Mixing properties of some maps with countable Markov partitions

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To the memory of Kolya Chernov

Abstract

In the previous works of the author and S.Newhouse ([9] and [10]) a class of piecewise smooth two-dimensional systems with countable Markov partitions was studied, and Bernoulli property was proved.
In this paper we consider 2-d maps F satisfying the same hyperbolicity and distortion conditions, and assume similar conditions for F^{-1}. We assume additionally that contraction of each map increases when points approach the boundary of its domain. For such systems we extend the results of [8], and prove exponential decay of correlations.

1 Statement of results

1. As in [9] , [10] we consider the following 2-d model. Let Q be the unit square. Let $\xi = \{E_1, E_2, \dots ,\}$ be a countable collection of closed curvilinear rectangles in Q. Assume that each E_i lies inside a domain of definition of a C^2 diffeomorphism f_i which maps E_i onto its image $S_i \subset Q$. We assume each E_i connects the top and the bottom of Q. Thus each E_i is bounded from above and from below by two subintervals of the line segments $\{(x,y) : y = 1,\ 0 \le x \le 1\}$ and $\{(x,y) : y = 0,\ 0 \le x \le 1\}$. Hyperbolicity conditions that we formulate below imply that the left and right boundaries of E_i are graphs of smooth functions $x^{(i)}(y)$ with $\left|\frac{dx^{(i)}}{dy}\right| \le \alpha$ where α is a real number satisfying $0 < \alpha < 1$.

The images $f_i(E_i) = S_i$ are narrow strips connecting the left and right sides of Q and that they are bounded on the left and right by the two subintervals of the line segments $\{(x,y) : x = 0,\ 0 \le y \le 1\}$ and $\{(x,y) : x = 1,\ 0 \le y \le 1\}$ and above and below by the graphs of smooth functions $Y^i(X), \mid \frac{dY^{(i)}}{dX} \mid \le \alpha$. We are saying that $E_i's$ are *full height* in Q while the $S_i's$ are *full width* in Q.

2. For $z \in Q$, let ℓ_z be the horizontal line through z. We define $\delta_z(E_i) = diam(\ell_z \bigcap E_i)$, $\delta_{i,max} = \max_{z \in Q} \delta_z(E_i)$, $\delta_{i,min} = \min_{z \in Q} \delta_z(E_i)$.
We assume the following

Geometric conditions

G1. For $i \ne j$ holds
$int\ E_i \cap int\ E_j = \emptyset$ and $int\ S_i \cap int\ S_j = \emptyset$.

G2. $mes(Q \setminus \cup_i\ int\, E_i) = 0$ where *mes* stands for Lebesgue measure.

G3. For some $0 < a \le b < 1$ and some $C_G \ge 1$ holds

$$C_G^{-1} a^i \le \delta_{i,min} \le \delta_{i,max} \le C_G b^i$$

Remark 1.1 *Condition [G3] is a simplified version of respective assumptions in [9] and [10] , but still allows the widths of E_i to oscillate exponentially.*

In the standard coordinate system for a map $F : (x,y) \to (F_1(x,y), F_2(x,y))$ we use $DF(x,y)$ to denote the differential of F at some point (x,y) and F_{jx}, F_{jy}, F_{jxx}, F_{jxy}, etc., for partial derivatives of F_j, $j = 1,2$.

Let $J_F(z) = \mid F_{1x}(z)F_{2y}(z) - F_{1y}(z)F_{2x}(z) \mid$ be the absolute value of the Jacobian determinant of F at z.

3. Next we assume
Hyperbolicity conditions
There exist constants $0 < \alpha < 1$ and $K_0 > 1$ such that for each i the map

$$F(z) = f_i(z) \text{ for } z \in E_i$$

satisfies

H1. $| F_{2x}(z) | + \alpha | F_{2y}(z) | + \alpha^2 | F_{1y}(z) | \leq \alpha | F_{1x}(z) |$

H2. $| F_{1x}(z) | - \alpha | F_{1y}(z) | \geq K_0$.

H3. $| F_{1y}(z) | + \alpha | F_{2y}(z) | + \alpha^2 | F_{2x}(z) | \leq \alpha | F_{1x}(z) |$

H4. $| F_{1x}(z) | - \alpha | F_{2x}(z) | \geq J_F(z) K_0$.

