AMSC 666 Assignment 2, Problem 1

Here $\| \cdot \|$ always denotes the 2-norm. We want to recover a blurred signal.

1. Let $f \in \mathbb{R}^n$ be a signal. The matrix $A \in \mathbb{R}^{n \times n}$ corresponds to a smoothing operator (see web page). We only have the blurred noisy signal $b = Af + e$ where $e$ is a tiny noise vector. We want to find an approximation $g$ of the original signal $f$.

(a) What happens when you use $g = A \backslash b$? Explain the size of $g$. Plot the singular values using `semilogy(s);axis tight;grid on;`

(b) Assume that $S \in \mathbb{R}^{n \times n}$ is diagonal with the singular values $\sigma_1 \geq \sigma_2 \geq \cdots$ on the diagonal. Let $B = SF + E$ where $E$ corresponds to “small noise”. We pick $k \leq n$. We define $G_1, \ldots, G_k$ by solving the first $k$ equations of $SG = B$, and let $G_{k+1} = \cdots = G_n = 0$. Denote this vector by $G_{(k)}$. Find an expressions for $\|SG_{(k)} - B\|^2$ and $\|G_{(k)}\|^2$ in terms of $B_j, \sigma_j$ (which we know). Find an expression for $\|G_{(k)} - F\|^2$ in terms of $F_j, E_j, \sigma_j$ (note that we don’t know $F_j, E_j$).

Show that $\|SG_{(k)} - B\|$ is decreasing and $\|G_{(k)}\|$ is increasing w.r.t. $k$ if $B_j \neq 0, \sigma_j \neq 0$.

Assume we know $\|E\| \approx \tilde{\varepsilon}$. The idea is to increase $k$ until $\|SG_{(k)} - B\| = \|Ag_{(k)} - b\| \leq \tilde{\varepsilon}$, since $\|Af - b\| \approx \tilde{\varepsilon}$, and increasing $k$ much further will give $g_{(k)}$ with a norm larger than $\|f\|$.

(c) The components of $e_j$ are normally distributed. We use `epsilon=epsilon*sqrt(chi2inv(.99,n))`, then $\|e\| \leq \tilde{\varepsilon}$ for 99% of the time. Find $\sigma_k, B_k$ and use this to determine the smallest $k$ such that $\|Ag_{(k)} - b\| \leq \tilde{\varepsilon}$. Plot $f$ and $g_{(k)}$ together. Experiment with other values of $k$, and try to find a better approximation $g_{(k)}$ for $f$.

(d) With $S, F, E$ as in (b): We now consider all vectors $G$ with $\|SG - B\| = \tilde{\varepsilon}$ and want to minimize $\|G\|$. Use the Lagrange multiplier $\lambda$ and take the partial derivatives of $\lambda \|SG - B\|^2 + \|G\|^2$ with respect to $G_j$. This gives the components of the solution vector $G^{(\lambda)}$ depending on $\lambda$ (which we still have to find). Show that for $\lambda \in [0, \infty)$ the values $\|SG^{(\lambda)} - B\|$ are decreasing from $\|B\|$ to 0, and $\|G^{(\lambda)}\|$ is increasing if $B_j \neq 0, \sigma_j \neq 0$. Hence we can determine $\lambda$ as the unique solution of $\|SG^{(\lambda)} - B\| = \tilde{\varepsilon}$.

(Note: $\|G^{(\lambda)}\|$ increases as $\tilde{\varepsilon}$ decreases. Hence we have actually minimized $\|G\|$ among all vectors $G$ with $\|SG - B\| \leq \tilde{\varepsilon}$. We want to make sure that $F$ is contained in this set, so we want $\|SF - B\| = \|E\| \leq \tilde{\varepsilon}$. So $\tilde{\varepsilon}$ should be a tight upper bound for $\|E\|$.)

(e) Find $\sigma_k, B_k$. Compute the unique $\lambda$ with $\|SG^{(\lambda)} - B\| = \|Ag^{(\lambda)} - b\| = \tilde{\varepsilon}$ using `fzero` (see web page). Plot $f$ and $g^{(\lambda)}$ together. Experiment with other values of $\lambda$, and try to find a better approximation $g^{(\lambda)}$ for $f$.  
