1 Approximation

- $2\pi$ periodic function: $v(t) = \sum_{k=-\infty}^{\infty} c_k e^{i k t}$
  function on $[-1, 1]$: $u(x) = \sum_{k=0}^{\infty} a_k T_k(x)$

- approximation method 1: orthogonal projection
  find $v_n \in \mathcal{T}_n$ such that $\|v - v_n\|_{L^2} = \min$: $v_n(t) = \sum_{k=-n}^{n} c_k e^{i k t}$
  find $u_n \in \mathcal{P}_n$ such that $\|u - u_n\|_{L^2} = \min$: $u_n(x) = \sum_{k=0}^{n} a_k T_k(x)$
  $\|v - v_n\|_{L^2} = 2\pi \sum_{|k| > n} |c_k|^2$, $\|v - v_n\|_\infty \leq \sum_{|k| > n} |c_k|$
  $\|u - u_n\|_{L^2} = \frac{n}{2} \sum_{k>n} |a_k|^2$, $\|u - u_n\|_\infty \leq \sum_{k>n} |a_k|$

- approximation method 2: interpolation
  find $q_n \in \begin{cases} \mathcal{T}_n^\text{cos} & \text{if } m = 2n \text{ even} \\ \mathcal{T}_n & \text{if } m = 2n + 1 \text{ odd} \end{cases}$ such that $q_n(t_j) = v(t_j)$ for equidistant nodes $t_j = j \frac{2\pi}{m}$, $j = 0, \ldots, m - 1$
  find $p_n \in \mathcal{P}_n$ such that $p_n(x_j) = u(x_j)$ for Chebyshev nodes $x_j = \cos(\frac{\pi j}{n})$, $j = 0, \ldots, n$
  $\|v - q_n\|_\infty \leq 2 \sum_{|k| > n} |c_k|$
  $\|u - p_n\|_\infty \leq 2 \sum_{k>n} |a_k|$

- smoothness implies decay of coefficients:
  if derivative $v^{(m)}(t)$ has bounded variation: $|c_k| \leq C k^{-m-1}$
  if derivative $u^{(m)}(x)$ has bounded variation: $|a_k| \leq C k^{-m-1}$
  if $v(t)$ is analytic: exists $\rho > 1$ such that $|c_k| \leq C \rho^{-k}$
  if $u(x)$ is analytic: exists $\rho > 1$ such that $|a_k| \leq C \rho^{-k}$

2 Convection-diffusion problem in 1D:

Find function $u(x)$ for $x \in \Omega = (a, b)$

$$-(c(x)u'(x))' + bu'(x) = f(x) \quad \text{in } \Omega, \ u = 0 \quad \text{on } \partial \Omega$$

variational formulation: find $u \in V$ such that

$$\forall v \in V : \quad a(u, v) = \ell(v)$$

where $V = H^1_0(\Omega)$, $a(u, v) = \int_{\Omega} [c(x)u'(x)v'(x) + bu'(x)v(x)] \, dx$, $\ell(v) = \int_{\Omega} f(x)v(x) \, dx$

Finite Element discretization:

subdivide $(a, b)$ into $N$ subintervals of length $h = (b - a)/N$

$V_h$ are piecewise linear functions with zero boundary values

We obtain stiffness matrix $A = A_{\text{diff}} + A_{\text{conv}}$ with

$$L := \|A\|_2 \leq c' h^d h^{-2}, \quad \gamma := \lambda_{\min} \left( \frac{1}{2} (A + A^\top) \right) \geq c h^d$$

$$\|A_{\text{diff}}\|_2 \leq C' h^d h^{-2}, \quad \|A_{\text{conv}}\|_2 \leq C'' h^d h^{-1}$$

Here $d$ is the dimension, in the 1-dimensional case we have $d = 1$. 
Large sparse linear system \( Au = b \)

Assume that symmetric part \( \frac{1}{2}(A + A^\top) \) is positive definite. (In this course we only consider this case, since it covers many equilibrium problems and nonsymmetric problems like the convection-diffusion problem.)

\[
L := \|A\|_2, \quad \gamma := \lambda_{\min} \left( \frac{1}{2}(A + A^\top) \right) > 0
\]

If \( A \) is symmetric positive definite: solving \( Au = b \) is equivalent to minimizing \( f(u) := \frac{1}{2} (Au, u) - (b, u) \).

\((u, v)_A := (Au, v)\) is inner product and \( \|u\|_A = (Au, u)^{1/2} \) is norm. Since

\[
\|u - u_*\|_A^2 = (u, u)_A - 2\left( (u_*, u)_A + (u, u_*)_A \right) - (b, u) = 2f(u) + (u_*, u_*)_A
\]

minimizing \( f(u) \) over a subspace is equivalent to minimizing \( \|u - u_*\|_A \) over a subspace.

