|\text{Fix}(\sigma_A)| = 1$ and $|\text{Fix}(\sigma_B)| = 0$, then σ_A and σ_B cannot commute

- $\sigma_{[2]}$ and $\sigma_{[3]}$ cannot commute

However, there is no set theoretic periodic point obstruction to the conjecture.

Commuting SFTs from commuting matrices

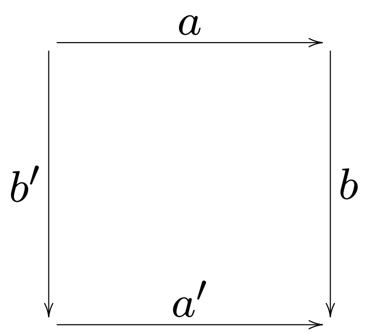
Note: $AB = BA$ does not guarantee that σ_A, σ_B can commute. (E.g. $[A] = 2, B = [3]$). However:

Proposition. Suppose A, B are commuting $\mathbb{Z}_+$ matrices. Then there are homeomorphisms S, T such that $ST = TS$ and $S^iT^j \sim \sigma_{A^iB^j}$ for $i, j > 0$.

The proposition follows from remarks on a construction of Nasu in his 1995 AMS Memoir, which created an elaborate "textile systems" apparatus for studying endomorphisms and automorphisms of an SFT. In this memoir and successor papers, Nasu achieved major results, especially on automorphisms of onesided SFTs.

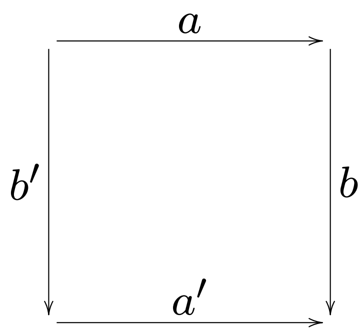
We go on to explain the Proposition.

Suppose A and B are $n \times n$ matrices over $\mathbb{Z}_+$, with $AB = BA$. View A and B as adjacency matrices for two directed graphs, with disjoint edge sets and a common vertex set $\{1, 2, \dots n\}$. Say e.g. an ab path from i to j is an A edge from i to some k followed by a B edge from that k to j. "$AB = BA$" means that for each pair i, j the number of ab paths from i to j equals the number of ba paths from i to j. Thus we can build a set $\mathcal{W}$ of Wang tiles

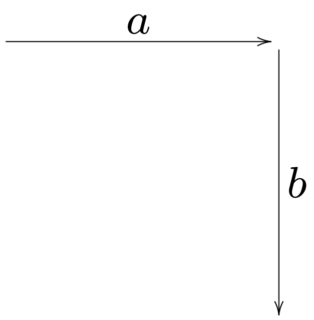


such that each ab path is the top/right of exactly one tile and each ba path is the left/bottom of exactly one tile. (In the tile pictured, a, a' are A-edges and b, b' are B-edges.)

