1. Introduction

Definition 1.1. Suppose G is a group. A G-set is a pair (X, ρ) where X is a set and $\rho : G \to A(X)$ is a group homomorphism.

1.2. Suppose (X, ρ) is a G-set. If $g \in G$ and $x \in X$, then we write gx for $\rho(g)(x)$. Note that, for $g, h \in G$, $(gh)x = \rho(gh)(x) = \rho(g)(\rho(h)(x)) = g(hx)$. Since we can use the notation gx instead of $\rho(g)(x)$, we do not always need to explicitly name the group homomorphism ρ. So, we often refer to a G-set simply as X instead of as the pair (X, ρ). If (X, ρ) is a G-set, then the map

$$a : G \times X \to X \text{ given by}$$
$$(g, x) \mapsto gx$$

is called the *action map*. Note that the action map determines ρ because $\rho(g)(x) = a(g, x)$ for all $x \in X$.

Note also that $a(gh, x) = \rho(gh)(x) = \rho(g)(\rho(h)(x)) = a(g, a(h, x))$, and that $a(e, x) = \rho(e)(x) = \mathrm{id}_X(x) = x$ for all $x \in X$.

Proposition 1.3. *Suppose X is a set, G is a group, and suppose $a : G \times X \to X$ is a map satisfying*

$$a(g, a(h, x)) = a(gh, x); \tag{1.3.1}$$
$$a(e, x) = x \tag{1.3.2}$$

for $g, h \in G$ and $x \in X$. For $g \in G$, set $\rho(g)(x) = a(g, x)$. Then (X, ρ) is a G-set with action map a.

Proof. Suppose $g \in G$ and $x \in X$. Then $\rho(g)\rho(g^{-1})(x) = a(g, a(g^{-1}, x)) = a(gg^{-1}, x) = a(e, x) = x$. Therefore $\rho(g) \circ \rho(g^{-1}) = \mathrm{id}$. So $\rho(g) \in A(X)$. We have $\rho(gh)(x) = a(gh, x) = a(g, a(h, x)) = \rho(g)(\rho(h)(x)) = (\rho(g) \circ \rho(h))(x)$. So $\rho(gh) = \rho(g) \circ \rho(h)$. Therefore, $\rho : G \to A(X)$ is a group homomorphism. □

Definition 1.4. Suppose X is a G-set. A sub-G-set is a subset Y of X such that, for all $y \in Y$ and $g \in G$, $gy \in Y$.

Proposition 1.5. *Suppose X is a G-set and $\{Y_i\}_{i \in I}$ are sub-G-sets. Then $\cap_{i \in I} Y_i$ is a sub-G-set.*

Proof. Suppose $y \in \cap_{i \in I} Y_i$ and $g \in G$. Then, for each $g \in G$, $gy \in Y_i$. So $gy \in \cap_{i \in I} Y_i$. □

Definition 1.6. Suppose X is a G-set and $x \in X$. The *stabilizer* of x is $G_x := \{g \in G : gx = x\}$. The *orbit* of x is the set $Gx := \{gx : g \in G\}$.

Proposition 1.7. *Suppose X is a G-set and $x \in X$.*

(1) *$G_x \leq G$;*
(2) *The orbit of x is a the intersection of all sub-G-sets of X containing x.*

Proof. (1): Clearly, $e \in G_x$, since $ex = x$. Suppose $g, h \in G_x$. Then $gh^{-1}x = gh^{-1}hx = gx = x$. So $gh^{-1} \in G_x$. Therefore $G_x \leq G$.

(2): First we show that Gx is a sub-G-set. To see this, suppose $gx \in Gx$ with $g \in G$ and $x \in X$. Then, if $h \in G$, $h(gx) = (hg)x \in Gx$. So Gx is a sub-G-set and clearly Gx contains x. On the other hand, suppose Y is a sub-G-set of X containing x. Then, for any $g \in G$, $gx \in Y$. So Y contains Gx. (2) follows. □

Lemma 1.8. *Suppose X is a G set. Let R be the set of all pairs $(x, y) \in X \times X$ such that $x = gy$ for some $g \in G$. Then*

(1) *R is an equivalence relation on X.*
(2) *If $x \in X$, then the equivalence class $[x]$ of x is the orbit Gx.*

Proof. (1): Write $x \sim y$ if $(x, y) \in R$. Then, for $x \in X$, $x \sim x$ since $x = ex$. If $x \sim y$, then $x = gy$ so $y = g^{-1}x$. So $y \sim x$. Similarly, if $x \sim y$ and $y \sim z$, then $x = gy$ and $y = hz$ for some $g, h \in G$. So $x = g(hz) = (gh)z$. So $x \sim z$.

(2): We have $y \in [x]$ if and only if $y = gx$ for some $g \in G$. By definition, this holds if and only if $y \in Gx$. □

Corollary 1.9. *Suppose X is a G-set. Then the set $\{Gx : x \in X\}$ of G-orbits of X is a partition of X.*

Proof. Follows directly from Lemma 1.8. □

Definition 1.10. Suppose X is a G-set. We write $G\backslash X$ for the set of G-orbits of X. By Lemma 1.8, this is the same as X/R (where R is the equivalence relation of Lemma 1.8). We say that X is a *transitive* G-set if X has exactly one orbit.

