HODGE DECOMPOSITION AND KÄHLER MANIFOLDS

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ABSTRACT. This is my manuscript on *RIT on Hodge Theory* organized by Professor Patrick Brosnan at UMD in 2023 Fall.

CONTENTS

1. SINGULAR/DE RHAM COHOMOLOGY AND DE RHAM THEOREM

1.1. **Singular Cohomology.** Let X be any topological space. Recall that a **singular k-simplex** is a continuous map $\sigma : \Delta^k \to X$, where Δ^k is the standard k-simplex given by

$$\Delta^k := \{(t_0, \dots, t_k) \in \mathbb{R}^{k+1} : \sum_{i=0}^{k} t_i = 1, \quad t_i \geqslant 0, \quad 0 \leqslant i \leqslant k\},$$

For $0 \leqslant i \leqslant k$, we define a singular $(k-1)$-complex by

$$\partial_i \sigma(t_0, \dots, t_{k-1}) = \sigma(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{k-1}), .$$

The singular k-chains $S_k(X)$ is the free abelian groups generated by the set of all singular k-simplices and we define $\partial : S_k(X) \to S_{k-1}(X)$ by

$$\partial(\sum_j n_j \sigma_j) = \sum_j n_j \partial \sigma_j, \quad \partial \sigma_j = \sum_{i=0}^{k} (-1)^i \partial_i \sigma_j.$$

One can verify direct that $\partial^2 = 0$ and thus defines a complex $(S_\bullet(X), \partial)$. The k-th **singular homology** of X is defined by

$$H_k(X) := \ker(\partial : S_k(X) \to S_{k-1}(X))/\mathrm{im}(\partial : S_{k+1}(X) \to S_k(X)).$$

Fact 1 (Poincaré Lemma)**.** *If U is a convex open set in $\mathbb{R}^n$, then $H_k(U) = 0$ for $k > 0$.*

This is proved by constructing a homotopy operator $h : S_k(X) \to S_{k+1}(X)$ via *cone construction* satisfying $\partial h \sigma + h \partial \sigma = \sigma$, for any $\sigma \in S_k(X)$.

Now, if R is a commutative ring, then we set $S^k(X, R) = \mathrm{Hom}_{\mathbb{Z}}(S_k(X), R)$ and $\delta^k : S^k(X, R) \to S^{k+1}(X, R)$ is the dual homomorphism. Again, we still have $\delta^2 = 0$ and thus the k-th **singular cohomology** with coefficients in R is given by

$$H^k(X, R) := \ker(\delta : S^k(X, R) \to S^{k+1}(X, R))/\mathrm{im}(\delta : S^{k-1}(X, R) \to S^k(X, R)).$$

By Poincaré lemma, we also have $H^k(U, R) = 0$ for any convex open set $U \subset \mathbb{R}^n$ and $k > 0$.

1.2. **Recap on Differential Forms and de Rham Cohomology.** Let M be a smooth manifold of dimension m, i.e.,

(i) M is a second countable, Hausdorff space,
(ii) there exists an open covering $\{U_\alpha\}_{\alpha\in I}$ and homeomorphism $\sigma_\alpha : U_\alpha \to \mathbb{R}^m$ onto some open set V_α such that the transition maps
$$\sigma_{\alpha\beta} := \sigma_\alpha \circ \sigma_\beta^{-1} : \sigma_\beta(U_\alpha \cap U_\beta) \to \sigma_\alpha(U_\alpha \cap U_\beta)$$
are smooth for any $\alpha, \beta \in I$.

Remark 1. Here, we always assume that M is second countable. The important consequence of second countability is that M admits a **smooth partition of unity**: for any open covering $\{U_\alpha\}_{\alpha\in I}$, there exists $\{\rho_\alpha \in C^\infty(M)\}_{\alpha\in I}$ (indexed over the same set I) such that $\operatorname{supp}\rho_\alpha \subset U_\alpha$, $\{\operatorname{supp}\rho_\alpha\}_{\alpha\in I}$ is locally finite[1], and $\sum_{\alpha\in I}\rho_\alpha = 1$.

We denote TM by its tangent bundle and T^*M by its cotangent bundle. For $0 \leqslant k \leqslant m$, $\Lambda^k T^*M$ is the k-th exterior bundle for T^*M. A smooth k-form is a smooth section $\alpha : M \to \Lambda^k T^*M$. Locally, given a local coordinate system $(x_1, \ldots, x_m)$ near p, we can write α locally as
$$\alpha = \sum_{|I|=k} \alpha_I dx_I,$$
where we use the multi-indices notation: $I = (i_1, \ldots, i_k) \in \mathbb{N}^k$ with $0 \leqslant i_1 < \cdots < i_k \leqslant m$ and $\alpha_I := \alpha_{i_1\cdots i_k}$, $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and $|I| := k$.

We denote $A^k(M)$ by the space of smooth k-forms on M. There is a natural differentiation $d : A^k(M) \to A^{k+1}(M)$, called the **exterior derivative**, which is locally defined by
$$d\alpha = \sum_{|I|=k}\sum_{j=1}^m \frac{\partial \alpha_I}{\partial x_j} dx_j \wedge dx_I.$$
By commutativity of mixed derivatives, one can easily see that $d^2 = 0$. This defines a complex
$$(A^\bullet(M), d) : 0 \to C^\infty(M) \xrightarrow{d} A^1(M) \xrightarrow{d} A^2(M) \to \cdots \xrightarrow{d} A^m(M) \to 0,$$
called the **de Rham complex**. The k-th cohomology of the complex is called the **de Rham cohmology**:
$$H^k_{dR}(M, \mathbb{R}) := \ker(d : A^k(M) \to A^{k+1}(M))/\operatorname{im}(d : A^{k-1}(M) \to A^k(M))$$
We usually call $\alpha \in \ker(d)$ a **closed form** and $\alpha \in \operatorname{im}(d)$ an **exact form**. Moreover, recall that
$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$
Hence, $H^\bullet_{dR}(M, \mathbb{R}) = \bigoplus_k H^k_{dR}(M, \mathbb{R})$ is a graded ring. If $f : M \to N$ is a smooth map between smooth manifolds, then it induces a $\mathbb{R}$-linear map $f^* : A^k(N) \to A^k(M)$ given by
$$(f^*\alpha)_p(v_1, \ldots, v_k) = \alpha_{(f(p))}(df_p(v_1), \ldots, df_p(v_k)), \quad \forall p \in M, \quad v_1, \ldots, v_k \in T_pM,$$
where $df_p : T_pM \to T_pN$ is the differential at p and we identify $\Lambda^k T^*_p M$ as the space of alternating k-linear form on T_pM. Moreover, we have
$$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta, \quad f^*d\alpha = df^*\alpha.$$
Particularly, f induces a ring homomorphism $f^* : H^\bullet_{dR}(N, \mathbb{R}) \to H^\bullet_{dR}(M, \mathbb{R})$.

Finally, we recall Poincaré Lemma for de Rham cohomology.

Lemma 1. *Let $U \subset \mathbb{R}^m$ be a convex open set. For $1 \leqslant k \leqslant m$, if $\alpha \in A^k(U)$ is a closed form, then there exists $\beta \in A^{k-1}(U)$ such that $\alpha = d\beta$. In other words, $H^k_{dR}(U, \mathbb{R}) = 0$.*

[1]This means that for any $p \in M$, there exists a neighborhood U of p such that $\operatorname{supp}\rho_\alpha \cap U = \varnothing$ for all but finitely many $\alpha \in I$

Sketch of Proof. The idea is again to construct a homotopy operator on the cochain level. WLOG, we may assume that $0 \in U$. If $\alpha = \sum_{|I|=k} f_I(x_1, \dots, x_m) dx_I$, then we define

$$K : A^{k+1}(U) \to A^k(U), \quad K\alpha = \sum_{|I|=k} \left(\int_0^1 t^k f_I(tx) dt \right) \sum_{j=1}^k (-1)^j x_{i_j} dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_j}} \cdots dx_{i_k}$$

By direct computation and fundamental theorem of Calculus, $\alpha = dK\alpha + Kd\alpha = d(K\alpha)$. □

1.3. **de Rham Theorem.** Now, we can state **de Rham Theorem**.

Theorem 1. *Let M be a smooth manifold. Then we have a natural isomorphism*

$$H^k(M, \mathbb{R}) \cong H^k_{dR}(M, \mathbb{R}).$$

In order to sketch some ideas on de Rham theorem, we need some preparation. Notice that via projection, Δ^k is affine equivalent to

$$\{(x_1, \dots, x_k) \in \mathbb{R}^k : \sum_{j=1}^k x_j \leqslant 1, \quad x_j \geqslant 0, \quad 1 \leqslant j \leqslant k\},$$

which we still denote by Δ^k. We say $\sigma : \Delta^k \to M$ is a **smooth singular k-simplex** if σ extends to some smooth maps on an open neighborhood of $\Delta^k \subset \mathbb{R}^k$. Proceeding as the construction in section 1.1, we can define the $C_k(M)$, the chain complex of smooth k-simplices and boundary operator $\partial : C_k(M) \to C_{k-1}(M)$ with $\partial^2 = 0$. We denote $H_k^\infty(M)$ by the corresponding homology group, called the **smooth singular homology** of M.

Given a k-form $\alpha \in A^k(M)$, $\sigma^*\alpha$ is a k-form defined on an open neighborhood of Δ^k, we can define

$$\int_\sigma \alpha := \int_{\Delta^k} \sigma^*\alpha.$$

We extend this $\mathbb{Z}$-linear to any $\sum_i n_i \sigma_i$ and get a group homomorphism

$$\int \alpha : C_k(M) \to \mathbb{R}, \quad \sigma \mapsto \int_\sigma \alpha.$$

Also, notice that $\int \alpha$ is also linear in α and thus we obtain a group homomorphism

$$\int : A^k(M) \to C^k(M, \mathbb{R}) := \mathrm{Hom}_{\mathbb{Z}}(C_k(M), \mathbb{R}), \quad \alpha \mapsto \left(\sigma \mapsto \int_\sigma \alpha \right).$$

By fundamental theorem of Calculus, it is easy to prove the following Stokes' theorem for chain:

$$\int_\sigma d\alpha = \int_{\partial\sigma} \alpha, \quad \forall \alpha \in A^{k-1}(M), \quad \sigma \in C_k(M).$$

As in singular cohomology, we denote $\delta : C^k(M, \mathbb{R}) \to C^{k+1}(M, \mathbb{R})$ by the adjoint of ∂ and define singular cohomology for smooth cochains by

$$H^k_{sm}(M, \mathbb{R}) := \ker(\delta : C^k(M, \mathbb{R}) \to C^{k+1}(M, \mathbb{R}))/\mathrm{im}(\delta : C^{k-1}(M, \mathbb{R}) \to C^k(M, \mathbb{R}))$$

Then Stokes' theorem reads $\int_\sigma d\alpha = \delta \int \alpha$ and thus this defines a group homomorphism

$$\int : H^k_{dR}(M, \mathbb{R}) \to H^k_{sm}(M, \mathbb{R}) \tag{1.1}$$

We prove that (1.1) is an isomorphism in the following steps.