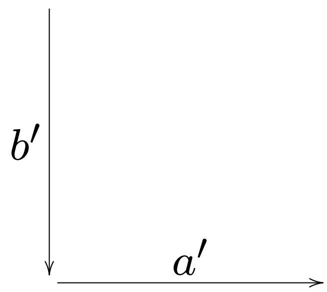
Thus each Wang tile



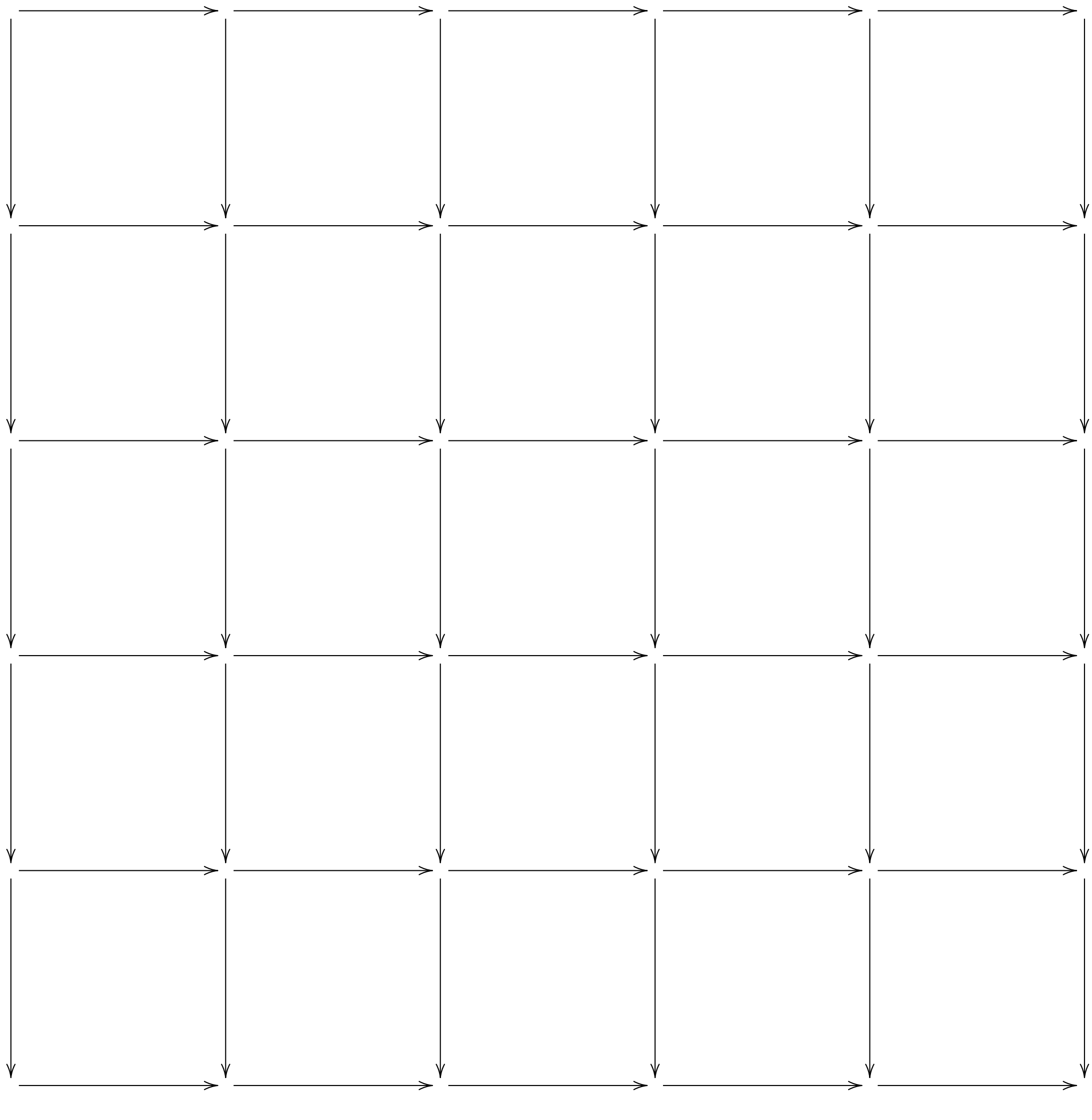
is determined by either of the paths



or

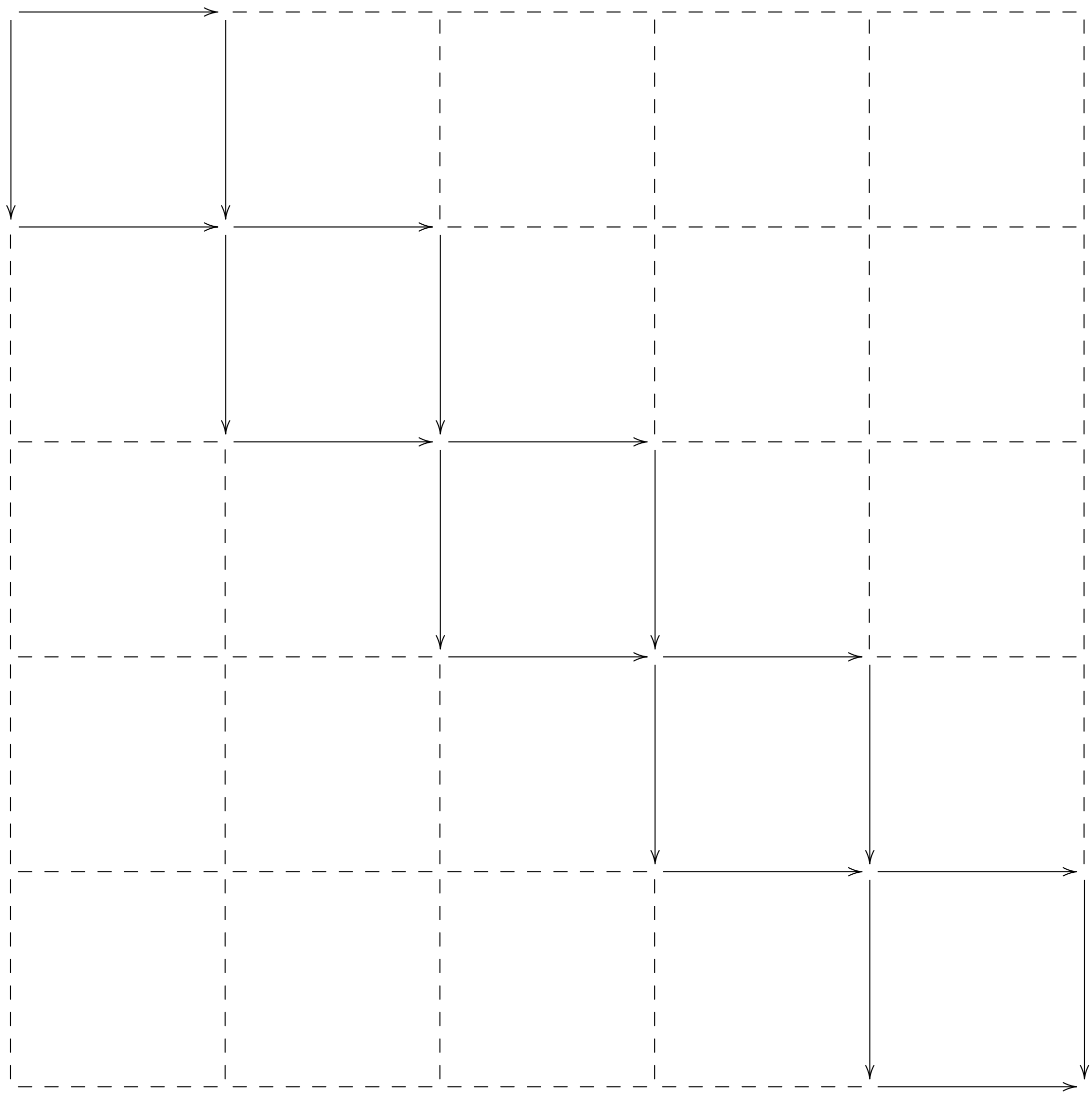


Now let the tile sides be unit length and let W be the space of infinite Wang tilings of the plane with $\mathcal{W}$, with tile corners on $\mathbb{Z}^2$.

