Assignment #1, due Monday, March 2

1. Let $F: \Omega \to \mathbb{R}^n$ with $\Omega \subset \mathbb{R}^n$. Show that $\|D^2F(x)\langle u,v \rangle\|_{\infty} \leq C \|u\|_{\infty} \|v\|_{\infty}$ with

$$C = \max_i \sum_{j,k} \left| \frac{\partial^2 F_i(x)}{\partial x_j \partial x_k} \right|$$

(b) We want to solve the nonlinear system $x_1 = \cos(x_1 + x_2)/3$, $x_2 = \sin(x_1 - x_2)/3$ using the Newton method with initial guess $x^0 = (0,0)$. Show that you can apply the Newton-Kantorovich theorem and (a). What does the theorem say about the location of $x_*$?

(c) Perform the Newton iteration from (b) in Matlab with 310 digits accuracy (see example code on the course web page). How many iterations do you need to obtain $\|x^k - x_*\|_{\infty} / \|x_*\|_{\infty} \leq 10^{-300}$? How can you see the convergence order in your computations?

(d) Repeat (c) for the Broyden iteration with $x^0 = (0,0)$, $B_0 = I$ (identity matrix). How many iterations do you need to obtain $\|x^k - x_*\|_{\infty} / \|x_*\|_{\infty} \leq 10^{-300}$? What convergence order do you observe numerically? Does $B_k$ converge to $F'(x_*)$? Check the convergence order of $\|(B_k - F'(x_*)) s_k\| / \|s_k\|$.

2. Let $h_0 \in (0, \frac{1}{2}]$ and $f(t) := \frac{1}{2} t^2 - t + h_0$. Let $\rho_- := 1 - \sqrt{1 - 2h_0}$.

(a) Consider the Newton iterates $t_k$ for the equation $f(t) = 0$ with initial guess $t_0 = 0$. Show that $t_k$ satisfies $t_{k-1} < t_k < \rho_-$. Show that this implies $\lim_{k \to \infty} t_k = \rho_-$. 

(b) Show that $t_k$ converges q-linearly for $h_0 = \frac{1}{2}$. Show that $t_k$ converges q-quadratically for $h_0 < \frac{1}{2}$.

(c) Let $a_0 := 1$, $a_{k+1} := a_k(1-h_k)$, $h_{k+1} := \frac{1}{2} h_k^2/(1-h_k)^2$. Show that $t_{k+1} - t_k = h_k a_k$ for $k = 0, 1, 2, \ldots$

Hint: Let $d_k := a_k h_k$, $\tau_0 := 0$, $\tau_{k+1} := \tau_k + d_k$. First show that $a_k - a_{k+1} = d_k$ and hence $\tau_k = 1 - a_k$. Then show that $d_k = \frac{1}{2} d_{k-1}^2/(1-\tau_k)$. On the other hand show from the definition of $t_k$ that $t_{k+1} = \frac{1}{2} t_k^2 - h_0 t_{k-1}$ which implies $\frac{1}{2} (t_{k+1} - t_k)^2 = \frac{1}{2} t_k^2 - h_0 t_{k-1} + h_0$ and hence $t_{k+1} - t_k = \frac{1}{2} (t_k - t_{k-1})^2/(1-t_k)$.

(d) Complete the proof of the Newton-Kantorovich theorem from class: Let $a_k = \omega_k^{-1}$, w.l.o.g. $\omega_0 = 1$. We showed that $\|x_{k+1} - x_k\| \leq d_k$ for $k = 0, 1, 2, \ldots$ with $d_k$ as defined in (c). Show that $\|x_0 - x_*\| < \rho_-$ for $k = 0, 1, \ldots$. Show that $x_k$ form a Cauchy sequence which converges to $x_*$ with $\|x_0 - x_*\| \leq \rho_-$. Show that the convergence is r-linear for $h_0 = \frac{1}{2}$, and r-quadratic for $h_0 < \frac{1}{2}$. 

3. Let $x_1 = \cos(x_1 + x_2)/3$, $x_2 = \sin(x_1 - x_2)/3$ using the Newton method with initial guess $x^0 = (0,0)$. Show that you can apply the Newton-Kantorovich theorem and (a). What does the theorem say about the location of $x_*$?

(c) Perform the Newton iteration from (b) in Matlab with 310 digits accuracy (see example code on the course web page). How many iterations do you need to obtain $\|x^k - x_*\|_{\infty} / \|x_*\|_{\infty} \leq 10^{-300}$? How can you see the convergence order in your computations?

(d) Repeat (c) for the Broyden iteration with $x^0 = (0,0)$, $B_0 = I$ (identity matrix). How many iterations do you need to obtain $\|x^k - x_*\|_{\infty} / \|x_*\|_{\infty} \leq 10^{-300}$? What convergence order do you observe numerically? Does $B_k$ converge to $F'(x_*)$? Check the convergence order of $\|(B_k - F'(x_*)) s_k\| / \|s_k\|$.