(1) Both H^k_{dR} and H^k_{sm} satisfies **Mayer–Vietoris property**: for two open sets $U, V \subset M$, then we have a long exact sequence in cohomology:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^k_{dR}(U \cup V) & \longrightarrow & H^k_{dR}(U) \oplus H^k_{dR}(V) & \longrightarrow & H^k_{dR}(U \cap V) & \longrightarrow & H^{k+1}_{dR}(M) & \longrightarrow \cdots \\ & \downarrow \int & & \downarrow \int & & \downarrow \int & & \downarrow \int & \\ \cdots \longrightarrow & H^k_{sm}(U \cup V) & \longrightarrow & H^k_{sm}(U) \oplus H^k_{sm}(V) & \longrightarrow & H^k_{sm}(U \cap V) & \longrightarrow & H^{k+1}_{sm}(M) & \longrightarrow \cdots \end{array}$$

an (1.1) on the corresponding domain forms the above commutative diagram. By 5-lemma, if (1.1) is an isomorphism on U, V, and $U \cap V$, then it is an isomorphism on $U \cup V$.

(2) If (1.1) is an isomorphism on a family of disjoint open sets, then it is an isomorphism on their union. This is obvious from $H^k_*(\bigcup_\alpha U_\alpha) = \bigoplus_\alpha H^k_*(U_\alpha)$ for $* = dR$ or sm.

(3) By Poincaré Lemma for both singular cohomology and de Rham cohomology, we know that $\int$ is an isomorphism for convex open set $U \subset \mathbb{R}^m$. By induction and $(U_1 \cup \cdots U_N) \cap V = (U_1 \cap V) \cup \cdots (U_N \cap V)$, it is an isomorphism for finite union of convex open sets.

(4) Let $f : M \to [0, \infty)$ be a proper map[2], i.e., preimage of compact sets are compact. Let $A_n = f^{-1}([n, n+1])$. We can cover A_n by finite union U_n of open sets which are diffeomorphic to convex sets in $\mathbb{R}^m$ which are contained in $f^{-1}([n - 1/2, n + 3/2])$. We then set $U = \bigcup_k U_{2k}$ and $V = \bigcup_k U_{2k+1}$ which are disjoint unions of convex open sets. By (2) and (4), (1.1) is an isomorphism on U, V, and $U \cap V$. Hence, (1.1) is an isomorphism on $M = U \cup V$.

(5) The final step is to show that the inclusion $C_k(M) \hookrightarrow S_k(M)$ is a chain homotopic equivalence. A well-known facts known as **Whitney approximation theorem** from differential topology asserts that any continuous map between smooth manifolds can be approximated by smooth one. Using this, for each singular k-simplex $\sigma : \Delta^k \to M$, one can construct a (continuous) homotopy $H : \Delta \times [0, 1] \to M$ so that $H(\cdot, 0) = \sigma$ and $H(\cdot, 1)$ is a smooth singular k-simplex. Once we construct such operator, one can easily deduce that $C_k(M) \hookrightarrow S_k(M)$ induces an isomorphism $H^\infty_k(M) \cong H_k(M)$ and thus $H^k_{sm}(M, \mathbb{R}) \cong H^k(M, \mathbb{R})$. The detail is quite tedious and can be found in John Lee's *Introduction to Smooth Manifolds*, Theorem 18.7.

Remark 2. The procedure above is known as **Mayer-Vietoris argument**. This was later generalized by Weil to Čech complex with respect an open covering. We give a short outline on the *modern proof* of **de Rham–Weil isomorphism**. Let $C^k_{M,\mathbb{R}}$ be the sheaf of singular k-cochains defined by $U \mapsto C^k(U, \mathbb{R})$, $\mathcal{A}^k_{M,\mathbb{R}}$ be the sheaf of smooth k-forms defined by $U \mapsto A^k(U)$, and $\mathbb{R}_M$ be the constant sheaf with stalk $\mathbb{R}$ on M. The exterior derivative and coboundary operator extends to a morphism of sheaves $d : \mathcal{A}^k_{M,\mathbb{R}} \to \mathcal{A}^{k+1}_{M,\mathbb{R}}$ and $\delta : C^k_{M,\mathbb{R}} \to C^{k+1}_{M,\mathbb{R}}$. Moreover, Poincaré Lemma for both cohomology theories and partition of unity shows that

$$0 \to \mathbb{R}_M \to \mathcal{A}^0_{M,\mathbb{R}} \xrightarrow{d} \mathcal{A}^1_{M,\mathbb{R}} \xrightarrow{d} \cdots,$$
$$0 \to \mathbb{R}_M \to C^0_{M,\mathbb{R}} \xrightarrow{\delta} C^1_{M,\mathbb{R}} \xrightarrow{\delta} \cdots.$$

are both acyclic resolution of $\mathbb{R}_M$. Hence,we have the isomorphism:

$$H^k(M) = H^k(\Gamma(C^\bullet_{M,\mathbb{R}}), \delta) \cong H^k(M, \mathbb{R}_M) \cong H^k(\Gamma(\mathcal{A}^\bullet_{M,\mathbb{R}}), d) = H^k_{dR}(M, \mathbb{R})$$

Remark 3. In fact, (1.1) is a ring homomorphism (for singular cohomology, ring structure is given by cup product) and is functorial.

[2]One can construct such function by partition of unity. It is well-known that for any topological manifold M, one can find a countable open covering $\{U_j\}_{j=1}^\infty$ with $\overline{U_i}$ compact. Let $\{\rho_j\}_{j=1}^\infty$ be the partition of unity subordinated to $\{U_j\}_{j=1}^\infty$. We then set $f = \sum_{j=1}^\infty j\rho_j$.

2. Hodge Theorem on Harmonic Forms

2.1. **Preliminaries.** Let us first recap some simple linear algebra. Let V be a $\mathbb{R}$-vector space of dimension m. Given an inner product $\langle\cdot,\cdot\rangle$ on V, this induces an inner product on $\Lambda^k V$ for $0 \leqslant k \leqslant m$ by first defining on monomials

$$\langle u_I, v_J\rangle = \det(\langle u_{i_k}, v_{j_l}\rangle), \quad u_I = u_{i_1} \wedge \cdots \wedge u_{i_k}, v_J = v_{j_1} \wedge \cdots \wedge v_{j_k} \in \Lambda^k V,$$

and then extending bilinearly to whole $\Lambda^k V$. Particularly, if $\{e_1, \ldots, e_m\}$ is an ONB for V, then $\{e_I : I = (i_1, \ldots, i_k), \quad 1 \leqslant i_1 < \cdots < i_k \leqslant m\}$ is an ONB for $\Lambda^k V$.Particularly, for $k = m$, we call the top form $dV := e_1 \wedge \cdots \wedge e_m$ a **Riemannian volume form** of V (with respect to the inner product).

Now, we define **Hodge $*$-operator** by

$$* : \Lambda^k V \to \Lambda^{m-k} V, \quad e_I \mapsto e_{I^c},$$

where $I = (i_1, \ldots, i_k)$ and I^c is the complement of I in $\{1, \ldots, m\}$ with the ordering so that

$$e_I \wedge *e_I = e_1 \wedge \cdots \wedge e_m.$$

Again, we extend $\mathbb{R}$-linearly to general k-vector $\alpha = \sum_{|I|=k} \alpha_I e_I, \beta = \sum_{|J|=k} \beta_J e_J$:

$$\alpha \wedge *\beta = \sum_{|I|=|J|=k} \alpha_I \beta_J e_I \wedge *e_J = \langle\alpha, \beta\rangle dV$$

Note that $*$ is independent of the choice of ONB *with the same orientation* and

$$*^2 = (-1)^{k(m-k)} = (-1)^{k(m-1)} \tag{2.1}$$

Previously, we define integration of differential forms with respect to a smooth singular simplex. Now, we review integration of differential forms on smooth manifolds. Let M be an oriented smooth manifold, i.e., M admits a smooth atlas whose transition functions has positive Jacobians, for $u \in A^m(M)$, we can define integration of u on M by

(1) First, if u vanishes outside a coordinate chart U and $u = f(x_1, \ldots, x_m) dx_1 \wedge \cdots \wedge dx_m$, then

$$\int_M u := \int_U f(x_1, \ldots, x_m) dx_1 \cdots dx_m.$$

(2) If we take $\{\rho_\alpha\}_{\alpha \in I}$ be the partition of unity subordinated to the oriented atlas, then we define

$$\int_M u := \sum_{\alpha \in I} \int_{U_\alpha} \rho_\alpha u.$$

The key upshot is that under oriented hypothesis, the change of variable formula shows that the integration is independent of the choice of coordinates. The key formula for us is **Stokes' formula**:

$$\int_M du = \int_{\partial M} u, \quad u \in A^{m-1}(M).$$

Now, for an oriented m-dimensional Riemannian manifold (M, g). By definition, for each $p \in M$, g_p is an inner product on each tangent space $T_p M$. By duality, g also induces an inner product on $V = T_p^* M$ and thus on $\Lambda^k T_p^* M$ for $0 \leqslant k \leqslant m$, which we denote by $\langle\cdot,\cdot\rangle$. Since M is oriented, we denote Riemannian volume form on M (with respect to g) by $dV_g \in A^m(M)$. We then define **Hodge $*$-operator** $* : A^k(M) \to A^{m-k}(M)$ by applying above construction to each $T_p^* M$. By construction,

$$\alpha \wedge *\beta = \langle\alpha, \beta\rangle dV_g, \quad \alpha, \beta \in A^k(M) \tag{2.2}$$

For simplicity, we now assume that M is closed[3] and endow $A^k(M)$ an L^2-inner product by

$$(\alpha,\beta)=\int_M\langle\alpha,\beta\rangle dV_g,\quad \forall\alpha,\beta\in A^k(M) \tag{2.3}$$

and we denote $\|\alpha\|=(\alpha,\alpha)^{1/2}$. We define **adjoint** $d^*: A^{k+1}(M)\to A^k(M)$ of d by

$$(d\alpha,\beta)=(\alpha,d^*\beta),\quad \alpha\in A^k(M),\beta\in A^{k+1}(M). \tag{2.4}$$

By (2.2), (2.3), and Stokes' formula, we get

$$(d\alpha,\beta)=\int_M\langle d\alpha,\beta\rangle dV_g=\int_M d\alpha\wedge *\beta=\int_M d(\alpha\wedge *\beta)-(-1)^k\alpha\wedge d(*\beta)=(-1)^{k+1}\int_M\alpha\wedge d(*\beta).$$

Then (2.1) implies

$$(d\alpha,\beta)=(-1)^{k+1}(-1)^{k(m-1)}\int_M\alpha\wedge **(d*\beta)=(-1)^{km+1}(\alpha,*d*\beta).$$

This shows that the adjoint d^* can be expressed into d and Hodge $*$-operator: $d^*=(-1)^{km+1}*d*$.