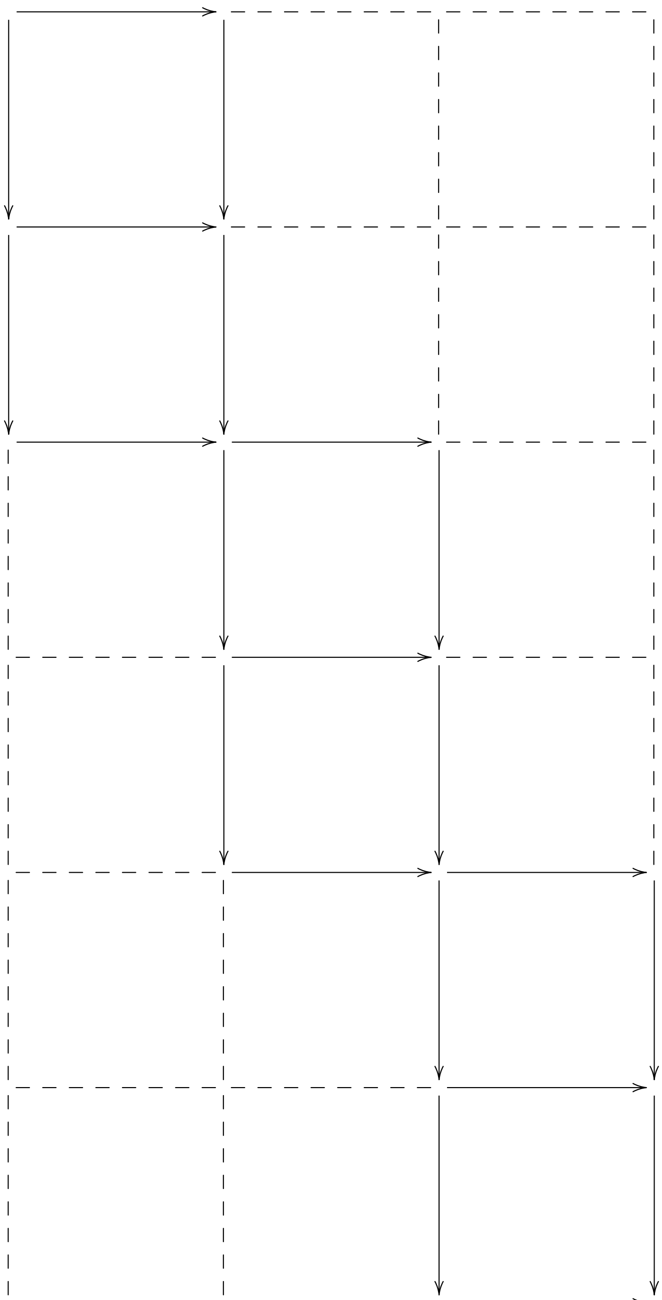


E.g., above is a finite piece of a point in W, with edge-name labels suppressed. For $\mathbf{v} \in \mathbb{Z}^2$, let $\alpha_{\mathbf{v}}$ denote the shift map on W in direction $\mathbf{v}$.

The bijections cited two slides back show the dashed-line sides below are determined by the solid diagonal squares. Thus $\alpha_{(1,-1)}$ is expansive and conjugate to the SFT σ_{AB}.



Likewise the solid squares below determine the rest, and $\alpha_{(1,-2)} \sim \sigma_{AB^2}$.



Given the commuting matrices A, B we showed how to embed $\sigma_{A^iB^j}$ into a commuting family of maps when $(i,j) = (1,1)$ or $(i,j) = (1,2)$. The argument is the same for $i > 0, j > 0$. The proof also works for onesided SFTs, for which Nasu has a converse: commuting onesided SFTs can be presented by commuting $\mathbb{Z}_+$ matrices.

Now we turn to algebraic invariants which can be realized by such commuting A, B, modulo passing to higher powers.

For a $k \times k$ $\mathbb{Z}_+$-matrix A, set $G_A = \varinjlim_A \mathbb{Z}^k$.

Regard G_A as an ordered group, with the natural order: G_A is the *dimension group* of A.

Proposition. Suppose σ_A is a mixing SFT and $\phi : G_A \to G_A$ is an isomorphism commuting with $\hat{A}$ all of whose eigenvalues are algebraic integers. There is a $\mathbb{Z}$ matrix B presenting the action of ϕ such that $BA = AB$. Suppose the spectral radius λ_B is a simple root of χ_B; $\lambda_B > 1$; and λ_B is the number by which B multiplies the Perron eigenvector of A. Then for all large i, B^i is positive, commutes with A.

The proof is routine dimensiongroupology and generalizes to finitely many commuting ϕ_j. This gives many families of commuting SFTs.