4. Some corollaries from Hyperbolicity conditions.

For a real number $0 < \alpha < 1$, we define the cones

$$K^u_\alpha = \{(v_1, v_2) : | v_2 | \leq \alpha | v_1 |\}$$

$$K^s_\alpha = \{(v_1, v_2) : | v_1 | \leq \alpha | v_2 |\}$$

and the corresponding cone fields $K^u_\alpha(z), K^s_\alpha(z)$ in the tangent spaces at points $z \in \mathbf{R}^2$.

The following proposition proved in [10] relates conditions H1-H4 above with the usual definition of hyperbolicity in terms of cone conditions. It shows that conditions H1 and H2 imply that the K^u_α cone is mapped into itself by DF and expanded by a factor no smaller than K_0 while H3 and H4 imply that the K^s_α cone is mapped into itself by DF^{-1} and expanded by a factor no smaller than K_0.

Unless otherwise stated, we use the *max* norm on $\mathbf{R}^2$, $| (v_1, v_2) | = \max(| v_1 |, | v_2 |)$.

Proposition 1.2 *Under conditions H1-H4 above, we have*

$$DF(K^u_\alpha) \subseteq K^u_\alpha \tag{1}$$

$$v \in K^u_\alpha \Rightarrow | DFv | \geq K_0 | v | \tag{2}$$

$$DF^{-1}(K^s_\alpha) \subseteq K^s_\alpha \tag{3}$$

$$v \in K^s_\alpha \Rightarrow | DF^{-1}v | \geq K_0 | v | \tag{4}$$

Remark 1.3 The first version of hyperbolicity conditions appeared in the paper of Smale [16]. It was developed in particular by Alexeev [4] and by Hirsch, Pugh and Shub, see [7], [11] . Cone conditions for billiard systems were first studied by Sinai, see [19].
Here we use hyperbolicity conditions from [10]. In [9] we used hyperbolicity conditions from [4] which implied the invariance of cones and uniform expansion with respect to the sum norm $\mid v \mid = \mid v_1 \mid + \mid v_2 \mid$.

5. A theorem about systems satisfying Geometric and Hyperbolicity conditions.

   The map

$$F(z) = f_i(z) \text{ for } z \in int\ E_i$$

is defined almost everywhere on Q. Let $\tilde{Q}_0 = \bigcup_i int\ E_i$, and, define $\tilde{Q}_n, n > 0$, inductively by $\tilde{Q}_n = \tilde{Q}_0 \bigcap F^{-1}\tilde{Q}_{n-1}$. Let $\tilde{Q} = \bigcap_{n \geq 0} \tilde{Q}_n$ be the set of points whose forward orbits always stay in $\bigcup_i int\ E_i$. Then, $\tilde{Q}$ has full Lebesgue measure in Q, and F maps $\tilde{Q}$ into itself.

The hyperbolicity conditions H1–H4 imply the estimates on the derivatives of the boundary curves of E_i and S_i which we described earlier. They also imply that any intersection $f_i E_i \bigcap E_j$ is full width in E_j. Further, $E_{ij} = E_i \bigcap f_i^{-1} E_j$ is a full height subrectangle of E_i and $S_{ij} = f_j f_i E_{ij}$ is a full width substrip in Q.

Given a finite string $i_0 \dots i_{n-1}$, we define inductively

$$E_{i_0 \dots i_{n-1}} = E_{i_0} \bigcap f_{i_0}^{-1} E_{i_1 i_2 \dots i_{n-1}}.$$

Then, each set $E_{i_0 \dots i_{n-1}}$ is a full height subrectangle of E_{i_0}.

Analogously, for a string $i_{-m} \dots i_{-1}$ we define

$$S_{i_{-m} \dots i_{-1}} = f_{i_{-1}}(S_{i_{-m} \dots i_{-2}} \bigcap E_{i_{-1}})$$

and get that $S_{i_{-m} \dots i_{-1}}$ is a full width strip in Q. It is easy to see that $S_{i_{-m} \dots i_{-1}} = f_{i_{-1}} \circ f_{i_{-2}} \circ \dots \circ f_{i_{-m}}(E_{i_{-m} \dots i_{-1}})$ and that $f_{i_0}^{-1}(S_{i_{-m} \dots i_{-1}})$ is a full-width substrip

of E_{i_0}.
We also define curvilinear rectangles $R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}}$ by

$$R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}} = S_{i_{-m}\dots i_{-1}} \bigcap E_{i_0\dots i_{n-1}}$$

If there are no negative indices then respective rectangle is full height in Q. For infinite strings, we have the following Proposition.