Definition 1. The **Hodge Laplacian** $\triangle: A^k(M)\to A^k(M)$ on k-forms is defined by by $\triangle = dd^*+d^*d$. A smooth k-form $\alpha\in A^k(M)$ is **harmonic** if $\triangle\alpha=0$ and we denote $\mathcal{H}^k(M)$ by the space of harmonic k-forms on M.

One can easily verify that for $M=\mathbb{R}^m$ with Euclidean metric $g=\sum_{j=1}^m dx_j^2$,

$$\triangle\alpha=-\sum_{|I|=k}\left(\sum_{j=1}^m\frac{\partial^2\alpha_I}{\partial x_j^2}\right)dx_I,\quad \forall\alpha=\sum_{|I|=k}\alpha_I dx_I\in A^k(M)$$

This justifies the name Laplcian for Hodge Laplacian $\triangle$. Also, notice that $\triangle$ is **self-adjoint**, i.e.,

$$(\triangle\alpha,\beta)=(\alpha,\triangle\beta),\quad \forall\alpha,\beta\in A^k(M).$$

2.2. **Hodge Theorem: Statement and Some Ideas of Proof.** Let M be an oriented, closed manifold. Given a cohomology class $[\alpha]\in H^k_{dR}(M,\mathbb{R})$, we wish to find a **canonical representative** within the class. If we endow M a Riemannian metric g, then we endow $A^k(M)$ a pre-Hilbert space structure by the L^2-norm (energy) by (2.3). One possibility is to require α to have *minimal energy* among the cohomology class $[\alpha]$. For any $\beta\in A^{k-1}(M)$ with $d\beta\neq 0$ and $t\in\mathbb{R}$, we find that

$$\begin{aligned}\|\alpha+td\beta\|^2&=\|\alpha\|^2+2t(\alpha,d\beta)+t^2\|d\beta\|^2\\ &=\|d\beta\|^2(t+(\alpha,d\beta)/\|d\beta\|^2)^2+\|\alpha\|^2-|(\alpha,d\beta)|^2/\|d\beta\|^2\leqslant\|\alpha\|^2\end{aligned}$$

iff $(\alpha,d\beta)=(d^*\alpha,\beta)=0$ for any $\beta\in A^{k-1}(M)$. Hence, $d^*\alpha=0$. On the other hand,

$$(\triangle\gamma,\gamma)=(dd^*\gamma,\gamma)+(d^*d\gamma,\gamma)=\|d\gamma\|^2+\|d^*\gamma\|^2,\quad \forall\gamma\in A^k(M).$$

Based on discussion above, we conclude that $\mathcal{H}^k(M)=\ker(d)\cap\ker(d^*)$ and

Proposition 1. *Let (M,g) be an oriented, closed Riemannian manifold, $[\alpha_0]\in H^k_{dR}(M,\mathbb{R})$ be a cohomology class. Then $\alpha\in[\alpha_0]$ has minimal energy if and only if $\alpha\in\mathcal{H}^k(M)$.*

Remark 4. Above proposition is the analogous to so called **Dirichlet principle** for harmonic functions. Let $\Omega\subset\mathbb{R}^m$ be a bounded open set. We define **Dirichlet energy** by

$$E[f]=\frac{1}{2}\int_\Omega|\nabla f|^2dx.$$

The Euler–Lagrange equation for E is exact the usual Laplace equation $\triangle f=0$.

[3] i.e., M is compact and $\partial M=\varnothing$.

Thus, the question in consideration becomes whether one can find a **harmonic representative** among each cohomology class? **Hodge theorem** asserts that the answer is affirmative:

Theorem 2. *Let (M, g) be an oriented, closed Riemannian manifold. For each cohomology class $[\alpha_0] \in H^k_{dR}(M, \mathbb{R})$, there exists a unique harmonic representative $\alpha \in \mathcal{H}^k(M)$ with $\alpha \in [\alpha_0]$. In other words,*

$$H^k_{dR}(M, \mathbb{R}) \cong \mathcal{H}^k(M), \quad [\alpha_0] \mapsto \alpha.$$

An important consequence of Theorem 2 is that it gives a quick proof for **Poincaré duality**.

Corollary 1 (Poincaré Duality)**.** *Let M be an oriented compact manifold. The bilinear pairing $A^k(M) \times A^{n-k}(M) \to \mathbb{R}$ give by $(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$ descends to a non-degenerate pairing on de Rham cohmology $H^k_{dR}(M, \mathbb{R}) \times H^{m-k}_{dR}(M, \mathbb{R}) \to \mathbb{R}$.*

Proof. By Stokes' theorem, it is clear that the pairing descends to the level of de Rham cohomology. We choose a Riemannian metric g on X and identify $H^k_{dR}(M, \mathbb{R}) \cong \mathcal{H}^k(M)$ and $H^{m-k}_{dR}(M) \cong \mathcal{H}^{m-k}(M)$ by Theorem 2. Notice that Hodge $*$-operator commutes with $\triangle$, since $\beta \in A^{m-k}(M)$, $**\beta = (-1)^{k(m-k)}\beta$ and thus

$$\begin{aligned}\triangle * \beta &= (-1)^{m(k-1)+1} d * d * *\beta + (-1)^{km+1} * d * d * \beta \\ &= (-1)^{m(m-k-1)+1} * d * d * \beta + (-1)^{m(m-k)+1} * * d * d\beta = *\triangle\beta.\end{aligned}$$

It follows that $* : \mathcal{H}^k(M) \to \mathcal{H}^{m-k}(M)$ is an isomorphism. Moreover, if $\alpha \in \mathcal{H}^p(M)$ and $\alpha \neq 0$, then

$$\int_M \alpha \wedge *\alpha = \|\alpha\|^2 > 0$$

Thus, the pairing is non-degenerate. induces an isomorphism $* : \mathcal{H}^k(M) \xrightarrow{\sim} \mathcal{H}^{m-k}(M)$. Combing with Hodge isomorphism, $* : H^k_{dR}(M, \mathbb{R}) \xrightarrow{\sim} H^{m-k}_{dR}(M, \mathbb{R})$. □

In fact, Theorem 2 is deduced from the following stronger statement.

Theorem 3 (Hodge Decomposition)**.** *Let (M, g) be an oriented, closed Riemannian manifold. We have an orthogonal decomposition with respect to* (2.3)*:*

$$A^k(M) = \mathcal{H}^k(M) \oplus \triangle(A^k(M)). \tag{2.5}$$

Proof of Theorem 2. Given any closed form $\alpha_0 \in A^k(M)$, we decompose α_0 uniquely by (2.5):

$$\alpha_0 = \alpha + \triangle\beta = \alpha + (dd^*\beta + d^*d\beta),$$

for some $\beta \in A^k(M)$. By assumption, $d\alpha_0 = 0 = dd^*d\beta$ and hence $d^*d\beta = 0$ since

$$0 = (dd^*d\beta, d^*d\beta) = \|d^*d\beta\|^2.$$

Therefore, $\alpha_0 = \alpha + d(d^*\beta)$ and α is the unique harmonic representative within $[\alpha_0]$. □

Notice that (2.5) is equivalent to solvability for inhomogeneous equation for Hodge Laplacian.

Theorem 4. *Given $\beta \in A^k(M)$, $\triangle\alpha = \beta$ is solvable iff $\beta \in \mathcal{H}^k(M)^\perp$.*

One direction is clear. If $\triangle\alpha = \beta$ for some $\alpha \in A^k(M)$, then for any $\gamma \in \mathcal{H}^k(M)$,

$$(\beta, \gamma) = (\triangle\alpha, \gamma) = (\alpha, \triangle\gamma) = 0.$$

The other inclusion requires some (nowadays standard) PDE techniques. Let us sketch the ideas of proof. The first step is some functional analytic formalism. Recall that $A^k(M)$ is a pre-Hilbert space with respect to L^2-inner product (2.3). Given $\beta \in A^k(M)$, if $\triangle\alpha = \beta$ is solvable, then

$$(\alpha, \triangle\gamma) = (\triangle\alpha, \gamma) = (\beta, \gamma), \quad \forall\gamma \in A^k(M),$$

which is a linear form on $\mathrm{im}(\triangle)$. The essence here is to construct α first in the *dual formulation*. Given $\beta \in \mathcal{H}^k(M)^\perp$, we define a linear form ℓ on the subspace $\mathrm{im}(\triangle) \subset A^k(M)$ by

$$\ell(\triangle\gamma) := (\beta, \gamma), \quad \forall \gamma \in A^k(M). \tag{2.6}$$

Notice that this is well-defined since $\beta \in \mathcal{H}^k(M)$: if $\gamma' \in A^k(M)$ with $\triangle\gamma' = \triangle\gamma$, then $\gamma' - \gamma \in \mathcal{H}^k(M)$ and thus $(\beta, \gamma) = (\beta, \gamma')$. The first difficulty is to prove the following estimate:

Proposition 2 (Closed Range)**.** *There exists $C > 0$ such that $\|\beta\| \leqslant C\|\triangle\beta\|$ for any $\beta \in \mathcal{H}^k(M)^\perp$.*

With Proposition 2, we prove that ℓ is a bounded linear form on $\mathrm{im}(\triangle)$. Indeed, since $\mathcal{H}^k(M) = \ker(\triangle)$ is a closed subspace, we denote $P : A^k(M) \to \mathcal{H}^k(M)$ by the projection. We set $\theta := \gamma - P(\gamma) \in \mathcal{H}^k(M)^\perp$. Then $\triangle\theta = \triangle\gamma$ and

$$|\ell(\triangle\gamma)| = |\ell(\triangle\theta)\| = |(\beta, \theta)| \leqslant \|\beta\|\|\theta\| \leqslant C\|\beta\|\|\triangle\theta\| = C\|\beta\|\|\triangle\gamma\|.$$

Hence, by Hahn–Banach theorem, ℓ can be extended to a bounded linear form ℓ on $A^k(M)$ with the same norm. In the terminology of PDE, we call a bounded linear operator ℓ on $A^k(M)$ satisfying (2.6) a **weak solution** of $\triangle\alpha = \beta$. The final analytic input is the following proposition.