When commuting matrices produce commuting SFTs, their dimension groups are the same; so, modulo determination of lower powers which commute, we won't get further commuting SFTs *directly* from commuting matrices.

SFTs σ_A and σ_B can commute without being algebraically related in any way I see:

EXAMPLE (Nasu 95): $\sigma_A T = T\sigma_A$, $T \sim \sigma_B$,

- $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

- $\chi_B(x) = (x+1)^2(x^3 - 2x^2 + x + 1)$.

(σ_A and T do not even have the same measure of maximal entropy.)

Nasu gave a complicated algorithm which, given an automorphism U of an irreducible SFT, will find a matrix B such that $\sigma_B \sim U$, *IF* U is SFT. The example above came from applying the algorithm to a particular automorphism.

It would be interesting to see any systematic construction of commuting SFTs which need not be algebraically related.

Question (Nasu 1989). Must an expansive automorphism of an irreducible SFT be itself SFT?

EXAMPLE (D. Fiebig, 1996) A reducible SFT S with an expansive automorphism U which is not SFT.

Here, S consists of two fixed points p, q and two connecting orbits from p to q. Concretely, the fixed points and connecting orbits are

$$p = \dots 000 \dots \quad , \qquad \dots 0002111 \dots \quad ,$$
$$q = \dots 111 \dots \quad , \qquad \dots 0003111 \dots \quad .$$

$U = S$, except that $U = S^{-1}$ on one of the connecting orbits. U is expansive and totally chain transitive but not SFT. D.F. (easily) also elaborated this example to positive entropy.

We have a result which at least, after all this time, addresses a meaningful case of the question.

THEOREM (B. 2004) A strictly sofic AFT (almost finite type) shift S cannot commute with a mixing SFT T.

Above, "mixing" can be replaced by "chain recurrent". A sofic shift is AFT if it a factor of an irreducible SFT by a biclosing map. (Krieger showed this map is canonical). The AFT sofic shifts enjoy various properties and seem to be the one big, natural class of nice sofic shifts.

We'll outline the proof of the theorem. Let S and T be as above. Notation: a map f is *totally P* if f^n has property P for all $n > 0$.

Step 1: LEMMA A.
An expansive automorphism of a mixing SFT is totally chain transitive.
(A subshift is totally chain transitive iff for all $n > 0$ the SFT built from its allowed words of length n is mixing.)

Step 2: setup with canonical cover.
We suppose S is strictly sofic AFT and $ST = TS$. Lemma A then implies S is mixing. Let $\pi : \widetilde{S} \to S$ be the canonical biclosing cover of S by a mixing SFT. By "canonical", T lifts to an automorphism $\widetilde{T}$ of $\widetilde{S}$.

Step 3: $\widetilde{T}$ is expansive.
Fibers of π are uniformly separated, because T is expansive. Points within fibers are uniformly separated because π is biclosing.

STEP 4: $\widetilde{T}$ is a mixing SFT.
Because $\widetilde{T}$ is an expansive automorphism of the mixing SFT $\widetilde{S}$, Lemma A implies that $\widetilde{T}$ is totally chain transitive. Now $\pi : \widetilde{T} \to T$ is a closing factor map from a totally chain transitive subshift onto a mixing SFT. A sofic argument of Kitchens adapts to this situation to show $\widetilde{T}$ must be a mixing SFT.
(That argument. Suppose $\widetilde{T}$ is not SFT. Then for some mixing SFT U containing $\widetilde{T}$ to $\widetilde{T}$, π extends to a closing factor map $U \to T$. Then $h(U) > h(T)$ because U is mixing SFT and T is a proper subsystem; and $h(U) = h(T)$ because closing maps are finite to one. Contradiction.)

STEP 5: the contradiction.
Now $\pi : \widetilde{T} \to T$ is a biclosing map of mixing SFTs, hence constant-to-one. But $\pi : \widetilde{S} \to S$ is 1-1 a.e. (as the canonical cover) but not everywhere (since S is strictly sofic). QED

The heart of the proof of Lemma A is the following lemma (surprisingly difficult ?).

