Fixed points in Toroidal Compactifications and Essential Dimension of Covers

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Outline

Essential dimension is a numerical invariant of algebraic objects invented in the late 90s by J. Buhler and Z. Reichstein. Roughly speaking, the essential dimension of an algebraic object is the number of parameters it takes to define it generically. Traditionally, the main focus in the study of essential dimension has been on the essential dimension of G-torsors for various groups G. But a recent paper of Farb, Kisin and Wolfson (FKW) shows that it can also be interesting to prove theorems about the essential dimension of congruence covers arising from Shimura varieties.

This talk is about my work with **Najmuddin Fakhruddin**. The goal are to:

- Give geometric proofs of the results of FKW, which were proved by purely arithmetic methods.
- Prove results about non-classical Shimura varieties (in particular, those of type E_7), which are unobtainable using the FKW methods (combined with present technology). We also prove results for reductions mod p of classical Shimura varieties and for certain quantum level-structures on the moduli space of curves.
- Point out a general conjecture about essential dimension in Hodge theory.

Pullback Dimension

Let $f : X \to Y$ be a (generically) finite morphism of (quasi-projective) varieties (over $\mathbb{C}$). I start with the following (nonstandard) definition.

Definition

The **pullback dimension** $\mathrm{pbd}(f)$ is the minimum dimension of a variety W fitting in a pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W. \end{array}$$

Essential Dimension

Definition

Suppose (for simplicity) that X and Y are irreducible. The **essential dimension** of $f : X \to Y$ is

$$\mathrm{ed}\, f := \min\{\mathrm{pbd}(f^{-1}U \to U) : \ U \text{ Zariski dense open subset of } Y.\}$$

For a prime number p, the **essential dimension of f at p** is $\mathrm{ed}(f; p) := \min\{\mathrm{pbd}(X_V \to V)\}$ where $V \to Y$ ranges over all generically finite morphisms from an (irreducible) variety V of degree prime to p.

So the essential dimension of $f : X \to Y$ can be thought of as the pullback dimension of f over the generic point (or function field) of Y, and the essential dimension of $f : X \to Y$ at p can be thought of as the pullback dimension of f over a prime-to-p closure of the function field $K(Y)$.

Incompressibility

We say $f : X \to Y$ is **incompressible** if $\mathrm{ed}\, f = \dim Y$ and p-**incompressible** if $\mathrm{ed}(f; p) = \dim Y$. So, speaking a little bit vaguely, a morphism $f : X \to Y$ is incompressible if there is no pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

with $\dim W < \dim Y$.

Essential Dimension of Groups

Most of the focus in the theory of essential dimension has been on the essential dimension of groups (both finite and infinite). This is the number of parameters it takes to define a G-torsor. When G is a finite group (or finite algebraic group), we can define it like this.

Definition

The essential dimension of G is the supremum of $\mathrm{ed}(X \to Y)$ where $X \to Y$ ranges over all G-torsors. Similarly, if p is a prime number, the essential dimension of G at p is the supremum of $\mathrm{ed}(X \to Y; p)$ where $X \to Y$ ranges over all G-torsors. We write $\mathrm{ed}\, G$ and $\mathrm{ed}(G; p)$ for the essential dimensions of G and of G at p respectively.

Here are some examples:

- (Babylonians) $\mathrm{ed}\, S_2 = 1$
- (Classical) $\mathrm{ed}\, S_3 = 1, \mathrm{ed}\, S_4 = 2.$
- (Klein, Joubert) $\mathrm{ed}\, S_5 = 2$ and $\mathrm{ed}\, S_6 = 3.$

More Examples

- (Buhler-Reichstein) $\mathrm{ed}\, A = r$ for A a finite abelian group of rank r.
- (Buhler-Reichstein) $\lfloor n/2 \rfloor \leq \mathrm{ed}\, S_n \leq n - 3$ for $n \geq 6$.
- (Duncan) $\mathrm{ed}\, S_7 = 4$.
- (Karpenko-Merkurjev) For a p-group G, $\mathrm{ed}\, G = \mathrm{ed}(G; p) = \min\{\dim V : V \text{ is a faithful } G\text{=representation}\}$.
- $\mathrm{ed}\, S_n =?$ for $n > 7$. Probably $n - 3$, but no one knows.
- $\mathrm{ed}\, \mathrm{PGL}_p =?$ Here p is prime and the group is, of course, infinite.
- $\mathrm{ed}(\mathbb{Z}/p^n) =?$ over a field of characteristic p. Should be n by conjecture of Ledet.