Proposition 3 (Elliptic Regularity)**.** *For any weak solution ℓ of $\triangle\alpha = \beta$ is actually smooth, i.e., there exists $\alpha \in A^k(M)$ such that $\ell(\gamma) = (\alpha, \gamma)$ for any $\gamma \in A^k(M)$.*

We find $\mathcal{H}^k(M)^\perp = \mathrm{im}(\triangle)$, assuming Proposition 2 and 3. We end with a few comments on them.

(1) Proposition 2 is also known as the **closed range** for if $(\beta_j)_{j=1}^\infty$ is a sequence such that $\triangle\beta_j \to \gamma$ in L^2-norm, then Proposition 2 implies that
$$\|\beta_j - \beta_k\| \leqslant C\|\triangle(\beta_j - \beta_k)\| \to 0, \quad j, k \to \infty.$$
Hence, $\{\beta_j\}_{j=1}^\infty$ is a Cauchy sequence in the pre-Hilbert space $A^k(M)$. A technical point here is that β_j converges **a priori** to the limit β_∞ in the completion of $A^k(M)$. By Riesz representation theorem, β_∞ is the weak solution of $\triangle\beta = \gamma$. By Proposition 3, $\beta_\infty \in A^k(M)$ and hence $\triangle\beta_\infty = \gamma$. In other words, $\mathrm{im}(\triangle)$ is closed.

(2) Both Proposition 2 and 3 depend heavily on the differential operator $\triangle$. The type of operators enjoy these facts are called **elliptic operators**. A prototype of ellipitc operator is of course the standard Laplacian on Euclidean space. Proposition 3 is the generalization of the classical facts that harmonic functions are actually smooth.

(3) The proof for closed range also shows that $\dim_{\mathbb{R}} \mathcal{H}^k(X) < \infty$.

(4) The actual proof of both Propositions requires some knowledge on Sobolev spaces, which generalizes the notion of derivatives to non-differentiable functions. For definitions and related results of Sobolev spaces, and details of actual proof to both Propositions, one can consult Griffths–Harris, Wells, and Warner.

3. Complex Manifolds and Kähler Metrics

3.1. An Interlude on Complex Linear Algebra.

We begin with a digression on linear algebra.

(1) Let W be a complex vector space with $\dim_{\mathbb{C}} W = n$. We choose a $\mathbb{C}$-basis $\{e_1, \ldots, e_n\}$ so that $W \cong \mathbb{C}^n$. Let $(z_1, \ldots, z_n) \in \mathbb{C}^n$ be the corresponding coordinates. By splitting z_j into real and imaginary parts
$$z_j = x_j + iy_j, \quad 1 \leqslant j \leqslant n,$$
we see that W has a (non–canonical) real vector space structure $W_{\mathbb{R}}$ of real dimension $2n$ with $\mathbb{R}$-basis $\{e_1, , \ldots, e_n, ie_1, \ldots, ie_n\}$. Multiplication by imaginary unit $w \mapsto iw$ can be identified as an $\mathbb{R}$-linear endomorphism $J : W_{\mathbb{R}} \to W_{\mathbb{R}}$ whose matrix representation with respect to above $\mathbb{R}$-basis is given by $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ and satisfies $J^2 = -Id_{W_{\mathbb{R}}}$.

(2) Conversely, given a real vector space V, a **linear complex structure** $J : V \to V$ is a $\mathbb{R}$-linear endomorphism with $J^2 = -Id_V$. We can then endow V a $\mathbb{C}$-vector space structure by

$$(a+ib)v := a + bJv, \quad \forall v \in V, \quad \forall a,b \in V.$$

Notice that if V admits a linear complex structure, then $\dim_{\mathbb{R}} V$ must be even[4]. Hence, a complex vector space is equivalent to a real vector space with a linear complex structure. Moreover, a $\mathbb{R}$-linear map $T : (V,J) \to (V',J')$ is $\mathbb{C}$-linear iff $J' \circ T = T \circ J$. Hence, the category of $\mathbb{C}$-vector space is equivalent to the category of $\mathbb{R}$-vector space equipped with a linear complex structure.

(3) Another way to obtain a complex vector space from a real one is extension by scalar. Let V be a real vector space with $\dim_{\mathbb{R}} V = m$. The **complexification** of V is defined by $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. which is a $\mathbb{C}$-vector space. If $\{e_1, \ldots, e_m\}$ is a $\mathbb{R}$-basis for V, then $\{e_1 \otimes 1, \ldots, e_m \otimes 1\}$ is a $\mathbb{C}$-basis for $V_{\mathbb{C}}$. This shows that $\dim_{\mathbb{C}} V_{\mathbb{C}} = m$. On the complexification $V_{\mathbb{C}}$ of V, we can define **(canonical) complex conjugation**, which is an anti $\mathbb{C}$-linear map given by

$$\bar{\cdot} : V_{\mathbb{C}} \to V_{\mathbb{C}}, \quad v \otimes z \mapsto v \otimes \bar{z}.$$

and extends additiviely. Notice that V embeds into a $\mathbb{R}$-linear subspace of $V_{\mathbb{C}}$ by $v \mapsto v \otimes 1$ which can be characterized as the fixed subspace $\{v' \in V_{\mathbb{C}} : \overline{v'} = v'\}$.

Remark 5. Let W be a $\mathbb{C}$- vector space. There are two ways to define complex conjugation.

(a) We can define a $\mathbb{C}$-vector space $\overline{W}$ which is the same underlying abelian group as W and conjugate complex multiplication $z \cdot w := \bar{z}w$ for $w \in W$ and $z \in \mathbb{C}$. Then $Id_W : W \to \overline{W}$ is an anti $\mathbb{C}$-linear isomorphism.

(b) By choosing a $\mathbb{C}$-basis $\{e_1, \ldots, e_n\}$ of W, we identify $W \cong \mathbb{C}^n$. However, $\mathbb{C}^n = \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{R}^n \oplus i\mathbb{R}^n$. Hence, we define $\overline{w} = v - iu$ if $w = v + iu$ according to the direct sum decomposition. Notice that this construction depends on the choice of basis and thus is not canonical.

(4) Conversely, given a complex vector space W with $\dim_{\mathbb{C}} W = m$ with an anti $\mathbb{C}$-linear involution $c : W \to W$, i.e., $c^2 = Id_W$, the fixed subspace $V := W^c := \{w \in W : c(w) = w\}$ is a $\mathbb{R}$-vector subspace $W_{\mathbb{R}}$, called the **real form** of W. One can easily show that $W = V \otimes_{\mathbb{R}} \mathbb{C}$. In other words, a complex vector space W is the complexification of some real vector space V iff we endow W an anti $\mathbb{C}$-linear involution $c : W \to W$.

Remark 6. Notice that the real form of a complex vector space is not unique. For instance, in representation theory, $\mathfrak{sl}(n,\mathbb{C})$ is both the complexification of $\mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{u}(n)$.

(5) Let (V,J) be a real vector space with linear complex structure of real dimension $2n$, which is equivalent to a complex vector space of complex dimension n by (2). If we complexify V into $V_{\mathbb{C}}$ and extend J to $V_{\mathbb{C}}$ by $J(v \otimes z) = J(v) \otimes z$, then we have **eigenspace decomposition**:

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}, \quad V^{1,0} := \{v' \in V_{\mathbb{C}} : Jv' = iv'\}, \quad V^{0,1} := \{v'' \in V_{\mathbb{C}} : Jv'' = -iv''\}.$$

Notice that $V^{1,0}, V^{0,1}$ are $\mathbb{C}$-linear space of (complex) dimension n and complex conjugation induces a $\mathbb{R}$-linear isomorphism $V^{1,0} \cong V^{0,1}$. Moreover, $(V,J) \cong (V^{1,0}, i)$ and $(\overline{V}, \overline{J}) \cong (V^{0,1}, i)$ as $\mathbb{C}$-vector space. Here, $(\overline{V}, \overline{J})$ means the conjugate $\mathbb{C}$-vector space[5] of (V,J).

(6) Let (V,J) be a real vector space with linear complex structure. We denote $V^* := \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$ be the $\mathbb{R}$-dual space of V. Then J induces a linear complex structure on V^* by

$$\langle v, Ju \rangle := \langle Jv, u \rangle, \quad v \in V, u \in V^*,$$

[4]If $\dim_{\mathbb{R}} V$ is odd, then there exists a real eigenvalue λ of J. However, $J^2 = -Id_V$ implies $\lambda^2 = -1$, a contradiction.

[5]Recall that for a $\mathbb{C}$-vector space W, $\overline{W}$ is also a $\mathbb{C}$-vector space with the same underlying abelian group structure as W and conjugate complex multiplication $z \cdot w := \bar{z}w$ for $w \in W$ and $z \in \mathbb{C}$.

where $\langle\cdot,\cdot\rangle : V \times V^* \to \mathbb{R}$ is the natural pairing. By functoriality of complexification,

$$(V^*)_{\mathbb{C}} := (V^*) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) =: (V_{\mathbb{C}})^*,$$

and the induced eigenspace decomposition on $(V^*)_{\mathbb{C}}$ is given by

$$(V^*)^{1,0} \cong \{u \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) : Ju = iu\} \cong (V^{1,0})^*,$$
$$(V^*)^{0,1} \cong \{u \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) : Ju = -iu\} \cong (V^{0,1})^*.$$

Finally, notice that $(V^*)^{1,0} \cong \mathrm{Hom}_{\mathbb{C}}((V, J), \mathbb{C})$.