Lemma. Suppose T is mixing SFT; S is an expansive automorphism of T; B is a closed open set; and $SB = B$. Then B is trivial.

Proof sketch. Let μ be the measure of max. entropy for T; it is S-invariant. Choose k large enough that $S' = S^k T$ and S lie in the same expansive component of the $\mathbb{Z}^2$ action, and therefore have the same Pinsker algebra with respect to μ, by a directional coding argument from B-Lind [Expansive subdynamics].

Now suppose the partition $\mathcal{B} = \{B, B'\}$ is non-trivial. Then the following list gives us the desired contradiction.

- $h(S', \mu, \mathcal{B}) = 0$.
  This holds because $h(S, \mu, \mathcal{B}) = 0$ (since $SB = B$) and S and S' have the same Pinsker algebra w.r.t. μ.

- $h(S', \mu, \mathcal{B}) = h(T, \mu, \mathcal{B})$.
  This holds since $SB = B$ implies for $n > 0$ that

$$\bigvee_{i=0}^{n} (S')^i B = \bigvee_{i=0}^{n} (TS^k)^i B = \bigvee_{i=0}^{n} T^i B \ .$$

- $h(T, \mu, \mathcal{B}) > 0$.
  This holds since μ is a K-automorphism.

QED

Periodic points of onto cellular automata.

A bonus section in the spirit of unanswered questions ...

Let f denote a surjective endomorphism of a full shift $\sigma_{[N]}$, i.e., an onto one-dimensional cellular automata.

Question.
Are the periodic points of f dense?
The answer is yes if f is right or left closing (B-Kitchens) or if f has a point of equicontinuity (Blanchard-Tisseur). Otherwise nothing is known.

The sequel follows experimental mathematics with Bryant Lee [in preparation], looking at periodic and preperiodic data for the action of f on points of given σ_N period.

(Martin, Odlyzko and Wolfram [1984] explained the pattern of jointly periodic points when f is a group endomorphism.)

Definition.
$\nu_k(f, \sigma_N) = |\{x \in \mathrm{Fix}(\sigma_N)^k : x \text{ is } f-\text{periodic}\}|$, and
$\nu(f, \sigma_N) = \overline{\lim}_k \, \nu_k(f, \sigma_N)^{1/k}$.

Note, $\nu_k(f, \sigma_N)$ does not change if f is replaced by $f\sigma^m$, for any m. Looking at a fairly large sample (including all span 4 onto automorphisms of the 2-shift), out to shift orbit periods of 19 to 26, we see no obvious difference between maps which are closing or not, or permutative or not. There are some rigorous arguments in certain classes to show $\nu(f, \sigma_N) > 1$, or $\nu(f, \sigma_N) = N$. We have no method for showing lower bounds to $\nu(f, \sigma_N)$ for any example.

Question.
Is $\nu(f, \sigma_N) > 1$ for every onto c.a. f?

Question.
Is $\nu(f, \sigma_N) \geq \sqrt{N}$ for every onto c.a. f?

For all large primes p, an onto c.a. f maps the set of points of period p into itself. So, the last question reflects a random maps heuristic: if a pattern doesn't force more periodicity, then we see i.o. at least about the periodicity we'd expect of a random map. An answer yes is consistent with our data, which are suggestive but (with the bound 26) certainly not compelling.

Conjecture.
There exist f such that $\nu(f, \sigma_N) < N$.

From our data, it seems obvious that the conjectured inequality is typical. (Equality holds in the algebraic case and some other classes.) But we can't give a proof for any example.

We know four ways to demonstrate $\nu(f, S_N)$ is large:

1. find a large shift fixed by f (or more generally by a power of f)

2. let f be a group endomorphism

3. use the algebra of a polynomial presenting f in very special cases [F.Rhodes, 1988]

4. finding equicontinuity points.

