Practice problems and solutions for Exam 2

1. Consider the (x, y) data points $(-1, 2)$, $(1, 1)$, $(2, 0)$. We want to fit the data with a function $g(x) = c_1 + c_2 x^2$

(a) Find the best least squares fit by hand.

Here $y = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. With $\phi_1(x) = 1$ and $\phi_2(x) = x^2$ we have $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 4 \end{bmatrix}$. The normal equations are $A^\top A c = A^\top y$:

$$\begin{bmatrix} 3 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

which gives $c_1 = 2$, $c_2 = -\frac{1}{2}$.

(b) Write a Matlab program which uses the backslash command to solve this problem.

```
x = [-1;1;2]; y=[2;1;0];
A = [x.^0,x.^2];
c = A\y;
```

2. We want to find x such that $x + x^5 = 3$.

(a) Perform one step of the bisection method with $a_0 = 1$, $b_0 = 2$. Find k such that $|b_k - a_k| \le 10^{-6}$.

$f(x) = x^5 + x - 3$, $f(a_0) = -1 < 0$, $f(b_0) = 31 > 0$, $c_0 = (a_0 + b_0)/2 = 1.5$, $f(1.5) = 1.5^5 + 1.5 - 3 = 1.5^5 - 1.5 > 0$, hence $[a_1, b_1] = [a_0, c_0] = [1, 1.5]$. So we have $x_* \in (1, 1.5)$.

We have $|b_k - a_k| = 2^{-k} |b_0 - a_0| = 2^{-k}$. We have

$$2^{-k} \le 10^{-6} \iff (-k) \log 2 \le \log(10^{-6}) \iff k \ge \frac{\log(10^6)}{\log 2} = 19.93,$$

hence we need $k \ge 20$.

(b) Perform one step of the secant method with $x_0 = 1$, $x_1 = 2$ to find x_2.

$$x_2 = x_1 - f(x_1) \frac{x_1 - x_0}{f(x_1) - f(x_0)} = 2 - 31 \frac{2 - 1}{31 - (-1)} = 2 - \tfrac{31}{32} = 1 + \tfrac{1}{32}$$

(c) Will the Newton method converge if we start with x_0 sufficiently close to the solution x_*? Explain.

We showed: If $f, f', f''$are continuous and $f'(x_*) \neq 0$, then the Newton method converges for x_0 sufficiently close to x_*. Here $f(x) = x^5 + x - 3$, $f'(x) = 5x^4 + 1$. We know from the intermediate value theorem that there is a root in the interval $(1, 2)$. Since $f'(x) \ge 1 > 0$ there is a unique root, and we must have $f'(x_*) > 0$. So all the assumptions of the theorem are satisfied.

3. Consider the nonlinear system

$$x_1 + x_1 x_2 + x_2 = 2, \qquad x_1 - x_2 - x_1 x_2^2 = 0$$

(a) Perform one step of the Newton method starting with initial guess $x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We have $\mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_1 + x_1 x_2 + x_2 - 2 \\ x_1 - x_2 - x_1 x_2^2 \end{bmatrix}$ with the Jacobian matrix $f'(\mathbf{x}) = \begin{bmatrix} 1 + x_2 & x_1 + 1 \\ 1 - x_2^2 & -1 - 2x_1 x_2 \end{bmatrix}$. For $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we get $\mathbf{y} = \mathbf{f}(\mathbf{x}^{(0)}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $A = f'(\mathbf{x}^{(0)}) = \begin{bmatrix} 2 & 2 \\ 0 & -3 \end{bmatrix}$. Solving the linear system $A\mathbf{d} = -\mathbf{y}$ gives $\mathbf{d} = \begin{bmatrix} -1/6 \\ -1/3 \end{bmatrix}$ and the new approximation $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \mathbf{d} = \begin{bmatrix} 5/6 \\ 2/3 \end{bmatrix}$.

(b) Write a Matlab program which uses the Newton method to find a solution, starting with initial guess $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The program should print out the approximation for $\mathbf{x}$ after each iteration.

```
f = @(x) [ x(1)+x(1)*x(2)+x(2)-2 ; x(1)-x(2)-x(1)*x(2)^2 ];
fp = @(x) [ 1+x(2), x(1)+1; 1-x(2)^2, -1-2*x(1)*x(2)];
x = [1;1];
while 1
  b = f(x); A = fp(x);
  d = -A\b;
  x = x + d                    % prints out x for each iteration
  if norm(d)<1e-14
    break
  end
end
```

4. Consider the integral $I = \int_0^2 x^3 dx$.

(a) Find the value of the **(i)** midpoint rule, **(ii)** trapezoid rule, **(iii)** Simpson rule (on the whole interval). Find an upper bound for the error $|Q - I| \leq \cdots$ using the error formulas in each case.
Here $a = 0$, $b = 2$ and $f(x) = x^3$, so we obtain

$$Q^{\text{Midpt}} = 2 \cdot f(1) = 2, \qquad Q^{\text{Trap}} = 2 \cdot \frac{f(0) + f(2)}{2} = 8, \qquad Q^{\text{Simpson}} = 2 \cdot \frac{f(0) + 4 \cdot f(1) + f(2)}{6} = 4$$

For the error estimates we have $f''(x) = 6x$ and $\max_{[0,2]} |f''(x)| = 12$ yielding

$$\begin{aligned} \left|Q^{\text{Midpt}} - I\right| &\leq \frac{(b-a)^3}{24} \max_{[0,2]} |f''(x)| = \frac{2^3}{24} \cdot 12 = 4 \\ \left|Q^{\text{Trap}} - I\right| &\leq \frac{(b-a)^3}{12} \max_{[0,2]} |f''(x)| = \frac{2^3}{12} \cdot 12 = 8 \\ \left|Q^{\text{Simpson}} - I\right| &\leq \frac{(b-a)^5}{90 \cdot 32} \max_{[0,2]} \left|f^{(4)}(x)\right| = 0 \end{aligned}$$

since we have $f^{(4)}(x) = 0$ in this case. Recall that the Simpson rule is actually exact if $f(x)$ is a polynomial of degree≤ 3.

(b) Find the value of the composite trapezoid rule Q_2^{Trap} with 2 subintervals of equal size.
Here $N = 2$ and $h = (b-a)/N = 1$ yielding

$$Q_2^{\text{Trap}} = h \cdot \frac{[f(0) + f(1)] + [f(1) + f(2)]}{2} = 1 \cdot \frac{0 + 2 \cdot 1 + 8}{2} = 5$$

(c) Find a value N such that we can guarantee $\left|Q_N^{\text{Trap}} - I\right| \leq 10^{-10}$ for the composite trapezoid rule with N intervals of equal size.
Using $\max_{[0,2]} |f''(x)| = 12$ we obtain

$$\left|Q_N^{\text{Trap}} - I\right| \leq \frac{1}{12} \cdot \frac{(b-a)^3}{N^2} \max_{x \in [a,b]} |f''(x)| = \frac{1}{12} \cdot \frac{2^3}{N^2} \cdot 12 = \frac{8}{N^2}$$

We need to choose N such that

$$\frac{8}{N^2} \leq 10^{-10} \quad \Longleftrightarrow \quad \sqrt{\frac{8}{10^{-10}}} \leq N,$$

i.e., $N \geq 10^5 \cdot \sqrt{8} \approx 282842.7$. Therefore we need $N \geq 282843$ to guarantee $\left|Q_N^{\text{Trap}} - I\right| \leq 10^{-10}$.