Proposition 1.4 *Any C^1 map F satisfying the above geometric conditions G1–G3 and hyperbolicity conditions H1–H4 has a "topological attractor"*

$$\Lambda = \bigcup_{\dots i_{-n}\dots i_{-1}} \bigcap_{k\geq 1} S_{i_{-k}\dots i_{-1}}$$

The infinite intersections $\bigcap_{k=1}^{\infty} S_{i_{-k}\dots i_{-1}}$ define C^1 curves $y(x)$, $|dy/dx| \leq \alpha$ which are the unstable manifolds for the points of the attractor. The infinite intersections $\bigcap_{k=1}^{\infty} E_{i_0\dots i_{k-1}}$ define C^1 curves $x(y)$, $|dx/dy| \leq \alpha$ which are the stable manifolds for the points of the attractor. The infinite intersections

$$\bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}}$$

define points of the attractor.

Proposition 1.4 is a well known fact in hyperbolic theory. For example it follows from Theorem 1 in [4]. See also [11]. The union of the stable manifolds has full measure in Q. The trajectories of all points in this set converge to Λ. That is the reason to call Λ a topological attractor.

6. Next we assume certain Distortion conditions.
   As we have a countable number of domains the derivatives of f_i grow. We formulate certain assumptions on the second derivatives. We use the distance function $d((x,y),(x_1,y_1)) = \max(|x-x_1|,|y-y_1|)$ associated with the norm $|v| = \max(|v_1|,|v_2|)$ on vectors $v=(v_1,v_2)$.

As above, for a point $z \in Q$, let l_z denote the horizontal line through z, and if $E \subseteq Q$, let $\delta_z(E)$ denote the diameter of the horizontal section $l_z \bigcap E$. We call $\delta_z(E)$ the $z-width$ of E.

In given coordinate systems we write $f_i(x,y) = (f_{i1}(x,y), f_{i2}(x,y))$. We use $f_{ijx}, f_{ijy}, f_{ijxx}, f_{ijxy}$, etc. for partial derivatives of $f_{ij}, j = 1,2$.

We define

$$| D^2 f_i(z) | = \max_{j=1,2,(k,l)=(x,x),(x,y),(y,y)} | f_{ijkl}(z) |.$$

Next we formulate distortion conditions which are used to control the fluctuation of the derivatives of iterates of F along unstable manifolds, and to construct Sinai local measures.

Suppose there is a constant $C_0 > 0$ such that the following
Distortion conditions D1 are satisfied

D1. $\sup_{z \in E_i, i \geq 1} \frac{| D^2 f_i(z) |}{| f_{i1x}(z) |} \delta_z(E_i) < C_0.$

7. An $F-$invariant Borel probablility measure μ on Q is called a $Sinai-Ruelle-Bowen$ measure (or SRB-measure) for F if μ is ergodic and there is a set $A \subset Q$ of positive Lebesgue measure such that for $x \in A$ and any continuous real-valued function $\phi : Q \to \mathbf{R}$, we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(F^k x) = \int \phi d\mu. \tag{5}$$

Existence of an SRB measure is a much stronger result than 1.4. It allows to describe statistical properties of trajectories in a set of positive phase volume.

Our conditions imply the following theorem proved in [9], [10].

Theorem 1.5 *Let F be a piecewise smooth mapping as above satisfying the geometric conditions G1–G3, the hyperbolicity conditions H1–H4 and the distortion condition D1.*

Then, F has an SRB measure μ supported on Λ whose basin has full Lebesgue measure in Q. Dynamical system (F,μ) satisfies the following properties.

(a) (F, μ) *is measure-theoretically isomorphic to a Bernoulli shift.*

(b) F *has finite entropy with respect to the measure* μ*, and the entropy formula holds*

$$h_\mu(F) = \int log|D^u F| d\mu \tag{6}$$

where $D^u F(z)$ *is the norm of the derivative of* F *in the unstable direction at* z*.*

(c)

$$h_\mu(F) = \lim_{n\to\infty} \frac{1}{n} \log | DF^n(z) | \tag{7}$$

where the latter limit exists for Lebesgue almost all z *and is independent of such* z*.*

8. Additional hyperbolicity and distortion conditions and statement of the main theorem.

When applying thermodynamic formalism to hyperbolic attractors one considers the function $\phi(z) = -\log(D^u F(z))$. Thermodynamic formalism is based on the fact that the pullback of $\phi(z)$ into a symbolic space determined by some Markov partition is a locally Hölder function.
In order to prove Hölder property of $\phi(z)$ we assume that the inverse map F^{-1} satisfies distortion conditions similar to D1 . As branches f_i^{-1} of F^{-1} are defined on strips S_i we consider crossections of S_i by vertical lines. Let $\xi_z(S_i)$ be the z-height of S_i, i.e. the height of the vertical crossections of S_i through $z \in S_i$. For $F^{-1}(z)$ the derivative $F_{2y}^{-1}(z)$ plays the same role as $F_{i1x}(z)$ for F.

