Solution for Midterm Exam AMSC/CMSC 666, Fall 2018

1. Let \( w(x) = x \). We define the inner product \((u, v) := \int_0^1 u(x)v(x) \, dx\) and norm \( \|u\| := (u, u)^{1/2} \).

(a) Find \( p \in \mathcal{P}_1 \) such that \( \|x^2 - p\| \) is minimal: let \( p(x) = c_1 + c_2 x \), write down a linear system \( Ac = b \) for \( c = [c_1^T c_2^T]^T \) and solve it.

Normal equations: \[
\begin{bmatrix}
(1, 1), & (x, 1), & (x, x)
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
(x^2, 1) \\
x^2, x
\end{bmatrix}
gives \begin{bmatrix}
\frac{1}{3} & \frac{1}{2} & \frac{1}{3}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{12}
\end{bmatrix}
and \begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
-\frac{3}{5}
\end{bmatrix}
\]

(b) Find \( \|x^2\|^2 \). Show that \( \|p\|^2 = c^T b \). Express \( \|x^2 - p\|^2 \) in terms of these two numbers.

\( \|x^2\|^2 = \frac{1}{3} \) and \( \|c_1 \cdot 1 + c_2 \cdot x\|^2 = c^T Ac = c^T b = \frac{33}{200} \). Since \((x^2 - p, p) = 0\) we have \( \|p\|^2 + \|x^2 - p\|^2 = \|x^2\|^2 \)

(“Pythagoras”), hence \( \|x^2 - p\|^2 = \|x^2\|^2 - \|p\|^2 = \frac{1}{3} - \frac{33}{200} = \frac{147}{200} \).

2. Consider the function \( u(x) = \begin{cases} 1 & x < 0 \\ \cos x & x \geq 0 \end{cases} \) on the interval \([-1, 1]\).

(a) Find a bound \( |a_k| \leq Ck^{-\beta} \) for the Chebyshev coefficients.

We have \( u'(x) = \begin{cases} 0 & x < 0 \\ -\sin x & x > 0 \end{cases} \) and \( u''(x) = \begin{cases} 0 & x < 0 \\ -\cos x & x > 0 \end{cases} \). Since \( u'' \) is piecewise continuous (with a jump at 0) we have \( u'' \in BV \), but \( u'' \notin BV \). Therefore we have \( m = 2 \) and the result from class gives \( |a_k| \leq Ck^{-m-1} \), i.e., \( \beta = 3 \).

(b) Use (a) to prove a sharp estimate

\[
\min_{p_n \in \mathcal{P}_n} \|u - p_n\|_{L^2([-1, 1])} \leq Cn^{-\alpha}
\]

*Hint:* Use the “obvious choice” \( p_n \in \mathcal{P}_n \). For which norms did we prove bounds \( \|u - p_n\| \leq \cdots \)? Use this to find the best value for \( \alpha \).

We define \( p_n := \sum_{k=0}^n a_k T_k \). We showed in class (i) \( \|u - p_n\|_{\infty} \leq \sum_{k=n+1}^\infty |a_k| \leq C' n^{-\beta+1} \) and (ii) \( \|u - p_n\|_{L^2_k} = \left(\sum_{k=n+1}^\infty |a_k|^2\right)^{1/2} \leq C'' n^{-\beta+\frac{1}{2}} \) where \( \|e\|_{L^2_k}^2 = \int_1^1 e(x)^2 \, dx (1-x^2)^{-1/2} \, dx \).

Using (i) we get for \( e := u - p_n \) the bound \( \|e\|_{L^2_k}^2 = \int_1^1 |e(x)|^2 \, dx \leq \int_1^1 \|e\|_{\infty}^2 \, dx = 2 \|e\|_{\infty}^2 \leq c \left(n^{-\beta+1}\right)^2 \)

Using (ii) we get the bound \( \|e\|_{L^2_k}^2 = \int_1^1 |e(x)|^2 \, dx \leq \int_1^1 |e(x)|^2 \, w(x) \, dx \leq c' \left(n^{-\beta+\frac{1}{2}}\right)^2 \) since \( w(x) = (1-x^2)^{-1/2} \geq 1 \).