In all but the first case we have $\nu(f, \sigma_N) = N$. The first trick can be used with some generality:

Proposition. Given the surjective c.a. f on σ_N and $\epsilon > 0$, there is an invertible c.a. ϕ such that $\log(\nu(\phi f, S_N) > h(S_N) - \epsilon$.

The proposition's proof appeals to extension theorems from symbolic dynamics [B-Krieger]. Now, some experimental results from Bryant's program. In row k, P denotes the number of points of shift-period k which are also periodic under the c.a. map, and L denotes the longest c.a.-period of a point of shift period k.

k	*Frac.* *Per.*	$P^{1/k}$	$L^{1/k}$	P	L
9	0.50	1.85	1.58	256	63
10	0.25	1.74	1.40	256	30
11	0.50	1.87	1.69	1,024	341
12	0.06	1.58	1.23	256	12
13	0.50	1.89	1.67	4,096	819
14	0.25	1.81	1.20	4,096	14
15	0.50	1.90	1.19	16,384	15
16	0.00	1.00	1.00	1	1
17	0.50	1.92	1.38	65,536	255
18	0.25	1.85	1.30	65,536	126
19	0.50	1.92	1.62	262,144	9,709
20	0.06	1.74	1.22	65,536	60
21	0.50	1.93	1.21	1,048,576	63
22	0.25	1.87	1.34	1,048,576	682
23	0.50	1.94	1.39	4,194,304	2,047

Above: $x_0 + x_1$ on the 2-shift

k	$Frac.$ $Per.$	$P^{1/k}$	$L^{1/k}$	P	L
9	.48	1.84	1.55	247	54
10	.53	1.87	1.82	548	410
11	.18	1.71	1.60	375	176
12	.17	1.73	1.40	722	60
13	.20	1.76	1.59	1639	416
14	.21	1.79	1.62	3482	882
15	.23	1.81	1.59	7589	1095
16	.11	1.74	1.63	7707	2688
17	.07	1.72	1.60	10354	3230
18	.07	1.73	1.37	20565	324
19	.06	1.72	1.64	32320	13471
20	.06	1.74	1.64	68996	21240
21	.03	1.69	1.56	68835	11865
22	.02	1.67	1.60	89609	32428
23	.01	1.64	1.48	94324	9108

$x_0 + x_1 x_2$ on the 2-shift:
degree 1, linear in the left variable

k	*Frac.* *Per.*	$P^{1/k}$	$L^{1/k}$	P	L
9	.30	1.75	1.27	157	9
10	.26	1.74	1.49	268	55
11	.38	1.83	1.57	793	143
12	.08	1.63	1.16	362	6
13	.15	1.72	1.63	1236	611
14	.12	1.72	1.51	2068	329
15	.09	1.70	1.50	3014	465
16	.09	1.72	1.50	6043	728
17	.10	1.75	1.68	14145	6783
18	.06	1.71	1.58	15753	4095
19	.07	1.74	1.60	38191	7619
20	.03	1.69	1.54	40396	5780
21	.01	1.65	1.48	37867	4011
22	.01	1.66	1.51	75309	9658
23	.01	1.67	1.57	144096	34477

$x_0x_1 + x_2$ composed with $x_0 + x_1x_2$ on $\sigma_{[2]}$: a map neither left nor right closing.

k	*Frac.* *Per.*	$P^{1/k}$	$L^{1/k}$	P	L
9	.054	1.44	1.22	28	6
10	.176	1.68	1.46	181	45
11	.070	1.57	1.55	144	132
12	.001	1.14	1.12	5	4
13	.055	1.60	1.49	456	182
14	.026	1.54	1.35	428	70
15	.034	1.59	1.45	1121	285
16	.007	1.47	1.47	485	480
17	.016	1.56	1.55	2109	1734
18	.006	1.50	1.41	1594	549
19	.004	1.50	1.45	2452	1197
20	.005	1.54	1.50	6165	3640
21	.001	1.47	1.36	3627	693
22	.003	1.54	1.46	14004	4147
23	.002	1.53	1.53	18746	18538