Distortion condition D2.

Suppose there is a constant $C_0 > 0$ such that

D2. $\sup_{z\in S_i, i\geq 1} \dfrac{| D^2 f_i^{-1}(z) |}{| f_{i2y}^{-1}(z) |} \xi_z(S_i) < C_0.$

In this paper we apply the same approach as in [8] to some models with F satisfying distortion conditions D1 and D2. We assume additionally that variation of $\log F_{1x}$ on initial rectangles E_i is uniformly bounded, and that contraction is sufficiently strong .

Bounded Initial Variation.

BIV. There exists $B_0 > 0$ such that for all i and all $\{z_1 = (x_1, y_1), z_2 = (x_2, y_2)\} \in E_i$ holds

$$| \log F_{1x}(z_1) - \log F_{1x}(z_2) | < B_0 \tag{8}$$

For rectangles $R_{i,j} = S_i \cap E_j$ we define the maximal height $H_{max}(R_{i,j}) = max_{z \in R_{i,j}} \xi_z(S_i)$, and the minimal width $W_{min}(R_{i,j}) = \min_{z \in R_{i,j}} \delta_z(E_j)$. We suppose the following condition of *strong contraction* holds.

Strong Contraction.

SC. There exists $M_0 > 0$ such that for all i, j holds

$$H_{max}(R_{i,j}) < M_0 W_{min}(R_{i,j}) \tag{9}$$

Examples where condition SC is satisfied can be constructed as follows. The widths of E_j decrease when E_j accumulate toward one of the vertical boundaries of Q, say toward $\{(x,y) : x = 0\}$. At the same time for each i the heights $\xi_z(S_i)$ converge to 0 for $\{z = (x,y) : x \to 0\}$, in such a way that condition SC is satisfied.

Let $\mathscr{H}_\gamma$ be the space of functions on Q satisfying Hölder property with exponent γ

$$| \phi(x) - \phi(y) | \leq c| x - y |^\gamma$$

We prove the following theorem

Theorem 1.6 *Suppose F satisfies conditions of Theorem 1.5 and also conditions D2, BIV and SC. Then (F, μ) has exponential decay of correlations for $\phi, \psi \in \mathscr{H}_\gamma$. Namely there exist $\eta(\gamma) < 1$ and $C = C(\phi, \psi)$ such that*

$$| \int \phi(\psi \circ F^n) d\mu - \int \phi d\mu \int \psi d\mu | < C\eta^n \tag{10}$$

Below in Section 4 we consider examples of systems satisfying conditions of theorem 1.6.

2 Hölder properties of $\log(D^u F(z))$ in the phase space.

The key step toward the proof of Theorem 1.6 is to establish that the pullback of the function $\log D^u F$ into respective symbolic space is Hölder continuous. Then one can follow Ruelle-Bowen approach ([12], [6]), in particular results of Sarig [13], and develop thermodynamic formalism for systems under consideration.
Hölder properties of the pullback of $\log D^u F$ into symbolic space follow from Hölder properties of $\log D^u F$ in the phase space. In this Section we establish such properties.
Although Markov partitions are partitions of the attractor, we need to check Hölder property on actual two-dimensional curvilinear rectangles $R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}}$. We call respective partition Markov as well.
In our models Markov partitions consist of full height rectangles E_i.
For any function $a(x,y)$ the variation of $a(x,y)$ over a rectangle R is defined as

$$var(a(x,y))|R = \sup_{(x_1,y_1)\in R,(x_2,y_2)\in R} | a(x_1,y_1) - a(x_2,y_2) | \tag{11}$$

The function $\log D^u F$ is **locally Hölder** if for $m \geq 0$, $n \geq 1$ the variation of $\log D^u F$ on $R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}}$ satisfies

$$var(\log D^u F)|R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}} < C\theta_0^{min(m,n)} \tag{12}$$

for some $C > 0$, $\theta_0 < 1$.

Proposition 2.1 *$\log D^u F$ is a locally Hölder function.*

The strategy of the proof is similar to the one in the proof of Proposition 5.1 in [8].
We prove Proposition 2.1 with some θ_0 and C determined by hyperbolicity and distortion conditions, and condition SC.

