HW01, due Wednesday, February 3
Math 406, Spring 2021

Reading: Read Chapter 1 of Crisman's text and begin reading Chapter 2.

Graded Problems: Work the following problems for a grade. Turn them in on Gradescope.

Some problems are taken from the Online Version of Crisman's text:

http://math.gordon.edu/ntic/

Each problem is worth 20 points.

1. Suppose S is a subset of the set $\mathbb{R}$ of real numbers. We say that S has the *well-ordering property* if, for any nonempty subset $T \subset S$, there exists an $n \in T$ such that, for all $m \in T$, $n \leq m$. In other words, S has the well-ordering property if every nonempty subset T of S has a smallest element. By Axiom 1.2.1, the set $\mathbb{P}$ of positive integers has the well-ordering property.

(a) Let $S \subset \mathbb{R}$, and suppose that there exists a strictly increasing function $f : S \to \mathbb{P}$. (Here by *strictly increasing* I mean that $x < y \Rightarrow f(x) < f(y)$.) Show that S has the well-ordering property.

(b) **(10 point bonus)** Does the converse to (a) hold? In other words, if a subset S of $\mathbb{R}$ has the well-ordering property, does it follow that there exists a strictly increasing function $f : S \to \mathbb{P}$. Prove or give a counterexample.

2. We say that a subset S of the set $\mathbb{R}$ is *closed under addition* if, for all $x, y \in S$, $x + y$ is also in S. Similarly, we say that S is *closed under multiplication* if, for all $x, y \in S$, $xy \in S$.

(a) Suppose S is a subset of $\mathbb{R}$, which is closed under addition and has the well-ordering property. Show that S is contained in the set $[0, \infty)$ of non-negative real numbers.

(b) Suppose S is a subset of the set $\mathbb{R}_+ = (0, \infty)$ of positive real numbers. And suppose further that S is closed under multiplication and has the well-ordering property. Show that $S \subset [1, \infty)$.

3. Suppose a and b are positive integers.

(a) Assuming that $a|b$, show that $a \leq b$.

(b) Assuming that $a|b$ and $b|a$, show that $a = b$.

4. Suppose a and b are positive integers, and set

$$S := \{as + bt : s, t \geq 0\}.$$

Suppose that $c, c+1, \ldots, c+a-1 \in S$ for some integer c.

Show that S contains all integers bigger than or equal to c.

5. As in Crisman's Exercise 1.6, compute the conductors for $\{3, 5\}$ and $\{4, 5\}$. Prove your answers.