Isolating zero dimensional dynamics on manifolds
(joint work with Scott Schmieding)

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Tomasz Downarowicz!

- A mathematician of boldness, vision, power and creativity

– the theory of entropy structure and symbolic extensions

– Law of Series

- Bird lover

Mountain adventurer

A leader, among mathematicians and friends

Photographer

Excellent applier of sun lotion!

Some definitions

• Let h be a homeomorphism of a compact metric space M. A compact subset Y is *isolated* (in M, by h) if it has a neighborhood U such that $Y = \cap_{n \in Z} U$.

• Y is *strongly isolated* if it has an isolating neighborhood U such that Y contains the intersection of U with the nonwandering set of h. We also say then that the subsystem $(Y, h|Y)$ is isolated (or strongly isolated).

• A system (X, T) is *isolated by h* if (X, T) is conjugate to $(Y, h|Y)$, where Y is some isolated set of h.

Some examples.

Example 1. Let h be a rotation of a disc. The fixed point at the center is not isolated.

Example 2. A hyperbolic fixed point of a smooth map is an isolated set.

Example 3. The Smale horseshoe is isolated. More generally, a basic set of an Axiom A diffeomorphism is isolated (a.k.a. locally maximal).

Example 4. Suppose h is a shift of finite type. Then its isolated subsystems are the shifts of finite type it contains. But they are not strongly isolated by h.

Question.

Which systems (X, T), with X zero dimensional compact metric, can be isolated (or strongly isolated) by a homeomorphism of an n-dimensional manifold M?

(If M has boundary, for definiteness we require Y to miss the boundary of M.)

My interest in this question started from a question David Fried asked me–can a sofic shift be an isolated system for a homeo/diffeo of a manifold?– when I was a postdoc.

That time predates even a familiar prehistoric image.

“Prehistoric version”

Motivation for the question?

• It is natural given the significance of isolation in hyperbolic dynamics and Conley index theory.

• A strongly isolated set is some topological dynamical analogue of a basic set (except, lacking structural stability) .

• As the question seems nontrivial–we may learn something by answering it.

Why “zero dimensional”?

• It is the initial dimension, for which one might hope for a more simple answer.

• Typically interested in domain the Cantor set; dynamical questions then separate from homeormorphism types of isolated sets.

Why consider “strongly isolated”?

For example, the 2-shift is isolated in the 3-shift. But it is dynamically entangled. Although points outside the two shift will move out of an isolating neighborhood, many of them will return regularly–dynamically, the 2-shift does not seem to be a separated dynamical part of the 3-shift.

We might consider this concretely in terms of another dynamical system.

In the neighborhood of Downarowicz

He's isolated

But not for long

Theorem

Suppose $(Y, h|_Y)$ is an invariant subsystem of (M, h) for a homeo h of an n-manifold M. Then it can be isolated by a homeomorphism of an $n+1$ dimensional manifold.

Proof Sketch

Let $\tilde{M} = M \times R/Z$.

1. Define $h_1 : \tilde{M} \to \tilde{M}, (x, t) \mapsto (h(x), t)$.

2. Let $h_2 : \tilde{M} \to \tilde{M}, (x, t) \mapsto (x, \psi(x, t)$ where

- $\psi(x, t)$ increases from 0 to 1, and
- $\psi(x, t) = t$ iff

(i) $t = 0$, (ii) $t = 1$ or (iii) $t = 1/2$ and $x \in Y$.

Let $h = h_2 \circ h_1$. Then $\{(y, 1/2) : y \in Y\}$ is isolated by $\tilde{h}$, with $(\tilde{Y}, \tilde{h})$ conjugate to $(Y, h|Y)$. $\square$

Corollary

Suppose (X, T) is a homeomorphism of the Cantor set (or any zero dimensional compact metric space) and $n \geq 3$. Then (X, T) can be isolated by a homeomorphism of an n-manifold.

Proof. Take X in the interior of the unit disc D. By a theorem of Moises, the embedding $T : X \to D$ can be extends to a homeomorphism $h : D \to D$. Then the previous theorem isolates the invariant system (X, T) in dimension $n > 2$.

So,

Question.

Which zero dimensional systems can be isolated by a surface homeomorphism?

A little context.

• There are constraints on the zeta functions of SFTs which are basic sets of Axiom A diffeos on surfaces (Franks, Fried). (But, every SFT can be isolated by a surface diffeomorphism (as a subsystem of some horseshoe.)

• Under some transversality assumptions on stable and unstable manifolds, a subshift isolated by a surface diffeomorphism or homeomorphism must have a rational zeta function (Fried, Mrozek).

We can't answer the question above , but we can give some clues, as follows.

Examples: Some systems isolated by surface homeomorphisms.

0. The identity map on a compact zero dimensional metric space.
(But we are interested more in transitive systems.)

I. Some special minimal subshifts.

II. A mixing strictly sofic shift.

III. A nonexpansive mixing positive entropy shift.

A constraint.

Let $G_T = C(X, Z)/(I - T)C(X, Z)$.

(X, T) is indecomposable if it is not the disjoint union of two systems.

Theorem If (X, T) is indecomposable zero dimensional and isolated by a surface homeomorphism, then there is a finitely generated subgroup K of G_T such that G_T/K is free. If that surface is orientable, then G_T is free.

There are many T for which the group G_T is free. Still, every countable abelian group is G_T for many zero dimensional systems (X, T). So, to be isolatable in dimension 2 is rather special. In particular,

Corollary. The following cannot be isolated by a surface homemorphism:
• An odometer.
• A zero dimensional system which has an odometer as a factor – e.g., a Toeplitz shift.