1. The sets $R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}}$ are bounded from above and below by some arcs of two unstable curves $\Gamma^u_{i_{-m}\dots i_{-1}}$, which are images of some pieces of the top

and bottom of Q, and from left and right by some arcs of two stable curves $\Gamma^s_{i_0 \dots i_{n-1}}$, which are preimages of some pieces the left and right boundaries of Q .
Let $Z_1, Z_2 \in R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}}$ be two points on the attractor. We connect Z_1, Z_2 by two pieces of their unstable manifolds to two points Z_3, Z_4 which belong to the same stable manifold. Let

$\gamma_1 = \gamma(Z_1, Z_3) \subset W^u(Z_1)$, $\gamma_2 = \gamma(Z_2, Z_4) \subset W^u(Z_2)$, $\gamma_3 = \gamma(Z_3, Z_4) \subset W^s(Z_3)$ be respective curves all located inside $R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}}$.
We estimate

$$| \log D^u F(Z_1) - \log D^u F(Z_2) | \leq | \log D^u F(Z_1) - \log D^u F(Z_3) | + \\ | \log D^u F(Z_3) - \log D^u F(Z_4) | + | \log D^u F(Z_4) - \log D^u F(Z_2) |$$

2. First we estimate $| \log D^u F(Z_1) - \log D^u F(Z_3) |$. We connect Z_1 and Z_3 by a chain of small rectangles $R \subset R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}}$ covering γ_1. Then $| \log D^u F(Z_1) - \log D^u F(Z_3) |$ is majorated by the sum of similar differences for points $z_1, z_2 \in W^u(Z_1) \cap R$.
Because of cone conditions we can choose rectangles $R = \Delta x \times \Delta y$ satisfying $| \Delta y | < \alpha | \Delta x |$. Let R be one of such rectangles.
As in the proof of Proposition 5.1 in [8] the estimate of $| \log D^u F(Z_1) - \log D^u F(Z_3) |$ is reduced to the estimate

$$| \log F_{1x}(z_1) - \log F_{1x}(z_2) | \tag{13}$$

for points $z_1, z_2 \in W^u(Z_1) \cap R$.
Let $\Gamma_1 \supset \gamma_1$ be the large piece of the same unstable manifold restricted to E_{i_0}. Using the mean value theorem we estimate the variation of $\log D^u F(z)$ on R as

$$const\ sup_{z \in R} \frac{| f_{1ij}(z) |}{| f_{1x}(z) |} | \Gamma_1 | \frac{\Delta x}{| \Gamma_1 |} \tag{14}$$

Note that differently from [8] here the ratios $\frac{| f_{ijkl} |}{| f_{i1x} |}$ can be unbounded, so in order to use distortion condition D1 we divide and multiply by $| \Gamma_1 |$. After we add over rectangles covering γ_1 we get an estimate

$$| \log D^u F(Z_1) - \log D^u F(Z_3) | < C_1 \frac{| \gamma_1 |}{| \Gamma_1 |} \tag{15}$$

Under f_{i_0} the curve Γ_1 is mapped onto a full width curve, and γ_1 is mapped onto a piece of $W^u(f_{i_0}(Z_1), E_{i_1 \dots i_{n-1}})$. The length of that curve is bounded by $c\frac{1}{K_0^{n-1}}$. Then applying again D1 we get

$$|\log D^u F(z_1) - \log D^u F(z_3)| < C_2 \frac{1}{K_0^n} \tag{16}$$

where C_2 is a uniform constant. Similar estimates hold for $Z_2, Z_4 \in \gamma_2$.

$$|\log D^u F(Z_2) - \log D^u F(Z_4)| < C_2 \frac{1}{K_0^n} \tag{17}$$

3. Next we estimate the variation of $\log | D^u F(z) |$ between points Z_3 and Z_4, which belong to the same stable manifold $W^s(Z_3) = W^s(Z_4) \subset R_{i_{-m}\dots i_{-1}, i_0 \dots i_{n-1}}$. BIV condition implies that expressions

$$|\log D^u F(Z_3) - \log D^u F(Z_4)| \tag{18}$$

are uniformly bounded on full height rectangles, so it is enough to consider subrectangles of $R_{i,j}$.
Hyperbolicity conditions imply (see [10]) that any unit vector in K^u_α at a point $z \in E_i$, in particular a tangent vector to $W^u(z)$, has coordinates $(1, a_z)$ with $| a_z | < \alpha$. Thus we need to estimate

$$\log | F_{1x}(Z_3) + a_{Z_3} F_{1y}(Z_3) | - \log | F_{1x}(Z_4) + a_{Z_4} F_{1y}(Z_4) | \tag{19}$$

This time instead of moving along $W^u(Z_1)$ we are moving along $W^s(Z_3)$, which connects Z_3 and Z_4. In that case we use $| \Delta x | < \alpha | \Delta y |$, so Δy variations are added.
As in [8] the proof of 19 is reduced to the estimates of two kinds.