(7) Again let (V, J) be a complex vector space. The decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ induces a decomposition on the exterior algebra (over $\mathbb{C}$):

$$\Lambda^k V_{\mathbb{C}} \cong \bigoplus_{k=p+q} \Lambda^{p,q} V, \quad \Lambda^{p,q} V := \Lambda^p V^{1,0} \otimes_{\mathbb{C}} \Lambda^q V^{0,1}.$$

For each $1 \leqslant p, q \leqslant n = \dim_{\mathbb{C}}(V, J)$, we identify $\Lambda^{p,q} V$ as a subspace of $\Lambda^k V_{\mathbb{C}}$ by

$$v_I \otimes u_K \mapsto v_I \wedge u_K.$$

and from $\overline{V^{1,0}} = V^{0,1}$, we see that $\overline{\Lambda^{p,q} V} \cong \Lambda^{q,p} V$.

(8) Let (V, g) be an Euclidean vector space. A linear complex structure J on V is **compatible** with g with $J \in \mathrm{O}(V, g)$, i.e., $g(v, w) = g(Jv, Jw)$ for any $v, w \in V$. In this case, we set

$$\omega(v, w) := g(Jv, w), \quad v, w \in V.$$

Notice that $\omega(w, v) = g(Jw, v) = g(J^2 w, Jv) = -g(w, Jv) = -\omega(v, w)$. Hence, $\omega \in \Lambda^2 V$ and

$$\omega(Jv, Jw) = g(J^2 v, Jw) = -g(v, Jw) = -\omega(w, v) = \omega(v, w), \quad \forall v, w \in V.$$

If we extend ω $\mathbb{C}$-linear to $\bigwedge^2 V_{\mathbb{C}}$, then for $v, w \in V^{1,0}$ or $V^{0,1}$,

$$\omega(v, w) = \omega(Jv, Jw) = \omega(\pm iv, \pm iw) = -\omega(v, w) \Longrightarrow \omega(v, w) = 0.$$

As a result, $\omega \in \Lambda^{1,1} V^* \cap \Lambda^2 V$, called the **hermitian form** of (V, g, J).

(9) Let (V, g, J) be an Euclidean space with compatible linear complex structure. We set

$$h(v, w) := g(v, w) - i\omega(v, w), \quad v, w \in V.$$

Then h is clearly $\mathbb{R}$-bilinear and $h(v, v) = g(v, v) > 0$ for $v \in V \backslash \{0\}$. Moreover,

$$h(w, v) = g(v, w) + i\omega(v, w) = \overline{g(v, w) - i\omega(v, w)} = \overline{h(v, w)}$$

and $h(Jv, w) = g(Jv, w) - i\omega(Jv, w) = \omega(v, w) + ig(v, w) = ih(v, w)$. Thus, h is a positive definite hermitian product on (V, J).

(10) Alternatively, one can extend g into a hermitian metric on $V_{\mathbb{C}}$ by

$$g_{\mathbb{C}}(v \otimes \mu, w \otimes \lambda) = \mu\bar{\lambda} g(v, w), \quad v, w \in V, \quad \mu, \lambda \in \mathbb{C}.$$

One can easily see that $g_{\mathbb{C}}(V^{1,0}, V^{0,1}) = 0$ and thus $V = V^{1,0} \oplus V^{0,1}$ is an orthogonal decomposition. However, under the isomorphism $(V, J) \cong (V^{1,0}, i)$, $h = 2g_{\mathbb{C}}\big|_{V^{1,0}}$.

(11) We now summarize above discussion in coordinates. Let $(x_1, \ldots, x_n)$ be a $\mathbb{C}$-basis for (V, J). Then $(x_1, y_1 := Jx_1, \ldots, x_n, y_n = Jx_n)$ is a $\mathbb{R}$-basis for V. Then

$$z_j := \frac{1}{2}\left(x_j - iy_j\right), \quad \bar{z}_j := \frac{1}{2}\left(x_j + iy_j\right), \quad 1 \leqslant j \leqslant n,$$

form bases for $V^{1,0}$ and $V^{0,1}$ respectively. Dually, if $(x^1, \ldots, x^n)$ is a $\mathbb{C}$-basis for (V^*, J), then $y^j = Jx^j$ is dual basis for y_j and

$$z^j := x^j + iy^j, \quad \bar{z}^j := x^j - iy^j, \quad 1 \leqslant j \leqslant n,$$

are dual bases for z_j and $\bar{z}_j$ respectively. Suppose that $h(x_i, x_j) = h_{ij}$. Then $g_{\mathbb{C}}(z_j, z_k) = \frac{1}{2}h_{jk}$ and

$$h(x_j, y_k) = h(x_j, Jx_k) = -ih_{jk}, \quad h(y_j, y_k) = h(Jx_j, Jx_k) = h(x_j, x_k) = h_{jk}.$$

Since $g = \operatorname{Re} h$ and $\omega = -\operatorname{Im} h$, we see that

$$\omega(x_j, x_k) = \omega(y_j, y_k) = -\operatorname{Im} h_{jk}, \quad \omega(x_j, y_k) = \operatorname{Re} h_{jk}$$
$$g(x_j, x_k) = g(y_j, y_k) = \operatorname{Re}(h_{jk}), \quad g(x_j, x_k) = \operatorname{Im}(h_{jk}).$$

Hence, we write

$$\omega = -\sum_{j<k} \operatorname{Im}(h_{jk})(x^j \wedge x^k + y^j \wedge y^k) + \sum_{j,k=1}^{n} \operatorname{Re}(h_{jk}) x^j \wedge y^k.$$

From $z^j \wedge \bar{z}^k = x^j \wedge x^k - i(x^j \wedge y^k + x^k \wedge y^j) + y^j \wedge y^k$, we see that

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{jk} z^j \wedge \bar{z}^k \in \Lambda^{1,1} V^* \cap \Lambda^2 V. \tag{3.1}$$

If we choose an ONB $(x_1, y_1 \ldots, x_n, y_n)$ for g, then $\omega = \frac{i}{2}\sum_{j=1}^{n} z^j \wedge \bar{z}^j = \sum_{j=1}^{n} x^j \wedge y^j$. We find that hermitian form determines the Riemannian volume form on (V, g, J):

$$\frac{\omega^n}{n!} = \left(\frac{i}{2}\right)^n (z^1 \wedge \bar{z}^1) \wedge \cdots \wedge (z^n \wedge \bar{z}^n)$$
$$= x^1 \wedge y^1 \wedge x^2 \wedge y^2 \wedge \cdots x^n \wedge y^n =: dV_g \in \Lambda^{n,n} V^* \cap \Lambda^{2n} V^*.$$

(12) As in the real case discussed in section 2, a hermitian product $g_{\mathbb{C}}$ on $V_{\mathbb{C}}$ induces hermitian products $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ on $\Lambda^k V^*_{\mathbb{C}}$ for all $0 \leqslant k \leqslant 2n$. We can then extend Hodge $*$-operator on (V, g) $\mathbb{C}$-linearly to $* : \bigwedge^k V_{\mathbb{C}} \to \bigwedge^{2n-k} V_{\mathbb{C}}$ which is characterized by

$$\alpha \wedge *\bar{\beta} = \langle \alpha, \beta \rangle_{\mathbb{C}} dV_g, \quad \forall \alpha, \beta \in \Lambda^k V_{\mathbb{C}}. \tag{3.2}$$

Since $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ is orthogonal with respect to $g_{\mathbb{C}}$, $\Lambda^k V^*_{\mathbb{C}} = \bigoplus_{p+q=k} \Lambda^{p,q} V^*$ is also an orthogonal decomposition. Moreover, notice that if $\gamma_j \in \Lambda^{p_j,q_j} V^*$ for $j = 1, 2$ with $p_1 + p_2 + q_1 + q_2 = 2n$ but $(p_1 + p_2, q_1 + q_2) \neq (n, n)$, then $\gamma_1 \wedge \gamma_2 = 0$. Hence, by (3.2),

$$* : \Lambda^{p,q} V^* \to \Lambda^{n-q,n-p} V^*.$$

3.2. **Complex Manifold and Kähler Metrics.** First, we recall the definition of holomorphic functions in several variables. Let $\Omega \subset \mathbb{C}^n$ be an open set, $(z_1, \ldots, z_n)$ be standard complex coordinates on $\mathbb{C}^n$. We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ via $z_j = x_j + iy_j$. Hence, $T_p\Omega$ has $\mathbb{R}$-basis $\{\partial/\partial x_1, \partial/\partial y_1, \ldots, \partial/\partial x_n, \partial/\partial y_n\}$ with linear complex structure $J : T_p\Omega \to T_\partial\Omega$ given by

$$J(\partial/\partial x_j) = \partial/\partial y_j, \quad J(\partial/\partial y_j) = -\partial/\partial x_j, \quad j = 1, \ldots n,$$

Applying the discussion in previous section, if we consider the complexification $(T_p\Omega)_{\mathbb{C}}$, then $(T_p\Omega)_{\mathbb{C}} = T_p^{1,0}\Omega \oplus T_p^{0,1}\Omega$ and

$$\partial/\partial z_j = \frac{1}{2}\left(\partial/\partial x_j - i\partial/\partial y_j\right), \quad \partial/\partial \bar{z}_j := \frac{1}{2}\left(\partial/\partial x_j + i\partial/\partial y_j\right) : j = 1, \ldots, n$$

are $\mathbb{C}$-basis for $T_p^{1,0}\Omega$ and $T_p^{0,1}\Omega$ respectively. Similarly, on contangent space $T_p^*\Omega$ has $\mathbb{R}$-basis $\{dx_1, dy_1, \ldots, dx_n, dy_n\}$ with linear complex structure

$$J(dx_j) = dy_j, \quad J(dy_j) = -dx_j, \quad j = 1, \ldots, n,$$

and the complexification $(T_p^*\Omega)_{\mathbb{C}} = T_p^{*1,0}\Omega \oplus T_p^{*0,1}\Omega$ with dual basis $\{dz_j := dx_j + idy_j\}_{j=1}^n$ and $\{d\bar{z}_j := dx_j - idy_j\}_{j=1}^n$, respectively.

For $f \in C^1(\Omega, \mathbb{C})$, we can write the differential $df_p \in \mathrm{Hom}_{\mathbb{R}}(T_p\Omega, \mathbb{C}) \cong (T_p^*\Omega)_{\mathbb{C}}$ as

$$df_p = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(p)dx_j + \sum_{j=1}^n \frac{\partial f}{\partial y_j}(p)dy_j = \sum_{j=1}^n \frac{\partial f}{\partial z_j}(p)dz_j + \frac{\partial f}{\partial \bar{z}_j}(p)d\bar{z}_j.$$

Definition 2. We say f is **holomorphic** if $df_p \in \Lambda^{1,0}T_p^*\Omega$ for any $p \in \Omega$. Equivalently,

(a) $df_p \in \mathrm{Hom}_{\mathbb{C}}(T_p\Omega, \mathbb{C})$, i.e., df_p is $\mathbb{C}$-linear.
(b) f satisfies Cauchy–RIemann equation $\partial f/\partial \bar{z}_j = 0$ on Ω for $j = 1, \ldots, n$.

