1 Polynomial interpolation

Consider the space of polynomials of degree at most $n$:

$$\mathcal{P}_n := \{a_0 + a_1 x + \cdots + a_n x^n \mid a_j \in \mathbb{R}\}.$$

For $n + 1$ distinct nodes $x_0, \ldots, x_n \in \mathbb{R}$ and given values $f_i$ there exists a unique polynomial $p \in \mathcal{P}_n$ such that

$$p(x_i) = f_i \quad i = 0, \ldots, n \quad (1)$$

2 Lagrange interpolation

The so-called Lagrange polynomials

$$\ell_j(x) := \prod_{k=0}^n \frac{x-x_j}{x_j-x_k} \quad (2)$$

obviously satisfy $\ell_j \in \mathcal{P}_n$ and

$$\ell_j(x_k) = \begin{cases} 0 & \text{for } k \neq j \\ 1 & \text{for } k = j \end{cases}$$

Hence we have the Lagrange interpolation formula

$$p(x) = \sum_{j=0}^n f_j \ell_j(x) \quad (3)$$

since this function obviously satisfies $p \in P_n$ and (1).

**Computational work:**

- evaluating (3) takes $O(n^2)$ operations for each $x$-value
- if we add a new data pair $(x_{n+1}, f_{n+1})$ we have to start from scratch, and we need $O(n^2)$ operations.

3 Modified Lagrange formula

With

$$\ell(x) := \prod_{j=0}^n (x-x_j)$$

$$w_j := \prod_{k=0}^n \frac{1}{(x_j-x_k)} \quad (4)$$
we obtain from (3) the modified Lagrange formula

$$p(x) = \ell(x) \sum_{j=0}^{n} \frac{w_j}{x-x_j} f_j$$

(5)

**Computational work:** Typically we want to evaluate $p(x)$ for many $x$-values.

- first: compute $w_0, \ldots, w_n$ with $O(n^2)$ operations
  then: for each $x$-value evaluate (5) using $O(n)$ operations
- if we add a new data pair $(x_{n+1}, f_{n+1})$:
  update $w_1, \ldots, w_n$: $n$ operations
  compute $w_{n+1}$: $n + 1$ operations
  evaluate (5): $O(n)$ operations
  so the total work is $O(n)$

We can use the updating idea to compute $w_0, \ldots, w_n$ as follows:

$$w_0 := 1$$

for $j = 1$ to $n$:

- for $k = 1$ to $j - 1$:
  $$w_k := (x_k - x_j) w_k$$
  $$w_j := \prod_{k=0}^{j-1} (x_j - x_k)$$
- for $j = 0$ to $n$:
  $$w_j := 1/w_j$$

4 Barycentric formula

Note that in the case $f_0 = \cdots = f_n = 1$ the interpolating polynomial must be $p(x) = 1$, hence

$$1 = \ell(x) \sum_{j=0}^{n} \frac{w_j}{x-x_j}$$

Solving this for $\ell(x)$ and substituting this in (5) gives the barycentric Lagrange formula

$$p(x) = \sum_{j=0}^{n} \frac{w_j}{x-x_j} f_j$$

$$\sum_{j=0}^{n} \frac{w_j}{x-x_j}$$

(6)

**Computational work:** same as for the modified Lagrange formula.

5 Numerical stability

For the evaluation in machine arithmetic the modified Lagrange formula gives the best result. For a “good choice of interpolation nodes” (such as Chebyshev nodes) the barycentric formula also gives good results, but for a “bad choice of interpolation nodes” (such as equidistant nodes) the barycentric formula may give a larger error than the modified Lagrange formula.

References