   (a) First we combine similar terms, and estimate sums of contributions

$$|\log F_{1x}(z_1) - \log F_{1x}(z_2)| \tag{20}$$

   over small rectangles R covering γ_3.
   Let $\Gamma_3 \supset \gamma_3$ be the large piece of the same stable manifold restricted to R_{i_{-1}, i_0}. As above in 14 by using the mean value theorem we estimate 20 as

$$const\ sup_{z \in R} \frac{| f_{1ij}(z) |}{| f_{1x}(z) |} | \Gamma_3 | \frac{\Delta y}{| \Gamma_3 |} \tag{21}$$

The sum of such contributions is estimated as

$$const\ sup_{z\in R_{i_{-1},i_0}} \frac{| f_{1ij}(z) |}{| f_{1x}(z) |} | \Gamma_3 | \frac{| \gamma_3 |}{| \Gamma_3 |} \tag{22}$$

Because of the strong contraction condition SC there exists a constant M_1 such that

$$| \Gamma_3 | < M_1 | \Gamma_1 | \tag{23}$$

We rewrite the above estimate as

$$const\ sup_{z\in R_{i_{-1},i_0}} \frac{| f_{1ij}(z) |}{| f_{1x}(z) |} | \Gamma_1 | M_1 \frac{| \gamma_3 |}{| \Gamma_3 |} \tag{24}$$

Using distortion condition D1 we get that 24 is bounded by

$$c_1 \frac{| \gamma_3 |}{| \Gamma_3 |} \tag{25}$$

Because of distortion condition D2 ratios of lengths on stable manifolds are preserved under the action of F^{-1} up to a constant. F^{-1} maps Γ_3 onto a stable curve of full height, and γ_3 is mapped onto a curve which is full height in $S_{i_{-m}\dots i_{-2}}$. As lengths on stable manifolds are contracted at least by K_0^{-1}, all terms from above contribute an estimate less than

$$c_2 \frac{1}{K_0^m} \tag{26}$$

(b) Next, as in [8] we need to estimate

$$| a_{Z_3} - a_{Z_4} | \tag{27}$$

We repeat the arguments of Lemma 5.2 from [8], and prove by induction that there exist $c_0 > 0$, $0 < \theta_0 < 1$ such that

$$| a_{Z_3} - a_{Z_4} | < c_0 \theta_0^m \tag{28}$$

As in Lemma 5.2 from [8], the proof of 28 is reduced to estimates of two types. Estimate of type 1 is obtained by taking sums of the following expressions over $W^s(z, R_{i_{-m}\dots i_{-1}, i_0 \dots i_{n-1}}$

$$\log F_{1x}(z_1) - \log F_{1x}(z_2) \tag{29}$$

where z_1 and z_2 are close points on $W^s(z, R_{i_{-m}\dots i_{-1}, i_0\dots i_{n-1}})$.
Such sums are estimated as above, and we get that respective sums are bounded by

$$c_3 \frac{1}{K_0^m} \tag{30}$$

Estimate of type 2 is the estimate of

$$\begin{aligned} &\mid \tfrac{F_{2y}}{F_{1x}}(Z_3)a(Z_3) - \tfrac{F_{2y}}{F_{1x}}(Z_4)a(Z_4) \mid \\ \leq &\mid \tfrac{F_{2y}}{F_{1x}}(Z_3)a(Z_3) - \tfrac{F_{2y}}{F_{1x}}(Z_3)a(Z_4) \mid \\ + &\mid \tfrac{F_{2y}}{F_{1x}}(Z_3)a(Z_4) - \tfrac{F_{2y}}{F_{1x}}(Z_4)a(Z_4) \mid \end{aligned} \tag{31}$$