We denote $\mathcal{O}(\Omega)$ by the set of holomorphic functions on Ω.

A C^1-map $F = (F_1, \ldots, F_m) : \Omega \to \mathbb{C}^m$ is called **holomorphic** if each $F_j \in \mathcal{O}(\Omega)$. Hence, $dF_p(T_p^{1,0}\Omega) \subset T_{F(p)}^{1,0}\mathbb{C}^m$ and $dF_p(T_p^{0,1}\Omega) \subset T_{F(p)}^{0,1}\mathbb{C}^m$. Now, we recall

Definition 3. A **complex manifold** X of (complex) dimesnon n is a smooth manifold of (real) dimension $2n$ with a **holomorphic atlas** , i.e., there exists an open covering $\{U_\alpha\}_{\alpha\in I}$ and homeomorphism $\sigma_\alpha : U_\alpha \to \mathbb{C}^n$ onto some open set V_α such that the transition maps

$$\sigma_{\alpha\beta} := \sigma_\alpha \circ \sigma_\beta^{-1} : \sigma_\beta(U_\alpha \cap U_\beta) \to \sigma_\alpha(U_\alpha \cap U_\beta)$$

are holomorphic, $\forall \alpha, \beta \in I$. We write $\sigma_\alpha = (z_1, \ldots, z_n)$, called **local complex coordinates** on U_α.

For $x \in X$, say $x \in U_\alpha$ for some $\alpha \in I$, we define **holomorphic tangent space** T_xX to be $T_{\sigma_\alpha(x))}V_\alpha$ with linear complex structure defined as above. Since transition maps are holomorphic, $d(\sigma_{\alpha\beta})_{\sigma_\beta(x)}$ is a $\mathbb{C}$-linear isomorphism and thus is independent of the choice of σ_α.

On the other hand, X has a underlying smooth manifold structure and thus T_xX has a underlying real vector space structure, denoted by $T_{x,\mathbb{R}}X$. The discussion above on open sets in $\mathbb{C}^n$ can be applied to X in a direct manner and generalize to bundle level:

$$\mathbb{C}T_{\mathbb{R}}X := T_{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}X \oplus T^{0,1}X, \quad TX \cong T^{1,0}X, \quad \overline{TX} \cong T^{0,1}X;$$
$$\mathbb{C}T_{\mathbb{R}}^*X = T^{*1,0}X \oplus T^{*0,1}X, \quad \Lambda^k(\mathbb{C}TX) = \bigoplus_{p+q=k} \Lambda^{p,q}T^*X.$$

When X is a complex manifold, we always denote $A^k(X)$ by the smooth sections of $\Lambda^k(\mathbb{C}TX)$, i.e., the complex-valued differential forms, and $A^k(X, \mathbb{R})$ by the real ones. A smooth section of $\Lambda^{p,q}T^*X$ is called a (p,q)**-form**. We denote $A^{p,q}(X)$ by the space of (p,q)-forms. We also have

$$A^k(X) = \bigoplus_{p+q=k} A^{p,q}(X).$$

For $\alpha \in A^{p,q}(X)$, we can locally write α with respect to a local complex coordinates

$$\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} dz_I \wedge d\bar{z}_J.$$

If we extend exterior derivative to complex-valued form $d : A^k(X) \to A^{k+1}(X)$ and restrict to $A^{p,q}$, then we can decompose $d = \partial + \bar{\partial}$, where

$$\partial : A^{p,q}(X) \to A^{p+1,q}(X), \quad \partial\alpha = \sum_{|I|=p,|J|=q} \sum_{j=1}^n \frac{\alpha_{IJ}}{\partial z_j} dz_j \wedge dz_I \wedge d\bar{z}_J$$
$$\bar{\partial} : A^{p,q}(X) \to A^{p,q+1}(X), \quad \bar{\partial}\alpha = \sum_{|I|=p,|J|=q} \sum_{j=1}^n \frac{\partial\alpha_{IJ}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J.$$

Since $d^2 = 0$ and $d^2 = \partial^2 + \bar{\partial}\partial + \partial\bar{\partial} + \bar{\partial}^2$ are in types $(p+2, 0)$, $(p+1, q+1)$, and $(p, q+2)$, we have

$$\partial^2 = \bar{\partial}^2 = \bar{\partial}\partial + \partial\bar{\partial} = 0.$$

Notice that for $p = 0$, $\ker(\bar\partial) = \mathcal{O}(X)$, the space of holomorphic functions on X. For $p > 0$, $\ker(\bar\partial) = \Omega^p(X)$, the space of **holomorphic p-forms** on X, the holomorphic section of $\Lambda^{p,0}T^*X$. Clearly, on a coordinate open set U, $\alpha \in \Omega^p(X)$ can be locally written as

$$\alpha|_U = \sum_{|I|=p} \alpha_I dz_I, \quad \alpha_I \in \mathcal{O}(U).$$

Hence, for each $0 \leqslant p \leqslant n$, we obtain a complex

$$(A^{p,\bullet}(X), \bar\partial) : 0 \to \Omega^p(X) \to A^{p,0}(X) \xrightarrow{\bar\partial} A^{p,1}(X) \xrightarrow{\partial} \cdots \to A^{p,n}(X) \to 0.$$

The q-th cohomology of the complex is called the **q-th Dolbeault cohomology** of X:

$$H^{p,q}(X) = \ker(\bar\partial : A^{p,q}(X) \to A^{p,q+1}(X))/\mathrm{im}(\bar\partial : A^{p,q-1}(X) \to A^{p,q}(X)). \tag{3.3}$$

As in the case of de Rham case, we call $\alpha \in A^{p,q}(X)$ is $\bar\partial$-closed if $\bar\partial\alpha = 0$ and $\bar\partial$-exact if $\alpha = \bar\partial\beta$ for some $\beta \in A^{p,q-1}(X)$.

Remark 7. An important fact is that we also have $\bar\partial$-Poincaré Lemma, also known as **Dolbeault–Grothendieck Lemma**, which says that on any open set $U \subset \mathbb{C}^n$ and $\alpha \in A^{p,q}(U)$ with $\bar\partial\alpha = 0$, then there exists a "suitable" open set $V \subset \mathbb{C}^n$ $\beta \in A^{p,q-1}(U)$ so that $\bar\partial\beta = \alpha$ on V. Hence, the corresponding complex on the sheaf level is exact.

$$0 \to \Omega^p_X \to \mathcal{A}^{p,0}_X \to \mathcal{A}^{p,1}_X \cdots to \mathcal{A}^{p,q}_X \to 0$$

Moreover, $\mathcal{A}^{p,q}_X$ is acyclic since we can multiply a (p,q)-forms by partition of unity. Hence, we obtain **Dolbeault theorem** which is complex analogue of de Rham theorem:

$$H^q(X, \Omega^p_X) \cong H^{p,q}(X).$$

Now, we discuss the metric structure on a complex manifold. A hermitian metric h on a complex manifold X is a smooth positive definite hermitian bundle metric on holomorphic tangent bundle TX. That is, in terms of local coordinates $(z_1, \dots, z_n)$ on a coordinate open set U, we can write

$$h = \sum_{j,k=1}^n h_{jk}(z) dz_j \otimes d\bar z_k, \quad h_{jk} \in C^\infty(U),$$

and $(h_{jk}(z))$ is a positive-definite hermitian matrix for each $x \in U$. Following the discussion as in previous section, h is equivalent to a Riemannian metric $g = \mathrm{Re}\, h$ on TX or $\omega = -\mathrm{Im}\, h \in A^{1,1}(X) \cap A^2(X, \mathbb{R})$ locally given by

$$\omega = \frac{i}{2} \sum_{j<k} h_{jk} dz_j \wedge d\bar z_k.$$

Definition 4. Let X be a complex manifold.

(i) A **hermitian manifold** is a pair (X, ω), where ω is a smooth, positive-definite real $(1,1)$-form, called a **hermitian metric**, **hermitian form**, or **fundamental $(1,1)$-form associated to h**.
(ii) A hermitian metric is called **Kähler** if $d\omega = 0$.
(iii) X is called a **Kähler manifold** if X admits a Kähler metric.

We know that if ω is a hermitian metric, then one can express Riemannian volume form by

$$dV_\omega = \frac{\omega^n}{n!}$$

Since ω is real, $d\omega = 0$, $\partial\omega = 0$, and $\bar\partial\omega = 0$ are equivalent. In local coordinates, $\partial\omega = 0$ means

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j}, \quad 1 \leqslant j, k, l \leqslant n.$$

Using this, one can show **existence of holomorphic normal coordinates** for Kähler metric.

Theorem 5. *Let (X, ω) be a hermitian manifold. Then ω is Kähler iff for any $x \in X$, there exists local holomorphic coordinates $(z_1, \dots, z_n)$ centered at x so that $h_{jk} = \delta_{jk} + O(|z|^2)$.*

The proof is quite standard so we omit (see Wells or Griffths–Harris, or many other textbooks). We end this section by discussing some examples and non–examples for Kähler manifolds.

Example 1. The most important example for us is **complex projective space** $\mathbb{CP}^n$. We have a natural Kähler metric given by **Fubini–Study metric**:

$$p^*\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log(|\xi_0|^2 + \cdots + |\xi_n|^2),$$

where $(\xi_0, \dots, \xi_n) \in \mathbb{C}^{n+1}$ and $p : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CP}^n$ is the projection. Let $z = (\xi_1/\xi_0, \dots, \xi_n/\xi_0)$ be the local coordinates on $U_0 = \{[\xi_0 : \cdots : \xi_n] \in \mathbb{CP}^n : \xi_0 \neq 0\} \cong \mathbb{C}^n$. Then ω_{FS} satisfies

$$\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log(1 + |z|^2), \quad \int_{\mathbb{CP}^n} \omega_{FS}^n = 1.$$

Example 2. Let (X, ω) be a Kähler manifold. If $\iota : Y \hookrightarrow X$ is a complex submanifold, then $\omega_Y := \iota^*\omega$ is still a positive definite real $(1, 1)$-form on Y. Moreover, since $d\iota^*\omega = \iota^*d\omega = 0$, ω_Y defines a Kähler metric on Y. Particularly, any non-singular smooth projective variety is Kähler.