Let's briefly get an idea of proofs. This will mostly be to indicate the ideas involved rather than go through the proofs.
First, the examples.

I. Some minimal shifts.

Take a Denjoy homeomorphism of the circle, a map with wandering open intervals and an irrational rotation factor. The complement of the union of those open intervals is a subshift, invariant in dimension 1 and thus isolatable in dimension 2. The group G_T can be chosen Z^k with $k \in [2, \infty]$.

II. Mixing nonexpansive positive entropy example.

Start with the toral automorphism T_A where $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Let q be the well known quotient map collapsing orbits of the torus involution $x \mapsto -x$. The quotient space is homeomorphic to S^2. The map q is 2-to-1 at all points of the torus except $(0,0), (0,1/2), (1/2,0), (1/2,1/2)$.

The toral automorphism has an SFT cover such that all points with multiple preimages are forwardly or backwardly asymptotic to $(0,0)$ (Adler-Weiss, Memoirs AMS 1970).

You can find (e.g. using the Adler-Weiss covering SFT) a mixing SFT subsystem W of the toral automorphism T_A $(0,0)$ such that

- W contains $\{(0,1/2),(1/2,0),(1/2,1/2)\}$
- W does not contain $\{(0,0)\}$
- W is invariant under $x \mapsto -x$.

Then qW is still zero dimensional (on W, q is locally constant 2-to-1 except at finitely many points) but cannot be expansive (e.g., if W were expansive then the generic number of preimage points of qW would be the minimum number).

Though not expansive, qW is rather close to being a shift of finite type.

III. A mixing strictly sofic shift.

This sofic shift, isolated by surface homeomorphism, is the quotient of a mixing SFT by a map which identifies two fixed points and nothing else.

This construction uses the “unwrapping” technique for realizing the natural extension of a selfmap of a graph as an attractor. The technique was introduced by Barge and Martin (1990), based on a theorem of Morton Brown, and further developed by Boyland, de Carvahlo and Hall (2013).

Under some delicate conditions on a continuous surface map $f\colon M \to M$ and a graph E contained in M, the inverse limit $\hat{f}\colon \hat{M} \to \hat{M}$ will have domain still homeomorphic to M, with the inverse limit of $f\colon E \to E$ an attractor.

We use M an annulus and $f\colon E \to E$ the factor of the tent map obtained by identifying its two fixed points. A mixing onesided SFT in the tent map system containing those two fixed points then produces a strictly sofic shift isolated in the attractor and hence in $\hat{M}$ by $\hat{f}$.

Altogether, it's a pretty tricky construction which we haven't been able to make at all general, which produces an isolated subsystem which is strictly sofic (barely).

I presume other strictly sofic isolated examples could be found which are subsystems containing prong points in expansive pseudoAnosov diffeos. These would also be quotients of SFTs by maps collapsing only finitely many points.

The constraint.

Let $G_T = C(X, Z)/(I - T)C(X, Z)$.

(X, T) is indecomposable if it is not the disjoint union of two systems.

Theorem If (X, T) is indecomposable and isolated by a surface homeomorphism, then there is a finitely generated subgroup K of G_T such that G_T/K is free. If that surface is orientable, then G_T is free.

The argument uses Conley index arguments of E.S.Thomas ("One dimensional minimal sets," 1973). Very schematically:

Suppose Y zero dimensional is isolated in (M, h) with M a surface. Let $\widehat{M}$ be the mapping torus, containing $\widehat{Y}$ as a one-dimensional isolated subset of the suspension flow ψ_t (i.e. for some neighborhood U of $\widehat{Y}$, $\cap_{t \in R} \psi_t U = \widehat{Y}$.

Then U contains an *isolating block* for $\hat{Y}$. This is a compact isolating neighborhood B of $\widehat{Y}$ whose boundary is the union of an entry set, an exit set, and a closed set which is the union of bounded length orbit intervals which begin at the entry set and end at the exit set.

There is an exact sequence in Cech cohomology

$$H^1(B) \to H^1(Y) \to H^2(b^-, a^-)\ .$$

Here b^- is the exit set and a^- is the subset of b^- backwardly asymptotic to Y. It's well known that the Cech group $H^1(\widehat{Y})$ is isomorphic to $C(Y, Z)/(I - T)C(Y, Z)$. The group $H^1(Y)$ is torsion free. $H^1(B)$ is finitely generated, because [Ruchalla] B is a Euclidean neighborhood retract with b^- a topological 2-manifold. (That proof fails for higher dimension.) Again using b^- is a topological 2-manifold, the right hand side is the direct sum of a free group and a finite group, and is free if M is orientable. Putting these together gives the constraint.

Two other results

Theorem Suppose two zero dimensional systems are flow equivalent and one can be isolated by a homeomorphism on an oriented surface. Then, so can the other.

Theorem Suppose a zero dimensional system of a surface homeomorphism h has a subsystem which is an odometer. Than all points in that odometer are limits of periodic points of h.

Open questions

There are various open questions reflecting the large gap between what we can do and what we can rule out.

E.g., although our constructions are very limited and the collection of zero dimensional systems which are factors of SFTs is very rich, we cannot rule out that every such factor can be isolated by a surface homeomorphism. (We do NOT expect they all can be.)

There are a couple of natural lines of investigation from here, but I'll close with one open question.

Tomasz, which mountain comes next?