In order to estimate the second term in 31 we split γ_3 into small pieces and get as above the estimate 30.
The first term is estimated using inductive assumption.
Note that we can assume

$$\frac{1}{K_0^2} + \alpha^2 < 1 \tag{32}$$

That is because (as proved in [10]) one can consider some power F^t instead of F, and still have conditions D1 and D2 (with different constants). By choosing appropriate power one can make K_0 arbitrary large. Then 32 will be satisfied. As exponential decay of correlations for F^t implies exponential decay of correlations for F, Theorem 1.6 follows from exponential decay of correlations for F^t.
So, differently from [8], here we do not need 32 as an additional condition H5.
Then as in [8] the total estimate is

$$\mid a_{F(Z_3)} - a_{F(Z_4)} \mid < c_4 \frac{1}{K_0^m} + \left(\frac{1}{K_0^2} + \alpha^2\right) c_0 \theta_0^m \tag{33}$$

As $K_0 > 1$ we can choose $\theta_0 < 1$ satisfying

$$\theta_0 > \frac{1}{K_0} \tag{34}$$

Also we can choose $\theta_0 < 1$ satisfying simulteneously

$$\frac{1}{K_0^2} + \alpha^2 < \theta_0 \tag{35}$$

Then if

$$c_0 > \frac{c_4}{\theta_0 - (\frac{1}{K_0^2} + \alpha^2)} \tag{36}$$

we get the left side of 33 less than $c_0 \theta_0^{m+1}$.

4. From 34, 28 and 26 we get

$$| \log D^u F(z_3) - \log D^u F(z_4) | < c_5 \theta_0^m \tag{37}$$

Combining 16, 17, 37 we conclude the proof of Proposition 2.1.

3 Proof of the main theorem

1. The following property, see [14], is useful for the study of the decay of correlations.
Let A be the matrix of admissible transitions for a countable shift. The matrix A satisfies **Big Images and Preimages** property if

BIP There is a finite set of states $i_1, i_2, \dots i_N$ such that for every state j in the alphabet there are k, l such that $a_{i_k j} a_{j i_l} = 1$.

Proposition 6.3 from [8] based on the results of Sarig ([13], [14]) gives sufficient conditions for exponential decay of correlations for Hölder (in particular smooth) functions restricted to the attractor. We state it as the following theorem.

Theorem 3.1 *Suppose there is a Markov partition of the attractor satisfying the following properties.*

(a) The matrix A of admissible transitions is topologically mixing and satisfies BIP property.

(b) $\Phi(x,y) = -log| D^u F |$ *is locally Hölder in the phase space.*

(c) *A function $\phi(x)$ cohomologous to the pullback of $\Phi(x,y)$ into symbolic space satisfies $P(\phi(x)) < \infty$.*

then Theorem 1.6 holds.

2. After Hölder property of the Markov partition is established conditions of the Theorem 3.1 are checked as in [8].
   As A is Bernoulli, property (a) is satisfied.
   Proposition 2.1 implies property (b).
   The same arguments which were used in the proof of Theorem 1.6 in [8], prove that property (c) is satisfied with $P(\phi) = 0$.
   That finishes the proof of Theorem 1.6.

4 One model with strong contraction

1. We fix some $A > 1$ and consider the following map $F = \{f_n\}, n = 0, 1\dots$ of the unit square Q into itself. The domain of f_n is the full height rectangle E_n bounded on the right by the vertical line $x = \frac{1}{A^n}$ and on the left by $x = \frac{1}{A^{n+1}}$, $n = 0, 1, \dots$. Coordinates f_{n1} and f_{n2} of f_n are given by

$$f_{n1}(x,y) = \frac{A^{2n+1}}{A-1} x\big(x - \frac{1}{A^{n+1}}\big) \tag{38}$$

$$f_{n2}(x,y) = \varepsilon_n y\big(x - \frac{1}{A^{n+1}}\big) + \delta_n \tag{39}$$

   If ε_n are small then the images $S_n = f_n(E_n)$ are narrow strips. From definition S_n are bounded on the left and on the right by some subintervals of the left and the right boundaries of Q.
   One can choose δ_n and small ε_n so that S_n are located in Q and do not intersect.