Example 3. A **complex torus** is a quotient $X := \mathbb{C}^n/\Lambda$, where Λ is a lattice of rank $2n$. Then X is a compact complex manifold. Moreover, any positive hermitian form $\omega = i\sum_{1\leqslant j<k\leqslant n} h_{jk}dz_j \wedge d\bar{z}_k$ with constant coefficients defines a Kähler metric on X.

Notice that $d\omega = 0$ imposes topological constraints on compact Kähler manifolds. Indeed, since $\mathrm{vol}_g(X) = \int_X \omega^n/n! > 0$, for $1 \leqslant k \leqslant n$, ω^k cannot be exact for $\int_X \omega^n/n! = 0$ by Stokes' formula. Hence, $[\omega^k] \neq 0$ in $H^{2k}_{dR}(X, \mathbb{R})$.

Example 4. Let $X = (\mathbb{C}^2\backslash\{0\})/\Gamma$, where $\Gamma := \{\lambda^n : n \in \mathbb{Z}\}$ acts on $\mathbb{C}^2$ by $(z_1, z_2) \mapsto (\lambda^n z_1, \lambda^n z_2)$. One can show that X is a compact complex manifold and X is diffeomorphic to $S^1 \times S^3$. Thus, $H^2(X, \mathbb{R}) = 0$ and hence, X cannot be Kähler.

4. Kähler Identities and Hodge Decomposition on Compact Kähler Manifolds

4.1. **Operators on Kähler Manifolds and their Commutation Relations.** Let (X, ω) be a hermitian manifold. As mentioned before, $g_{\mathbb{C}}$ induces a hermitian inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ on $\Lambda^k(\mathbb{C}T^*_xX)$ and we can define Hodge $*$-operator with respect to ω by

$$* : A^{p,q}(X) \to A^{n-q,n-p}(X),$$

which is a $\mathbb{C}$-linear isometry and satisfies $\alpha \wedge *\bar{\beta} = \langle\alpha, \beta\rangle_{\mathbb{C}}dV_\omega$. If X is compact, then we can endow a L^2-inner product on $A^{p,q}(X)$ by

$$(\alpha, \beta) := \int_X \langle\alpha, \beta\rangle_{\mathbb{C}}dV_\omega, \quad \forall\alpha, \beta \in A^{p,q}(X).$$

Thus, we can define adjoint $d^* = - * d * : A^{k+1}(X) \to A^k(X)$ as before as well as

$$\partial^* = - * \bar{\partial} * : A^{p+1,q}(X) \to A^{p,q}(X), \quad \bar{\partial}^* = - * \partial * : A^{p,q+1}(X) \to A^{p,q}(X).$$

From $d = \partial + \bar{\partial}$, we also have $d^* = \partial^* + \bar{\partial}^*$. Hence, we can define Hodge Laplacian $\triangle = dd^* + d^*d$ as well as ∂-Laplacian and $\bar{\partial}$-Laplacian:

$$\triangle_\partial := \partial^*\partial + \partial\partial^*, \quad \triangle_{\bar{\partial}} := \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*.$$

Now, we define the spaces of harmonic (p, q)-forms for Hodge Laplacians $\mathcal{H}^{p,q}(X) := \mathcal{H}^k(X) \cap A^{p,q}(X)$ and for ∂,$\bar{\partial}$-Laplcian:

$$\mathcal{H}^{p,q}_\partial(X) = \{\alpha \in A^{p,q}(X) : \triangle_\partial\alpha = 0\}, \quad \mathcal{H}^{p,q}_{\bar{\partial}}(X) := \{\alpha \in A^{p,q}(X) : \triangle_{\bar{\partial}}\alpha = 0\}$$

By definition, $\overline{\triangle_{\bar\partial}} = \triangle_{\partial}$ and thus $\overline{\mathcal{H}^{p,q}_{\bar\partial}(X)} = \mathcal{H}^{q,p}_{\partial}(X)$. Granting the fact that $\triangle$, $\triangle_{\partial}$, and $\triangle_{\bar\partial}$ are elliptic operators, we can proceed exactly same as Hodge theorem for compact Riemannian manifolds to show the following Hodge decompostion for for a compact hermitian manifolds.

$$A^k(X) = \mathcal{H}^k(X) \oplus \triangle(A^k(X)), \quad A^{p,q}(X) = \mathcal{H}^{p,q}_{\bar\partial}(X) \oplus \triangle_{\bar\partial}(A^{p,q}(X)), \quad A^{p,q}(X) = \mathcal{H}^{p,q}_{\partial}(X) \oplus \triangle_{\partial}(A^{p,q}(X)),$$

which is orthogonal with respect to L^2-norm on $A^k(X)$ and $A^{p,q}(X)$. Thus, we have $H^k(X,\mathbb{C}) \cong \mathcal{H}^k(X)$ and $H^{p,q}_{\bar\partial}(X) \cong \mathcal{H}^{p,q}_{\bar\partial}(X)$. Also, we have $\dim_{\mathbb{C}} \mathcal{H}^k(X), \dim_{\mathbb{C}} \mathcal{H}^{p,q}_{\bar\partial}(X), \dim_{\mathbb{C}} \mathcal{H}^{p,q}_{\partial}(X) < infty$.

Proposition 4 (Kodaira–Serre Duality). *Let (X,ω) be a compact hermitian manifold. The bilinear pairing*

$$A^{p,q}(X) \times A^{n-p,n-q}(X) \to \mathbb{C}, \quad (\alpha,\beta) \mapsto \int_X \alpha \wedge \beta$$

descends to a non-degenerate paring on $H^{p,q}_{\bar\partial}(X) \times H^{n-p,n-q}_{\bar\partial}(X) \to \mathbb{C}$. Particularly, $H^{p,q}_{\bar\partial}(X) \cong (H^{n-p,n-q}_{\bar\partial}(X))^*$.

Proof. For $\alpha \in A^{p,q}(X)$, $\gamma \in A^{n-p,n-q-1}(X)$, since $\alpha \wedge \gamma \in A^{n,n-1}(X)$, we have

$$d(\alpha \wedge \gamma) = \bar\partial(\alpha \wedge \gamma) = \bar\partial\alpha \wedge \gamma + (-1)^{p+q}\alpha \wedge \bar\partial\gamma.$$

Hence, if β, β' are $\bar\partial$-closed and $\beta' = \beta + \bar\partial\gamma$, then by Stokes' theorem,

$$\int_X \alpha \wedge \beta' = \int_X \alpha \wedge \beta + \int_X \alpha \wedge \bar\partial\gamma = \int_X \alpha \wedge \beta + (-1)^{p+q} \int_X d(\alpha \wedge \gamma) = \int_X \alpha \wedge \beta.$$

Similarly, the pairing is independent of representative of the Dolbeault cohomology class $[\alpha] \in H^{p,q}_{\bar\partial}(X)$. Therefore, the pairing descends to the Dolbeault cohomology level. Similar to the proof of Poincaré duality, one notice that $*\triangle_{\bar\partial} = \triangle_{\partial}*$ and thus

$$* : \mathcal{H}^{p,q}_{\bar\partial}(X) \to \mathcal{H}^{n-q,n-p}_{\partial}(X).$$

Since $\overline{\mathcal{H}^{n-q,n-p}_{\partial}(X)} = \mathcal{H}^{n-p,n-q}_{\bar\partial}(X)$, $\alpha \mapsto *\bar\alpha$ maps $\mathcal{H}^{p,q}_{\bar\partial}(X) \to \mathcal{H}^{n-p,n-q}_{\bar\partial}(X)$. Finally, notice that $\mathcal{H}^{p,q}_{\bar\partial}(X) \to \mathcal{H}^{p,q}_{\bar\partial}(X) \to \mathbb{C}$ is non-degenerate since $\int_X \alpha \wedge *\alpha = \|\alpha\|^2 > 0$ if $\alpha \neq 0$. Therefore, the result follows from $\mathcal{H}^{p,q}_{\bar\partial}(X) \cong H^{p,q}_{\bar\partial}(X)$. □

However, for compact hermitian manifolds (X,ω),

(a) $\mathcal{H}^k(X)$ may not respect the bidgree decomposition.
(b) $\mathcal{H}^{p,q}_{\bar\partial}(X)$, $\mathcal{H}^{p,q}_{\partial}(X)$, and $\mathcal{H}^{p,q}(X)$ might be different.

Both issues will be resolved when ω is a Kähler. The key is the **Kähler identities** we now discuss.

Now, we assume that (X,ω) is a Kähler manifold. We define **Lefschetz operator**

$$L : A^{p,q}(X) \to A^{p+1,q+1}(X), \quad \alpha \mapsto \omega \wedge \alpha. \tag{4.1}$$

and its adjoint $\Lambda := - * L* : A^{p+1,q+1}(X) \to A^{p,q}(X)$ satisfying

$$\langle L\alpha, \beta\rangle_{\mathbb{C}} = \langle \alpha, \Lambda\beta\rangle, \quad \forall \alpha \in A^{p,q}(X), \beta \in A^{p+1,q+1}(X). \tag{4.2}$$

Theorem 6 (Kähler Identities). *Let (X,ω) be a Kähler manifold. Then*

$$[\bar\partial^*, L] = i\partial, \quad [\Lambda, \bar\partial] = -i\partial^*; \tag{4.3}$$

$$[\partial^*, L] = -i\bar\partial, \quad [\Lambda, \partial] = i\bar\partial^*. \tag{4.4}$$

Proof. Notice that (4.4) follows from (4.3) by taking complex conjugation and $[\Lambda, \bar\partial] = -i\partial^*$ follows from $[\bar\partial^*, L] = i\partial$ by taking adjoint.