Example 1.11. Suppose $G = \mathbf{O}(2)$ and $X = \mathbb{R}^2$. Then G actions on X by multiplication. If $\mathbf{v} = (x, y) \in \mathbb{R}^2$, then the G-orbit of $\mathbf{v}$ is the circle of radius $|\mathbf{v}|$ centered at the origin. We have $G_0 = G$. On the other hand, if $\mathbf{v} \neq 0$, then the stabilizer $G_{\mathbf{v}}$ is the group K of order 2 generated by the reflection in the line from the origin though $\mathbf{v}$.

Definition 1.12. Suppose G is a group and H is a subgroup. Define maps $L : H \to E(G)$, $R : H \to E(G)$ and $I : H \to E(G)$ as follows:

(1) $L(h)(g) = hg$;
(2) $R(h)(g) = gh^{-1}$;
(3) $I(h)(g) = hgh^{-1}$.

Proposition 1.13. *Suppose H is a subgroup of a group G. The maps L, R and I defined above are all group homomorphisms from H to $A(G)$. Consequently, each defines an action of H on G. The action defined by L is called the* left *action, the action defined by R is called the* right *action and the action defined by I is called the* inner *action.*

Proof. We have $L(hk)(g) = hkg = L(h)(L(k)g)$. So $L(hk) = L(h) \circ L(k)$. So $L(hh^{-1}) = L(e) = \mathrm{id}_G$. Thus $L(h)^{-1} = L(h)^{-1}$. So $L : G \to A(G)$, and L is a group homomorphism.

We have $R(hk)(g) = g(hk)^{-1} = gk^{-1}h^{-1} = R(h)(R(k)g) = (R(h) \circ R(k))(g)$. So $R(hk) = R(h) \circ R(k)$. Since $R(e) = \mathrm{id}_G$, this shows that $R(G) \subset A(G)$ and that $R : G \to A(G)$ is a group homomorphism.

The proof for I is similar. □

Remark 1.14. Suppose $H \leq G$ and $g \in G$. The H-orbit of g under the action L is the right coset Hg. The H-orbit of g under the action R is the right coset gH. If $H = G$, then the H-orbit of g under the action I is the conjugacy class of G.

Definition 1.15. Suppose G is a group and X and Y are G-sets. A morphism of G-sets is a map $f : X \to Y$ such that, for $g \in G$ and $x \in X$, $f(gx) = gf(x)$.

Proposition 1.16. *Suppose G is a group and X, Y and Z are G-sets.*

(1) *If $\alpha : X \to Y$ and $\beta : Y \to Z$ are morphisms of G-sets, then so is $\beta \circ \alpha$.*
(2) *If $\alpha : X \to Y$ is a morphism of G-sets which is one-one and onto then $\alpha^{-1} : Y \to X$ is also a morphism of G-sets.*

Proof. (1): For $x \in X$ and $g \in G$, we have $(\beta \circ \alpha)(gx) = \beta(\alpha(gx)) = \beta(g\alpha(x)) = g\beta(\alpha(x)) = g(\beta \circ \alpha)(x)$.

(2): Set $\beta = \alpha^{-1}$. Pick $y \in Y$ and set $x = \beta(y)$. Then $\beta(gy) = \beta(g\alpha(x)) = \beta(\alpha(gx)) = gx = g\beta(y)$. □

1.17. Suppose X and Y are two G-sets. An *isomorphism* of G-sets from X to Y is a morphism $\alpha : X \to Y$ of G-sets which is one-one and onto. By Proposition 1.16, if $\alpha : X \to Y$ is an isomorphism of G-sets, then so is $\alpha^{-1} : Y \to X$. We say that two G-sets X and Y are *isomorphic* and write $X \cong Y$ if there exists an isomorphism of G-sets from X to Y. Clearly, $X \cong X$, and, by Proposition 1.16, if $X \cong Y$ and $Y \cong Z$, then $X \cong Z$.

1.18. Suppose G is a group and $H \leq G$. Then for $x, y \in G$, we have $x(yH) = (xy)H$. So we can define a map

$$a : G \times G/H \to G/H \text{ given by}$$
$$(x, yH) \mapsto x(yY).$$

It is very easy to see that this map satisfies the conditions of Proposition 1.3. So it defines an action of G on G/H. This is the only action we will consider on G/H (unless otherwise specified). Clearly G/H is a transitive G-set with this action.

Theorem 1.19 (Orbit-Stabilizer Theorem)**.** *Suppose X is a transitive G-set and $x \in X$. Then there is a map $\varphi : G/G_x \to X$ satisfying $\varphi(gG_x) = gx$. Moreover, φ is an isomorphism of G-sets.*

Proof. Suppose $g_1, g_2 \in G$ and that $g_1G_x = g_2G_x$. Then $g_1 = g_2h$ for some $h \in G_x$. So $g_1x = (g_2h)x = g_2(hx) = g_2x$. Therefore, we can define a map $\varphi : G/G_x \to X$ by setting $\varphi(gG_x) = gx$.

The map φ is surjective because X is transitive. So, if $y \in X$, there exists $g \in G$ such that $\varphi(gG_x) = gx = y$.

To see that φ is a morphism of G-sets, suppose $a, b \in G$. Then $\varphi(a(bG_x)) = \varphi((ab)G_x) = abx = a(bx) = a\varphi(bG_x)$.

Finally, to show that φ is one-one, suppose $\varphi(aG_x) = \varphi(bG_x)$. Then $ax = bx$. So $x = a^{-1}bx$. So $a^{-1}b \in G_x$. Therefore, $b \in aG_x$. So $aG_x = bG_x$. □