A Convex Separation Theorems

Notation: For vectors $\vec{x}, \vec{y} \in \mathbb{R}^M$ let $\vec{x} \cdot \vec{y} = x_1y_1 + \cdots + x_My_M$ denote the inner product; let $\|\vec{x}\| := (\vec{x} \cdot \vec{x})^{1/2}$ denote the Euclidian norm.

We first state the basic convex separation theorem:

**Theorem 6.** Assume that the set $C \subset \mathbb{R}^M$ is closed, convex, and does not contain the origin $\vec{0}$.
Then there exists $\vec{y} \in \mathbb{R}^M$ and $\alpha > 0$ such that

$$\vec{y} \cdot \vec{x} \geq \alpha \text{ for all } \vec{x} \in C$$

**Proof. Idea:**

**Step 1:** Choose $\vec{y}$ as the point in $C$ which is closest to the origin.

**Step 2:** Show: For $\vec{x} \in C$ we have $\vec{y} \cdot (\vec{x} - \vec{y}) \geq 0$ and hence $\vec{y} \cdot \vec{x} \geq \|\vec{y}\|^2 > 0$.

**Step 1:** Consider the ball $B_r := \{\vec{x} \in \mathbb{R}^M \mid \|\vec{x}\| \leq r\}$ and pick $r$ such that $X := C \cap B_r$ is nonempty. This set is closed and bounded, hence compact. Therefore the function $\vec{x} \mapsto \|\vec{x}\|$ attains its minimum at a point $\vec{y} \in X$. For $\vec{x} \notin B_r$ we have $\|\vec{x}\| \geq r \geq \|\vec{y}\|$. For $\vec{x} \in C \cap B_r$ we also have $\|\vec{x}\| \geq \|\vec{y}\|$. Hence

$$\forall \vec{x} \in C \quad \|\vec{x}\| \geq \|\vec{y}\|. \quad (66)$$

**Step 2:** For $\vec{x} \in C$ all the points on the line segment connecting $\vec{x}$ and $\vec{y}$ are in $C$ since it is convex set:

$$\forall t \in [0, 1] \quad \vec{y} + t(\vec{x} - \vec{y}) \in C \quad (67)$$

From (66), (67) we get by multiplying out

$$\|\vec{y} + t(\vec{x} - \vec{y})\|^2 \geq \|\vec{y}\|^2$$

$$\|\vec{y}\|^2 + 2t\vec{y} \cdot (\vec{x} - \vec{y}) + t^2 \|\vec{x} - \vec{y}\|^2 \geq \|\vec{y}\|^2$$

$$\forall t \in (0, 1) \quad 2\vec{y} \cdot (\vec{x} - \vec{y}) + t \|\vec{x} - \vec{y}\|^2 \geq 0$$

Hence $\vec{y} \cdot (\vec{x} - \vec{y}) \geq 0$, i.e.,

$$\forall \vec{x} \in C : \quad \vec{y} \cdot \vec{x} \geq \|\vec{y}\|^2 > 0$$

We will use the following corollary:

**Theorem 7.** Assume that $V$ is a subspace of $\mathbb{R}^M$, and that the set $K \subset \mathbb{R}^M$ is convex, closed, and bounded. If $V \cap K = \emptyset$ there exists $\vec{y} \in \mathbb{R}^M$ and $\alpha > 0$ such that

$$\vec{y} \cdot \vec{z} = 0 \quad \text{for all } \vec{z} \in V \quad (68)$$

$$\vec{y} \cdot \vec{z} \geq \alpha \quad \text{for all } \vec{z} \in K \quad (69)$$

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Proof. Let $\mathcal{C} = \mathcal{K} - \mathcal{V} = \{x - y \mid x \in \mathcal{K}, y \in \mathcal{V}\}$. This set is convex. Since $\mathcal{V}$ is closed and $\mathcal{K}$ is closed and bounded the set $\mathcal{C}$ is closed.

[Note: We need that $\mathcal{K}$ is bounded. The sum of two closed sets is not always closed! Consider e.g. $\mathcal{K} = \{(x, y) \in \mathbb{R}^2 \mid y \geq e^x\}$ and $\mathcal{V} = \{(x, 0) \mid x \in \mathbb{R}\}$.]

Since $\mathcal{V} \cap \mathcal{K} = \emptyset$ we have $0 \notin \mathcal{C}$. By Theorem 6 there exists $\vec{y} \in \mathbb{R}^M$ and $\alpha > 0$ such that $\vec{y} \cdot \vec{z} \geq \alpha$ for all $\vec{z} \in \mathcal{C}$. Hence

$$\forall \vec{x} \in \mathcal{K}, \ \forall \vec{z} \in \mathcal{V} : \quad \vec{y} \cdot (\vec{x} - \vec{z}) \geq \alpha$$

By taking $\vec{z} = 0$ we obtain (69). Now let $\vec{x} \in \mathcal{K}$ and use $\lambda \vec{z}$ instead of $\vec{z}$ with $\lambda \in \mathbb{R}$. This yields

$$\forall \lambda \in \mathbb{R} : \quad \lambda(\vec{y} \cdot \vec{z}) \leq \vec{y} \cdot \vec{x} - \alpha$$

Therefore we must have $\vec{y} \cdot \vec{z} = 0$ and we obtain (68). $\square$