AMSC 666 Practice problems for final exam

1. We want to solve the convection-diffusion problem on $\Omega = (-1, 1)$ with the diffusion coefficient $c(x) = \begin{cases} \varepsilon & x < 0 \\ 1 & x \geq 0 \end{cases}$, $\varepsilon = 0.1$, the convection $\beta = 4$, and the right-hand side function $f(x) = 1$. We use a finite element method with piecewise linear functions on a uniform mesh with mesh size $h = 2/N$.

(a) We obtain a linear system $Au = b$. Find the matrix A and the right hand side vector b for $N = 4$.

(b) We solve the linear system with the GMRES(1) method. We want to achieve a residual $\left\|r^{(k)}\right\|_2 \leq \delta$ with a given tolerance δ. What happens with the number of iterations if N goes to infinity?

(c) What happens with the number of iterations in (b) if we have a fixed N, but now let ε go to zero?

2. Let $A = \begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix}$.

(a) Find $\gamma > 0$ such that $(Av, v) \geq \gamma \|v\|_2^2$ for all $v \in \mathbb{R}^2$.
Find L such that $\|Av\|_2 \leq L \|v\|_2$ for all $v \in \mathbb{R}^2$.

(b) Let $b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. We want to solve a linear system $Ax = b$ using the GMRES(1) method, starting with the initial guess $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Find $x^{(1)}$.

(c) Find k such that $\left\|r^{(k)}\right\|_2 \leq 10^{-5}$ where $r^{(k)} := b - Ax^{(k)}$.

3. Let $A = \begin{bmatrix} 0 & 4 \\ -2 & 4 \end{bmatrix}$. We want to solve a linear system $Ax = b$ using Richardson iteration with a fixed value $\alpha > 0$.

(a) Can you find $\gamma > 0$ such that $(Av, v) \geq \gamma \|v\|_2^2$ for all $v \in \mathbb{R}^2$? Explain!

(b) Find the eigenvalues of A. Find $\alpha_* > 0$ such that $0 < \alpha < \alpha_*$ implies convergence, and for $\alpha \geq \alpha_*$ there are initial guesses $x^{(0)}$ for which the Richardson iteration does not converge.
Hint: Let $e^{(k)} := x^{(k)} - x_*$. Show that $e^{(k+1)} = M_\alpha e^{(k)}$ with a certain matrix M_α. Use a basis of eigenvectors.

4. We consider a linear system $Ax_* = b$ with the matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$.

(a) For which $\alpha \in \mathbb{R}$ does the Richardson iteration $x^{(k+1)} = x^{(k)} + \alpha(b - Ax^{(k)})$ converge? For which $\alpha \in \mathbb{R}$ do we get $\left\|x^{(k+1)} - x_*\right\|_A \leq q \left\|x^{(k)} - x_*\right\|_A$ with the smallest q?

(b) We perform k steepest descent iterations starting with $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. How many iterations do we need to obtain $\left\|x^{(k)} - x_*\right\|_A / \|x_*\|_A \leq 10^{-6}$?

5. We want to find a local minimum $x_* \in \mathbb{R}^2$ of the function $f(x) = x_1^4 - x_1^2 + x_2^2$. We start at the initial guess $x^{(0)} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$.

(a) Find the steepest descent direction d. Let $g(\lambda) := f(x^{(0)} + \lambda d)$ and find $g'(0)$.

(b) Find the Newton direction $\tilde{d}$. Let $g(\lambda) := f(x^{(0)} + \lambda \tilde{d})$ and find $g'(0)$.

(c) Modify the Newton method by adding αI with α sufficiently large to $F'(x^{(0)})$. Check that you now obtain a descent direction.

(d) Let $D = \left\{x \in \mathbb{R}^2 \mid x_1 \geq \frac{1}{2}\right\}$. Let $F(x) = \operatorname{grad} f(x)$ and show $(F(y) - F(x), y - x) \geq \gamma \|y - x\|_2^2$ for all $x, y \in D$ with some $\gamma > 0$.