1. We want to solve the convection-diffusion problem on $\Omega = (-1,1)$ with the diffusion coefficient $c(x) = \begin{cases} \varepsilon & x < 0 \\ 1 & x \geq 0 \end{cases}$, $\varepsilon = 0.1$, the convection $\beta = 4$, and the right-hand side function $f(x) = 1$. We use a finite element method with piecewise linear functions on a uniform mesh with mesh size $h = 2/N$.

(a) We obtain a linear system $Ax = b$. Find the matrix $A$ and the right hand side vector $b$ for $N = 4$.

(b) We solve the linear system with the GMRES(1) method. We want to achieve a residual $\|r^{(k)}\|_2 \leq \delta$. What happens with the number of iterations if $N$ goes to infinity?

(c) What happens with the number of iterations in (b) if we have a fixed $N$, but now let $\varepsilon$ go to zero?

2. Let $A = \begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix}$.

(a) Find $\gamma > 0$ such that $(Av,v) \geq \gamma \|v\|_2^2$ for all $v \in \mathbb{R}^2$. Find $L$ such that $\|Av\|_2 \leq L \|v\|_2$ for all $v \in \mathbb{R}^2$.

(b) Let $b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. We want to solve a linear system $Ax = b$ using the GMRES(1) method, starting with the initial guess $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Find $x^{(1)}$.

(c) Find $k$ such that $\|r^{(k)}\|_2 \leq 10^{-5}$ where $r^{(k)} := b - Ax^{(k)}$.

3. Let $A = \begin{bmatrix} 0 & 4 \\ -2 & 4 \end{bmatrix}$. We want to solve a linear system $Ax = b$ using Richardson iteration with a fixed value $\alpha > 0$.

(a) Can you find $\gamma > 0$ such that $(Av,v) \geq \gamma \|v\|_2^2$ for all $v \in \mathbb{R}^2$? Explain!

(b) Find the eigenvalues of $A$. Find $\alpha_\ast > 0$ such that $0 < \alpha < \alpha_\ast$ implies convergence, and for $\alpha \geq \alpha_\ast$ there are initial guesses $x^{(0)}$ for which the Richardson iteration does not converge. 

Hint: Let $e^{(k)} := x^{(k)} - x_\ast$. Show that $e^{(k+1)} = M_\alpha e^{(k)}$ with a certain matrix $M_\alpha$. Use a basis of eigenvectors.

4. We consider a linear system $Ax_\ast = b$ with the matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$.

(a) For which $\alpha \in \mathbb{R}$ does the Richardson iteration $x^{(k+1)} = x^{(k)} + \alpha (b - Ax^{(k)})$ converge? For which $\alpha \in \mathbb{R}$ do we get $\|x^{(k+1)} - x_\ast\|_A \leq q \|x^{(k)} - x_\ast\|_A$ with the smallest $q$?

(b) We perform $k$ steepest descent iterations starting with $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. How many iterations do we need to obtain $\|x^{(k)} - x_\ast\|_A / \|x_\ast\|_A \leq 10^{-6}$?

5. We want to find a local minimum $x_\ast \in \mathbb{R}^2$ of the function $f(x) = x_1^4 - x_1^2 + x_2^2$. We start at the initial guess $x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(a) Find the steepest descent direction $d$. Let $g(\lambda) := f(x^{(0)} + \lambda d)$ and find $g'(0)$.

(b) Find the Newton direction $\tilde{d}$. Let $g(\lambda) := f(x^{(0)} + \lambda \tilde{d})$ and find $g'(0)$.

(c) Modify the Newton method by adding $\alpha I$ with $\alpha$ sufficiently large to $F'(x^{(0)})$. Check that you now obtain a descent direction.

(d) Let $D = \{ x \in \mathbb{R}^2 \mid x_1 \geq \frac{1}{3} \}$. Let $F(x) = \text{grad} f(x)$ and show $(F(y) - F(x), y - x) \geq \gamma \|y - x\|_2^2$ for all $x, y \in D$ with some $\gamma > 0$. 