Farb-Kisin-Wolfson Proto-theorem

Write A_g for the moduli space of principally polarized abelian varieties (ppavs) of dimension g, and $A_g[N]$ for the moduli space of such things with full level N structure. That is $A_g[N]$ parameterizes pairs consisting of a ppav A and a (symplectic) isomorphism $\phi : H^1(A, \mathbb{Z}/N) \xrightarrow{\cong} (\mathbb{Z}/N)^{2g}$. For $N \geq 3$, $A_g[N]$ is a fine moduli space.

Theorem (FKW)

Pick an integer $N \geq 3$ and a prime p not dividing N. Then the canonical forget-the-structure map

$$A_g[pN] \to A_g[N]$$

is p-incompressible.

More FKW

The above theorem is proved by arithmetic methods involving integral models for A_g and $A_g[N]$. The main techniques are Serre-Tate coordinates and in the end it is about how abelian varieties degenerate from characteristic 0 to characteristic p.
The method is very flexible so it generalizes from the case where the base is $A_g[N]$ to the case where the base is any variety Z inside of (or even finite over) $A_g[N]$ which has a suitably good reduction mod p. From that FKW get the following generalizations.

Theorem (FKW)

With the notation of the previous theorem, but with M_g the moduli space of genus g curves, the morphism $M_g[pN] \to M_g[N]$ is p-incompressible.

And, to avoid technical details, I give a vague form of the next main theorem of FKW.

Theorem (FKW)

The same type of p-incompressibility results hold for various Shimura varieties of Hodge type.

The point here is that Hodge type Shimura varieties are the Shimura varieties that can basically be put inside of A_g.

Work with Fakhruddin

Our original goal was to recover the theorems of FKW by geometric methods. The idea, which we owe to a suggestion from Zinovy Reichstein, was to use the **fixed point method**. As I'll explain below, this is a general "classical" method of proving incompressibility of a G-torsor $f : X \to Y$ by finding fixed points for an abelian subgroup of G in a suitable compactification of X.

Work with Fakhruddin

For locally symmetric spaces, we can prove p-incompressibility of congruence covers whenever the associated Hermitian symmetric domain is a tube domain with 0-dimensional rational boundary component. So with our methods:

- We recover the results of FKW for A_g.
- We also extend their results to certain Shimura varieties of type E_7 studied by Baily. These are not of Hodge type.
- We recover the results of FKW for M_g.
- For both A_g and M_g in finite characteristic we can prove incompressibility.
- We can prove incompressibility of certain "quantum" cover of $M_g[N]$ studied by
- **HOWEVER**, FKW works for many compact Shimura varieties. Our method can't possibly work for these because there cannot be fixed points in a compactification.
- Similarly, we can't prove incompressibility for E_6-type Shimura varieties, which are not tube. For these, incompressibility is open.

Fixed Point Method

I state the fixed-point method in the form from Reichstein—Youssin. For this, G is a finite group and $f : X \to Y$ is a G-torsor. $\bar{X}$ is a G-equivariant partial compactification of X, p is a prime and $H = (\mathbb{Z}/p)^r$ for some $r \geq 0$.

Theorem (Reichstein—Youssin)

Suppose H has a smooth fixed-point on $\bar{X}$. Then $\mathrm{ed}(f; p) \geq r$.

The theorem is not that difficult to prove once you know one more result of Reichstein–Youssin and Kollár-Szabó.