Hence we obtain \( \alpha = \beta - \frac{1}{2} = \frac{5}{2} \).

3. Let \( w(x) = x \). We want to approximate integrals of the form \( I(f) := \int_0^1 u(x) \, w(x) \, dx \) by a quadrature formula \( Q_n(u) := w_1 u(x_1) + \cdots + w_n u(x_n) \).

(a) Find \( x_1, w_1 \) so that the quadrature formula \( Q_1 \) is exact for the highest polynomial degree \( k \). What is \( k \)?

The orthogonal polynomial \( p_1(x) = x + c_0 \) satisfies \( 0 = (p_1, 1) = (x, 1) + (1, 1)^T c_0 = \frac{1}{2} + \frac{1}{2} c_0 \), hence \( c_0 = -\frac{1}{2} \). Solving \( p_1(x_1) = 0 \) gives \( x_1 = \frac{2}{3} \). We must have \( Q[1] = I[1] \), hence \( w_1 \cdot 1 = \frac{1}{2} \). Gaussian quadrature \( Q_n \) is exact for \( \mathcal{P}_2n-1 \), hence \( k = 2n - 1 \).

(b) Let \( Q_n \) denote the Gaussian quadrature formula with \( n \) nodes. Assume that \( u(x) \) is continuous and prove that \( \lim_{n \to \infty} Q_n(u) = I(u) \). State all theorems you use for the proof.

*Hint:* Consider polynomials \( p_k \in \mathcal{P}_k \) approximating \( u \).

(1) The Weierstrass approximation theorem states: For a continuous function \( u \) there exists a sequence \( p_k \in \mathcal{P}_k \) such that \( \|u - p_k\|_{\infty} \to 0 \) as \( k \to \infty \).

(2) Gaussian quadrature \( Q_n \) is exact for \( \mathcal{P}_k \) with \( k = 2n - 1 \), and we have \( w_j > 0 \) for \( j = 1, \ldots, n \).

(3) We showed for a quadrature rule \( Q[f] := w_1 f(x_1) + \cdots + w_n f(x_n) \) approximating \( I[f] := \int_a^b f(x) \, w(x) \, dx \): If the rule is exact for \( \mathcal{P}_k \) and \( w_j \geq 0 \) we have \( |Q[f] - I[f]| \|f - p_k\|_{\infty} \) for any \( p_k \in \mathcal{P}_k \).

With \( p_k \) from (1) we get from (2), (3) that \( \|Q_k[u] - I[u]\| \leq 2 \|I[1]\| \|u - p_k\|_{\infty} \to 0 \) as \( k \to \infty \).

(c) Design a quadrature rule \( \int_0^1 f(x) \, dx \approx Q[f] := W_0 f(0) + W_1 f(X_1) \) which is exact for \( \mathcal{P}_2 \). Show that you can obtain such a rule from (a). *Hint:* Write \( p \in \mathcal{P}_2 \) as \( p(x) = p(0) + x \cdot q(x) \). First show that your rule works if \( p(0) = 0 \). Then choose \( W_0 \) such that it works for all \( p \in \mathcal{P}_2 \).

Let \( p \in \mathcal{P}_2 \), then \( p(x) = p(0) + x \cdot q(x) \) with \( q \in \mathcal{P}_1 \). First assume \( p(0) = 0 \). From (a) we get \( \int_0^1 p(x) \, dx = \int_0^1 x \cdot q(x) \, dx = w_1 q(x_1) = \frac{w_1}{2} p(x_1) \). Hence we choose \( X_1 := x_1 = \frac{2}{3} \) and \( W_1 = \frac{w_1}{2} = \frac{3}{4} \) and get \( \int_0^1 p(x) \, dx = W_1 p(x_1) \) for all \( p \in \mathcal{P}_2 \) with \( p(0) = 0 \). We can integrate the function 1 exactly if we define \( W_0 = 1 - W_1 = \frac{1}{4} \). Then we have for arbitrary \( p \in \mathcal{P}_2 \) that \( \int_0^1 p(x) \, dx = \int_0^1 p(0) \, dx + \int_0^1 x \cdot q(x) \, dx = Q[p(0) \cdot 1] + Q[x \cdot q(x)] = Q[p] \).