2. From 38, 39 we get the following partial derivatives.

$$f_{n1x} = \frac{A^{2n+1}}{A-1}\big(2x - \frac{1}{A^{n+1}}\big) \tag{40}$$

$$f_{n1y} = 0 \tag{41}$$

$$f_{n2x} = \varepsilon_n y \tag{42}$$

$$f_{n2y} = \varepsilon_n\big(x - \frac{1}{A^{n+1}}\big) \tag{43}$$

$$f_{n1xx} = \frac{2}{A-1}A^{2n+1} \tag{44}$$

$$f_{n2xy} = \varepsilon_n \tag{45}$$

$$f_{n1xy} = f_{n1yy} = f_{n2yy} = f_{n2xx} = 0 \tag{46}$$

For $x \in E_n$ satisfying $\frac{1}{A^{n+1}} < x < \frac{1}{A^n}$ we get

$$f_{n1x} \geq \frac{A^n}{A-1} \tag{47}$$

For any $\alpha < 1$, if we choose ε_n decreasing, and ε_0 sufficiently small, then the above formulas imply Hyperbolicity Conditions with

$$K_0 = \frac{1}{A-1} \tag{48}$$

From 44 and 47 we get

$$\frac{\mid f_{n1xx} \mid}{\mid f_{n1x}^2 \mid} < c \tag{49}$$

but

$$\frac{\mid f_{n1xx} \mid}{\mid f_{n1x} \mid} > A^n \tag{50}$$

is unbounded. We get that Distortion Conditions of [8] are not satisfied.
At the same time, as all ratios $\frac{\mid f_{nijk} \mid}{\mid f_{n1x}^2 \mid}$, except for $i = 1, j = k = x$, are small and decreasing with n, we get that Distortion Condition D1 is satisfied. Respectively F has an SRB measure μ, and Theorem 1.5 holds for the map F.
As f_{n1x} do not depend on y BIV condition is satisfied.
We assume

(a) $A > 1$ is sufficiently close to 1

(b) ε_0 is sufficiently small.

Then D1 and 43 imply that condition SC is satified.

3. To check condition D2 we evaluate partial derivatives of f_n^{-1}.
   Jacobian of f_n equals

$$J_n = \frac{A^{2n+1}}{A-1}\big(2x - \frac{1}{A^{n+1}}\big)\varepsilon_n\big(x - \frac{1}{A^{n+1}}\big) \tag{51}$$

   Partial derivatives depend on coordinates $(u,v) \in S_n$, but we write them in terms of coordinates $(x,y) \in E_n$.

$$f_{n1u}^{-1} = J_n^{-1}\varepsilon_n\big(x - \frac{1}{A^{n+1}}\big) \tag{52}$$

$$f_{n1v}^{-1} = 0 \tag{53}$$

$$f_{n2u}^{-1} = -J_n^{-1}\varepsilon_n y \tag{54}$$

$$f_{n2v}^{-1} = J_n^{-1}\frac{A^{2n+1}}{A-1}\big(2x - \frac{1}{A^{n+1}}\big) \tag{55}$$

   Next we evaluate second partials of f_n^{-1} by using formulas

$$f_{n1uu}^{-1} = f_{n1ux}^{-1}\frac{\partial x}{\partial u} + f_{n1uy}^{-1}\frac{\partial y}{\partial u} \tag{56}$$

   etc. Then we get

$$f_{n1uu}^{-1} = \big(\frac{A-1}{A^{2n+1}}\big)^2\frac{-2}{\big(2x - \frac{1}{A^{n+1}}\big)^3} \tag{57}$$

$$f_{n1uv}^{-1} = f_{n1vv}^{-1} = f_{n2vv}^{-1} = 0 \tag{58}$$

$$f_{n2uu}^{-1} = y\big(\frac{A-1}{A^{2n+1}}\big)^2\frac{4x - \frac{3}{A^{n+1}}}{\big(2x - \frac{1}{A^{n+1}}\big)^3\big(x - \frac{1}{A^{n+1}}\big)^2} + \big(\frac{A-1}{A^{2n+1}}\big)^2\frac{y}{\big(2x - \frac{1}{A^{n+1}}\big)^2\big(x - \frac{1}{A^{n+1}}\big)^2} \tag{59}$$

$$f_{n2uv}^{-1} = -\frac{(A-1)}{\varepsilon_n A^{2n+1}}\frac{1}{\big(x - \frac{1}{A^{n+1}}\big)^2\big(2x - \frac{1}{A^{n+1}}\big)} \tag{60}$$

   To check that D2 is satisfied we divide second derivatives by $(f_{n2v}^{-1})^2$. Using 55 we get that second derivatives are multiplied by

$$\varepsilon_n^2\big(x - \frac{1}{A^{n+1}}\big)^2 \tag{61}$$

Then the above formulas imply that for $\frac{1}{A^{n+1}} < x < \frac{1}{A^n}$ condition D2 is satisfied.
Thus all conditions of Theorem 1.6 are satisfied, and our models have exponential decay of correlations.
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