Going down

Theorem (Going down)

Suppose $H = (\mathbb{Z}/p)^r$ and $f : X \dashrightarrow Y$ is a rational map of H-varieties with Y proper. Then, if H has a smooth fixed point on X, it must also have a fixed point on Y.

The proof of this is a very nice induction on dimension. It's obvious for $\dim X = 0$ or even 1 really. And, if $\dim X > 1$, we can blow up a smooth fixed point p to get an exceptional divisor $E \cong \mathbb{P}^{\dim X - 1}$. But then we get a birational map $E \dashrightarrow Y$ and we're done by induction.

Producing Fixed points: Idea

Our initial idea was to produce fixed points in compactifications of locally symmetric varieties using the toroidal compactifications of Ash-Mumford-Rapoport-Tai. This “works” when there are 0-dimensional strata in the toroidal compactifications, or, equivalently, when the domain is tube type and their are 0-dimensional rational boundary components. Roughly speaking, the reason is that, then the "torus" which is compactified to produce the toroidal compactification has a fixed point.

Producing Fixed Points: Theorem

But we found it's nicer and more general to abstractify AMRT with a general theorem about toroidal singularities and fixed points. For this, by a *toroidal singularity* S we mean the completion of the local ring of a toric variety at a torus fixed. We write S^0 for the complement of the boundary divisor.

Theorem

Suppose $f : X \to Y$ *is a finite étale cover which is Galois with group* G*. Suppose there exists a map* $g : S^0 \to Y$ *such that the composite*

$$\pi_1(S^0) \to \pi_1(Y) \to G$$

is a finite abelian group A*. And suppose* g *extends to a morphism* $\bar{g} : S \to \bar{Y}$ *where* $\bar{Y}$ *is a partial compactification of* Y*.*
Then any G*-equivariant partial compactification* $\bar{X}$ *of* X *admitting a proper morphism* $\bar{f} : \bar{X} \to \bar{Y}$ *has an* A*-fixed point.*

Producing Fixed Points: Application

We apply the above theorem to the case where $X \to Y$ is a cover of locally symmetric spaces. So

- $D = G/K$ is a Hermitian symmetric domain.
- $\Gamma \in G(\mathbb{Q})$ is an arithmetic subgroup.
- $\Delta \trianglelefteq \Gamma$ is a finite index normal subgroup.
- $X \to Y$ is the map $\Delta \backslash D \to \Gamma \backslash D$.
- $\bar{Y}$ is the Baily-Borel compactification.
- S is one of the following (very similar) objects:
  - A polydisk Δ^r equipped with a period map $(\Delta^*)^r \to Y$. By Borel, this extends to $\Delta^r \to \bar{Y}$.
  - A toroidal neighborhood in an AMRT compactification of Y.

Conjecture

Suppose H *is a torsion-free integral variation of Hodge structure on a smooth connected quasi-projective complex variety B. Let d denote the dimension of the image of the period map. Then there exists an integer N such that, if p is a prime number and n is an integer with $p^n \geq N$, then*

$$\mathrm{ed}(\mathsf{H} \otimes \mathbb{Z}/p^n \to B; p) \geq d.$$

Partial evidence:

- FKW. Just take H to be the canonical variation on A_g. And similarly in general.
- Our theorems for exceptional Shimura varieties. Plus our fixed point theorem can be used to get weaker lower bounds.
- As M. Nori explained to us, if you don't tensor with $\mathbb{Z}/p^n$, then the conjecture (properly interpreted) follows directly from the theorem of the fixed part.

Sketch of Nori's Observation

Suppose we have U Zariski open in B and a morphism $\pi : U \to W$ such that $(\mathsf{H}_{\mathbb{Z}})_U \cong \pi^* L$ for some $\mathbb{Z}$ local system L. To simplify the notation we can replace B with U and then we just have $\mathsf{H}_{\mathbb{Z}} \cong \pi^* L$ where L is pulled back from $\pi : B \to W$.

But then the local system H is constant on the fibers of π. So, by the theorem of the fixed part, the period map has to be constant. This shows that $\dim W$ is at least the dimension of the image of the period map.