$x_0 + x_1$ followed by $x_0 + x_1 x_2$

k	*Frac.* *Per.*	$P^{1/k}$	$L^{1/k}$	P	L
9	.148	1.61	1.58	76	63
10	.088	1.57	1.52	91	70
11	.070	1.57	1.46	144	66
12	.057	1.57	1.36	236	42
13	.035	1.54	1.53	287	273
14	.020	1.51	1.39	330	105
15	.012	1.49	1.44	404	255
16	.023	1.58	1.54	$1,525$	$1,008$
17	.028	1.62	1.52	$3,758$	$1,377$
18	.009	1.54	1.53	$2,386$	$2,250$
19	.003	1.49	1.47	$2,091$	$1,672$
20	.001	1.44	1.31	$1,635$	240
21	.004	1.54	1.48	$9,650$	$4,326$
22	.001	1.47	1.40	$4,896$	$1,848$
23	.002	1.54	1.53	$23,461$	$19,297$

$x_0 + x_1$ composed with the involution which flips x_0 when $x[-2,2] = 10x_0 11$.

k	*Frac. Per.*	$P^{1/k}$	$L^{1/k}$	P	L
9	.1796	1.65	1.48	92	36
10	.0263	1.39	1.17	27	5
11	.1782	1.70	1.53	365	110
12	.0122	1.38	1.30	50	24
13	.1049	1.68	1.53	860	260
14	.0056	1.38	1.37	93	84
15	.0340	1.59	1.43	1,117	225
16	.0154	1.54	1.40	1,010	224
17	.0135	1.55	1.45	1,770	612
18	.0037	1.46	1.33	980	180
19	.0078	1.54	1.50	4,125	2,242
20	.0011	1.42	1.32	1,227	280
21	.0008	1.42	1.39	1,731	1,092
22	.0006	1.43	1.27	2,829	220
23	.0008	1.46	1.44	6,833	4,462

$x_{-1} + x_0 x_1 + x_2$, linear in both end variables but not a group endomorphism.

k							
9	1.99	1.99	1.86	1.68	1.82	1.99	1.99
10	1.98	1.99	1.76	1.70	1.82	1.99	1.99
11	1.99	1.99	1.70	1.65	1.68	1.99	1.99
12	1.99	1.99	1.51	1.65	1.61	1.99	1.99
13	2.00	2.00	1.70	1.57	1.63	2.00	2.00
14	1.99	1.99	1.74	1.65	1.70	1.99	1.99
15	1.99	1.99	1.71	1.68	1.70	1.99	1.99
16	1.99	1.99	1.74	1.67	1.70	1.99	1.99
17	2.00	2.00	1.67	1.53	1.71	2.00	2.00
18	1.99	1.99	1.71	1.56	1.65	1.99	1.99
19	2.00	2.00	1.73	1.54	1.72	2.00	2.00

$\nu_k^o(\cdot, S_2)$ for the first 7 span-4 onto maps of $\sigma_{[2]}$. The other 59 give similar data (except with fewer automorphisms).
(ν_k^o counts points of least shift period k.)

k							
9	1.89	1.76	1.74	1.90	1.90	1.91	1.91
10	1.80	1.58	1.25	1.83	1.78	1.81	1.91
11	1.66	1.69	1.75	1.73	1.78	1.85	1.92
12	1.71	1.75	1.78	1.84	1.68	1.85	1.84
13	1.73	1.72	1.79	1.73	1.72	1.87	1.93
14	1.66	1.61	1.73	1.73	1.63	1.81	1.91
15	1.66	1.71	1.60	1.73	1.74	1.85	1.92
16	1.68	1.64	1.74	1.71	1.72	1.79	1.93
17	1.69	1.53	1.73	1.68	1.72	1.84	1.91
18	1.68	1.46	1.69	1.68	1.71	1.83	1.91
19	1.67	1.61	1.68	1.67	1.69	1.81	1.93

$\nu_k^o(\cdot, S_2)$ for the first seven sporadic span 5 maps on $\sigma_{[2]}$ [Hedlund-Appel-Welch, 1963]. The other 47 give similar data.