Math 608E Fall 2012
Homework 1
Due, November 1

Problem 1. Suppose (V, g, J) is a hermitian space. So V is a finite dimension real vector space, g is a positive definite inner product on V and J is an endomorphism of V satisfying $J^2 = -1$. Suppose $\dim V = n = 2m$.

Recall that we have $\omega = g(Jx, y)$ for $x, y \in V$. And, using ω, we define vol $= \omega^m/m!$. The metric g induces a metric on $\wedge^* V^*$ which we write as $\alpha \otimes \beta \mapsto (\alpha, \beta)$.

(a) Show that $(*\alpha, *\beta) = (\alpha, \beta)$ for $\alpha, \beta \in \wedge^* V^*$.

(b) Recall that I defined $\gamma : \wedge^* V^* \to \wedge^* V^*$ by setting $\gamma\alpha = (-1)^{\binom{p}{2}}\alpha$ for $\alpha \in \wedge^p V^*$, and that I defined $\sigma = *\gamma$. Showt that $\sigma^2 = (-1)^{\binom{n}{2}} = (-1)^m$.

Problem 2. Recall that complex projective n-space is the space $\mathbb{P}^n = \{[Z_0, \ldots, Z_n]$ of lines in $\mathcal{C}^n$.

(a) Explain why $\omega := i\partial\bar{\partial} \log(\sum_{i=0}^n |\frac{Z_i}{Z_k}|^2)$ defines a 2-form on $\mathbb{P}^n$ which is indendent of k.

(b) Show that ω is real and closed.

(c) Show that ω is the 2-form corresponding to a Kähler metric g on $\mathbb{P}^n$. This is called the *Fubini-Study* metric.

Problem 3. The point of this problem is to show that we can define complex analytic manifolds in terms of ringed spaces along the same lines as is done for schemes in Hartshorne's Algebraic Geometry. (See page 72.)

A ringed space consists of a topological space X together with a sheaf $\mathcal{O}_X$ of rings on X. A morphism from a ringed space $(X, \mathcal{O}_X)$ to a ringed space $(Y, \mathcal{O}_Y)$ consists of a pair $(f, f^\#)$ where $f : X \to Y$ is a continuous morphism of topological spaces and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings. A ringed space $(X, \mathcal{O}_X)$ is a *locally ringed space* if, for every point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring. In other words, it is a ring with a unique maximal ideal $\mathfrak{m}_{X,x}$.

Any morphism of $(f, f^\#)$ from a ringed space $(X, \mathcal{O}_X)$ to a ringed space $(Y, \mathcal{O}_Y)$ induces ring homormorphisms $f^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ on the stalks. We say that $(f, f^\#)$ is a morphism of locally ringed spaces if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed spaces and, for any $x \in X$, the morphism $f^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring homomorphism. That is, $\mathfrak{m}_{Y,f(x)}$ is the inverse image of $\mathfrak{m}_{X,x}$ under the map $f^\#$.

(a) Define composition of morphisms of ringed spaces (in what should be a fairly obvious way) and show that you get a category.

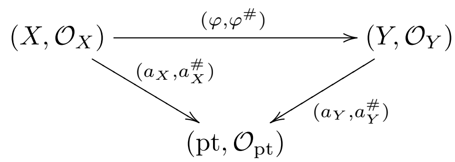
(b) Suppose X is an open subset of $\mathbb{C}^n$, and let $\mathcal{O}_X$ denote the sheaf of analytic functions on X. Show that $(X, \mathcal{O}_X)$ is a locally ringed space. What is the maximal ideal $\mathfrak{m}_{X,x}$?

(c) Suppose that X is as above and Y is an open subset of $\mathbb{C}^m$. Let $\varphi : X \to Y$ be a holomorphic map from X to Y. Define $\varphi^\# : \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ by $f \mapsto f \circ \varphi$. Show that $(\varphi, \varphi^\#)$ is a morphism of ringed spaces.

(d) Show that the morphism of ringed spaces above is, in fact, a morhphism of locally ringed spaces.

(e) Write pt for the ringed space consisting of a single point and let $\mathcal{O}_{\mathrm{pt}}$ denote the constant sheaf $\mathbb{C}$. If X is an open subset of $\mathbb{C}^n$ as above define $a_X : X \to \mathrm{pt}$ to be the map taking every element of X to a point. Show that, if $\varphi : X \to Y$ is as above, then the diagram of locally ringed spaces

(1)



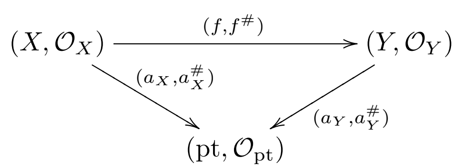
commutes.

(f) The maps a_X and the commutativity of the diagram (1) means essentially is that $\mathcal{O}_X$ and $\mathcal{O}_Y$ are sheaves of $\mathbb{C}$-algebras, and $\varphi^\#$ is a morphism of sheaves of $\mathbb{C}$-algebras. Explain why this is so.

(g) Suppose A and B are local $\mathbb{C}$-algebras with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$ respectively. Suppose that the residue fields $A/\mathfrak{m}_A$ and $B/\mathfrak{m}_B$ are both isomorphic to $\mathbb{C}$ (via the canonical maps from $\mathbb{C}$ to the residued fields). Show that any $\mathbb{C}$-algebra homomorphism $h : A \to B$ is automatically a local ring homomorphism. That is, show that $h^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

(h) Suppose X and Y are open subsets of $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively as above. Suppose $(f, f^\#) : (X, \mathcal{O}_X) \to \mathcal{O}_Y$ is any morphism of ringed spaces making the diagram

(2)



commute. Show that $(f, f^\#)$ is automatically a morphism of locally ringed spaces.

(i) Suppose $(f, f^\#)$ is a morphism of the ringed space $X \subset \mathbb{C}^n$ to $Y \subset \mathbb{C}^m$ as above. Show that f is, in fact, holomorphic (considered as a map from the open subset X of $\mathbb{C}^n$ to the open subset Y of $\mathbb{C}^m$). Then show that $f^\#$ is given by $\lambda \mapsto \lambda \circ f$.

(j) Define a space ringed by $\mathbb{C}$-algebras to be a ringed space $(X, \mathcal{O}_X)$ equipped with a map $(a_X, a_X^\#) : (X, \mathcal{O}_X) \to (\mathrm{pt}, \mathcal{O}_{\mathrm{pt}})$. These form a category where the morphisms are commutative diagrams as in (2). Suppose $(X, \mathcal{O}_X)$ is a space ringed by $\mathbb{C}$-algebras and U is an open subset of X. Explain why the restriction of $\mathcal{O}_X$ to U naturally induces on U the structure of a space ringed by $\mathcal{O}_X$-algebras.

(k) Define a complex manifold of dimension n to be a metrizable space $(X, \mathcal{O}_X)$ ringed by $\mathbb{C}$-algebras covered by open sets $U \in I$ such that, for each U, $(U, \mathcal{O}_U)$ is isomorphic to $(V, \mathcal{O}_V)$ where V is open in $\mathbb{C}^n$ and $\mathcal{O}_V$ is the $\mathbb{C}$-algebra of holomorphic functions on V. Explain why this definition is equivalent to the one given in class. (Or to one in a text.)