Richardson iteration:

Residual \( r^{(k)} := b - Au^{(k)} \)

\( u^{(k+1)} = u^{(k)} + \alpha_k r^{(k)} \)

Three ways to choose \( \alpha_k \):

1. Choose fixed \( \alpha > 0 \). Note: method will not converge if \( \alpha \) is too large.

2. Choose \( \alpha_k \) such that \( \|r^{(k+1)}\|_2 \) is minimal: 1 step min. res. method aka GMRES(1) method.

\[
\text{minimize } \|r^{(k+1)}\|_2 = \|r^{(k)} - \alpha_k Ar^{(k)}\|_2, \text{ hence } (r^{(k)} - \alpha Ar^{(k)}), Ar^{(k)} = 0 \text{ and }
\alpha_k = \frac{(r^{(k)}, Ar^{(k)})}{(Ar^{(k)}, Ar^{(k)})}
\]

3. If \( A \) is symmetric pos. def.: Choose \( \alpha_k \) such that \( f(u^{(k)}) \) is minimal: steepest descent method aka CG(1) method.

\[
\text{minimize } \|u_* - (u^{(k)} + \alpha_k r^{(k)})\|_A, \text{ hence 0 = (u_* - (u^{(k)} + \alpha_k r^{(k)}), r^{(k)})}_A = (r^{(k)} - \alpha_k Ar^{(k)}, r^{(k)}) \text{ and }
\alpha_k = \frac{(r^{(k)}, r^{(k)})}{(Ar^{(k)}, r^{(k)})}
\]

Error estimates in the three cases:

1. \( \|u^{(k+1)} - u_*\|_2 \leq q \|u^{(k)} - u_*\|_2 \) with \( q = (1 - 2\alpha\gamma + \alpha^2 L^2)^{1/2} \). We have \( q < 1 \) if \( \alpha \in (0, \frac{2\gamma}{L^2}) \)

2. \( \|r^{(k+1)}\|_2 \leq q_s \|r^{(k)}\|_2 \) with \( q = (1 - K^{-1})^{1/2} \) with \( K = \frac{L^2}{\gamma^2} \).

Improved estimates: If \( A \) is symmetric: \( q = 1 - \frac{2}{\kappa + 1}, \kappa = \frac{L}{\gamma} = \text{cond}_2(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \)

If \( A \) is nonsymmetric: \( A = H + S \) where \( H = \frac{1}{2} (A + A^\top) \) is symmetric pos. def., \( S := \frac{1}{2} (A - A^\top) \) is antisymmetric

\[
q_s = (1 - K^{-1})^{1/2}, \quad K = \frac{\|H\|_2}{\gamma} + \left( \frac{\|S\|_2}{\gamma} \right)^2
\]

3. \( \|u^{(k+1)} - u_*\|_A \leq q \|u^{(k)} - u_*\|_A \) with \( q = 1 - \frac{2}{\kappa + 1}, \kappa = \frac{L}{\gamma} = \text{cond}_2(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \).

Full CG method will NOT be on final exam.

3 Nonlinear problems and minimization of functions

Nonlinear equation: find \( x \in \mathbb{R}^n \) such that \( F(x) = 0 \).

Minimization problem: find \( x \in \mathbb{R}^n \) such that \( f(x) \) has local minimum at \( x_* \).

Note: necessary condition is \( F(x) = 0 \) where \( F(x) = \text{grad } f(x) \).
Methods for nonlinear equation $F(x) = 0$

- Richardson iteration: either fixed $\alpha$, or choose $\alpha_k$ to minimize $\|r^{(k)}\|_2 = \|F(x^{(k)})\|_2$. Convergence of order 1 if $F$ satisfies positivity condition, Lipschitz condition.

- Newton method
  - Local convergence of order 2 (if $F'(x_*)$ is nonsingular).

- Quasi-Newton method (e.g. Broyden method): NOT ON FINAL EXAM

Methods for finding local minimum of $f(x)$

Assume that local minimum is “not degenerate”, i.e., $H(x_*)$ is positive definite.

General framework: Current guess $x_k$

- Find search direction $d_k$:
  - Construct positive definite matrix $B_k$
  - Solve $B_k d_k = -F(x_k)$ (this implies $F(x_k)^\top d_k < 0$, i.e., $d_k$ is “descent direction”)

- “Line search”: find $\lambda_k$ such that decrease $f(x_k + \lambda_k d_k) - f(x_k)$ is “sufficiently negative”:
  - Armijo condition: Pick $\alpha \in (0, 1)$, e.g., $\alpha = .1$.
    \[
    f(x_k + \lambda_k d_k) < f(x_k) + \alpha \lambda_k F(x_k)^\top d_k
    \]

Algorithm to choose $\lambda_k$: Let $\lambda_k = 1$. While (1) is not satisfied: $\lambda_k := \lambda_k / 2$.

Methods for constructing positive definite matrix $B_k$:

1. $B_k = I$: Steepest descent method
2. $B_k = F'(x_k) + \alpha I$: If $F'(x_k)$ is positive definite use $\alpha = 0$. If $F'(x_k)$ is not positive definite: pick $\alpha$ sufficiently large such that $B_k = F'(x_k) + \alpha I$ is positive definite.
   - Note: if we are sufficiently close to $x_*$ we will use $\alpha = 0$ and $\lambda_k = 1$, and we have quadratic convergence: $\|x_{k+1} - x_*\| \leq C \|x_k - x_*\|^2$.
3. Quasi-Newton method, e.g. BFGS (NOT ON FINAL EXAM)