Now, we sketch the proof $[\bar\partial^*, L] = i\partial$ for the case wheb $X \subset \mathbb{C}^n$ is a bounded open set with standard Kähler metric $\omega = i\sum_{j=1}^n dz_j \wedge d\bar z_j$. For $\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} dz_I \wedge d\bar z_J \in A^{p,q}(\mathbb{C}^n)$,

$$\partial\alpha = \sum_{|I|=p,|J|=q} \sum_{k=1}^n \frac{\partial \alpha_{IJ}}{\partial z_k} dz_k \wedge dz_I \wedge d\bar z_J, \quad \bar\partial\alpha = \sum_{|I|=p,|J|=q} \sum_{k=1}^n \frac{\partial \alpha_{IJ}}{\partial \bar z_k} d\bar z_k \wedge dz_I \wedge d\bar z_J.$$

Notice that for $v = \sum_{|K|=p,|L|=q} v_{KL} dz_K \wedge d\bar{z}_L$, $\langle u, v\rangle_{\mathbb{C}} = \sum_{|I|=p,|J|=q} u_{IJ}\overline{v_{IJ}}$ and thus

$$(u,v) = \int_X \sum_{|I|=p,|J|=q} u_{IJ}\overline{v_{IJ}} dV$$

where $dV = \omega^n/n! = 2^n(dx_1 \wedge dy_1 \wedge \cdots dx_n \wedge dy_n)$. Then one can directly compute that

$$\bar{\partial}^*\alpha = -\sum_{|I|=p,|J|=q}\sum_{k=1}^n \frac{\partial \alpha_{IJ}}{\partial z_k}\iota_{\partial/\partial\bar{z}_k} dz_I \wedge d\bar{z}_J =: -\sum_{k=1}^n \iota_{\partial/\partial\bar{z}_k}\frac{\partial\alpha}{\partial z_k}.$$

Here, $\iota_{\partial/\partial\bar{z}_k}\alpha$ is the interior multiplication of $\partial/\partial\bar{z}_k$ into α. Then we get

$$[\bar{\partial}, L]\alpha = -\sum_{k=1}^n \iota_{\partial/\partial\bar{z}_k}\left(\frac{\partial}{\partial z_k}(\omega\wedge\alpha)\right) + \omega \wedge \sum_{k=1}^n \iota_{\partial/\partial\bar{z}_k}\frac{\partial\alpha}{\partial z_k}.$$

Since ω has constant coefficients, $\frac{\partial}{\partial z_k}(\omega\wedge\alpha) = \omega \wedge \frac{\partial\alpha}{\partial z_k}$ and therefore

$$[\bar{\partial}^*, L]\alpha = -\sum_{k=1}^n \iota_{\partial/\partial\bar{z}_k}\left(\omega\wedge\frac{\partial\alpha}{\partial z_k}\right) - \omega\wedge\left(\iota_{\partial/\partial\bar{z}_k}\frac{\partial u}{\partial x_k}\right) = -\sum_{k=1}^n \left(\iota_{\partial/\partial\bar{z}_k}\right)\wedge\frac{\partial u}{\partial z_k}.$$

Since $\iota_{\partial/\partial\bar{z}_k}\omega = -idz_k$, we get

$$[\bar{\partial}^*, L]\alpha = i\sum_{k=1}^n dz_k \wedge \frac{\partial u}{\partial z_k} = i\partial u.$$

Finally, for general (X,ω) and any $x \in X$, if we choose holomorphic normal coordinates $(z_1, \ldots, z_n)$ centered at x as in Theorem 5, above calculation go through with error term

$$[\bar{\partial}^*, L]\alpha = i\partial\alpha + O(|z|),$$

for (p,q)-form α supported in a neighborhood of x. Particularly, $[\bar{\partial}^*, L]\alpha(x) = i\partial\alpha(x)$, for $x \in X$. □

Corollary 2. *If (X,ω) is Kähler, then*

$$[\partial, \bar{\partial}^*] = [\bar{\partial}, \partial^*] = 0 \tag{4.5}$$

$$\triangle = 2\triangle_{\bar{\partial}} = 2\triangle_{\partial}, \tag{4.6}$$

and $\triangle$ commutes with $*, \partial, \bar{\partial}, \partial^*, \bar{\partial}^*, L, \Lambda$.

Proof. We have $[\partial, \bar{\partial}^*] = -i[\partial, [\Lambda, \partial]]$ and Jacobi identity implies

$$-[\partial.[\Lambda, \partial]] + [\Lambda, [\partial, \partial]] + [\partial, [\partial, \Lambda]] = 0.$$

Hence, $-2[\partial, [\Lambda, \partial]] = 0$ and $[\partial, \bar{\partial}^*] = 0$. The second relation $[\bar{\partial}, \partial^*] = 0$ is the adjoint of the first. Next,

$$\triangle_{\bar{\partial}} = [\bar{\partial}, \bar{\partial}^*] = -i[\bar{\partial}, [\Lambda, \partial]].$$

Since $[\partial, \bar{\partial}] = 0$, Jacobi identity implies $-[\bar{\partial}, [\Lambda, \partial]] + [\partial, [\bar{\partial}, \Lambda]] = 0$. Hence,

$$\triangle_{\bar{\partial}} = [\partial, -i[\bar{\partial}, \Lambda]] = [\partial, \partial^*] = \triangle_{\partial}.$$

From (4.5),we have $\triangle = [\partial + \bar{\partial}, \partial^* + \bar{\partial}^*] = \triangle_\partial + \triangle_{\bar{\partial}} + [\partial, \bar{\partial}^*] + [\bar{\partial}, \partial^*] = \triangle_\partial + \triangle_{\bar{\partial}}$. Finally, $[\partial, \triangle_\partial] = [\partial^*, \triangle_\partial] = [\bar{\partial}, \triangle_{\bar{\partial}}] = [\bar{\partial}^*, \triangle_{\bar{\partial}}] = 0$ and $[\triangle, *] = 0$ are immediate. Furthermore, $[\partial, L] = \partial\omega = 0$ together with Jacobi identity implies

$$[L, \triangle_\partial] = [L, [\partial, \partial^*]] = -[\partial, [\partial^*, L]] = i[\partial, \bar{\partial}] = 0.$$

By taking adjoint, $[\triangle_\partial, \Lambda] = 0$. □

4.2. **Hodge Theory on Compact Kähler Manifolds.** Now, we assume that (X,ω) is a comapct Kähler manifold. The identity $\triangle = 2\triangle_{\bar\partial}$ shows that $\triangle$ is homogeneous with respect to bidegree, $\mathcal{H}^{p,q}_{\bar\partial}(X) = \mathcal{H}^{p,q}(X)$, and that there is an orthogonal decomposition

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X). \tag{4.7}$$

As $\overline{\triangle_{\bar\partial}} = \triangle_\partial = \triangle_{\bar\partial}$, we have $\mathcal{H}^{p,q}(X) = \overline{\mathcal{H}^{p,q}(X)}$. Using Hodge theorem for de Rham and Dolbeault cohomology, we get **Hodge decomposition** on compact Kähler manifolds:

$$H^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(X), \tag{4.8}$$

$$H^{p,q}_{\bar\partial}(X) \cong \overline{H^{q,p}_{\bar\partial}(X)}. \tag{4.9}$$

A priori, it is not clear that the decomposition is independent of the choice of Kähler metrics. We now show the following result known as $\partial\bar\partial$**-Lemma** which will deduce that this is the case.

Lemma 2. *Let (X,ω) be a compact Kähler manifold. For a d-closed (p,q)-form α, TFAE*

(a) α is d-exact.
(b) α is ∂-exact.
(b)′ α is $\bar\partial$-exact
(c) α is $\partial\bar\partial$-exact, i.e., there exists $v \in A^{p-1,q-1}(X)$ such that $\alpha = \partial\bar\partial v$.
(d) $\alpha \in \mathcal{H}^{p,q}(X)^\perp$.

Proof. (c)⇒(a),(b),(b)′ and (a), or (b) or (b′)⇒ (d) are obvious. It suffices to show that (d)⇒ (c). As $d\alpha = 0$, we have $\partial\alpha = 0 = \bar\partial\alpha = 0$. Since $\alpha \in \mathcal{H}^{p,q}(X)^\perp$, there exists $\beta \in A^{p,q-1}(X$ such that $\alpha = \bar\partial\beta$. By Hodge decomposition for $\triangle_\partial$:

$$A^{p,q-1}(X) = \mathcal{H}^{p,q-1}(X) \oplus \mathrm{im}(\triangle_\partial),$$

we can write $\beta = h + (\partial\partial^* + \partial^*\partial)u$ for some $u \in A^{p,q-1}(X)$. Let $v := \partial^* u \in A^{p-1,q-1}(X)$ and $w = \partial^* u \in A^{p+1,q-1}(X)$. Therefore, by (4.5),

$$\alpha = \bar\partial\partial v + \bar\partial\partial^* w = -\partial\bar\partial v - \partial^*\bar\partial w.$$

However, as $\partial u = 0$ and $\partial^*\bar\partial w \in \ker\partial^\perp$, $\partial^*\bar\partial w = 0$ and hence $\alpha = \bar\partial\partial v$. □

Corollary 3. (4.8) *is independent of the choice of Kähler metric.*

Proof. Let ω' be another Kähler metric on X. We denote $\mathcal{H}^{p,q}(X,\omega)$ and $\mathcal{H}^{p,q}(X,\omega')$ be the harmonic forms with respect to ω and ω' respectively. Given a Dolbeault cohomology class $[\alpha_0] \in H^{p,q}_{\bar\partial}(X)$, we denote $\alpha \in \mathcal{H}^{p,q}(X,\omega)$ and $\alpha' \in \mathcal{H}^{p,q}(X,\omega')$ be the corresponding harmonic representative of $[\alpha_0]$. By definition, there exists $\gamma \in A^{p,q-q}(X)$ such that $\alpha = \alpha' + \bar\partial\gamma$. However, $d\bar\partial\gamma = d(\alpha - \alpha') = 0$ shows that $\bar\partial\gamma \in \mathcal{H}^{p,q}(X)^\perp$ by Hodge decomposition for $\triangle$. Hence, $\bar\partial\triangle \in \mathrm{im}(\triangle)$ and thus α, α' represent the same de Rham cohomology class. □

We denote **Betti number** and **Hodge number** by

$$b_k = \dim_{\mathbb{C}} H^k(X,\mathbb{C}), \quad h^{p,q} := \dim_{\mathbb{C}} H^{p,q}_{\bar\partial}(X).$$

Then (4.8), (4.9), and Kodaira–Serre duality implies

$$b_k(X) = \sum_{p+q=k} h^{p,q}, \quad h^{p,q}(X) = h^{q,p}, \quad h^{p,q}(X) = h^{n-p,n-q}(X).$$

Particular, this gives another topological constraints for compact Kähler manifolds

Corollary 4. *If X is a compact manifold, then $b_{2k+1}(X)$ is even.*

Proof. This follows from $b_{2k+1}(X) = 2\sum_{p=0}^{k} h^{p,k+1-p}(X)$. □